Einstein's equation and Schwarzschild metric

Exercise 1 *No gravity in 2D and 3D spacetimes*

1) Let's consider a *ND* spacetime, one temporal dimension and *N-1* spacial dimensions, $N\geq 2$. Use the Einstein's equation to show that, in *ND*, the Ricci tensor is identically null in vacuum.

2) 2D (t,x): there is only one independent Riemann curvature tensor component. Demonstrate the nullity of this component.

6) 3D (t,x,y): there are six independent Riemann curvature tensor components. The demonstration can be done in the local inertial frame with rectangular coordinates. Demonstrate the nullity of the Riemann components.

Only in 4D, or more dimensions, gravity can exist.

Exercise 2 *Black and White Holes*

Schwarzschild metric: $$ds^2=(1-r_S/r)c^2dt^2-\frac{1}{1-r_S/r}dr^2-r^2d\Omega^2 \quad \text{with} \quad r_S=\frac{2GM}{c^2}$$

The Schwarzschild metric works only if $r_S< r <+\infty$. We are looking for others coordinates with no singularity at $r=r_S$. To probe the whole spacetime let's use geodesics to find other coordinates and other metrics for the spherical body.

1) Find the equations $ct=f(r)+cst$ of the radial inward light rays in the Schwarzschild system.

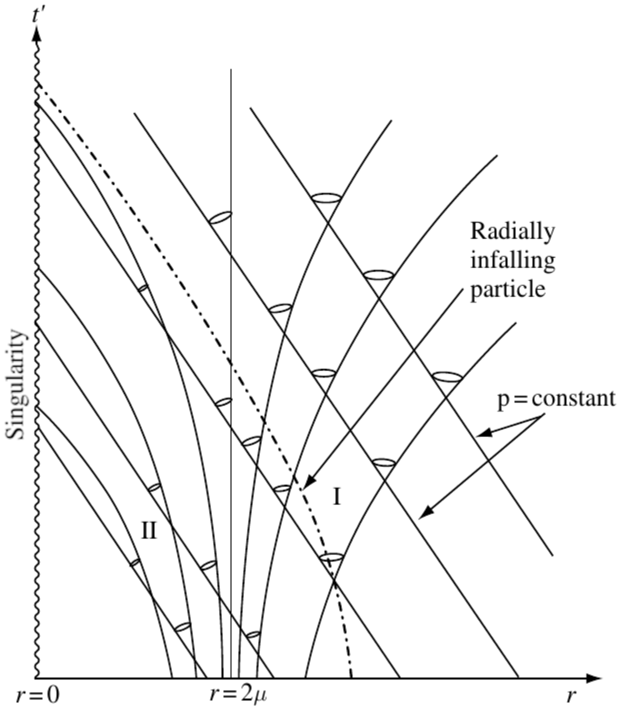
2) The integration constant can define each geodesic as a line coordinate, then let's set $p=cst$ as a new coordinate. Find the following metric:

$$ds^2=\left(1-\frac{r_S}{r}\right)dp^2-2\,dp\,dr-r^2d\Omega^2$$

This is the *Eddington* coordinate and metric. This metric is regular at $r=r_S$ and works for the whole range $0< r <+\infty$. The singularity at $r=r_S$ is not essential.

3) But it is unusual to use a null coordinate (light geodesic), so, it is also useful to use the related timelike coordinate $ct'=p-r$ called the *Advanced Eddington coordinate*. Find the following metric:

$$ds^2=\left(1-\frac{r_S}{r}\right)c^2dt'^2-2\frac{r_S}{r}c\,dt'\,dr-\left(1+\frac{r_S}{r}\right)dr^2-r^2d\Omega^2$$



Again it is regular and $0< r <+\infty$. Draw the minkowski diagram.

4) But what happens if the *outgoing* radial light rays are considered? The *Retarded Eddington coordinate* $ct^*=q+r$ is then set, a new metric is established and one finds in the minkowski diagram that the outgoing radial null geodesics are continuous straight lines at 45° but the ingoing rays are tending to $t^*=+\infty$ at $r=r_S$. So, the surface $r=r_S$ act as a one-way membrane only from inside to outside: this is a W*hite Hole*. The whole Schwarzschild spacetime is now revealed!

Hobson, *General relativity, An Introduction for Physicists*, p254.
https://it.wikipedia.org/wiki/Coordinate_di_Eddington-Finkelstein