Lectures on Relativity

Non-Inertial Special Relativity and General Relativity

Licence 3 - Master - Engineering schools

2025 - Mathieu ROUAUD

Foreword

This document compiles the courses taught over one semester at *Universitad Mayor San Simon* in Cochabamba, Bolivia. There are many excellent books and PDFs on general relativity, but here is what characterizes and can sets this one apart:

1. Numerous exercises with detailed answers accompany the lectures.

2. Special relativity is often studied only in the inertial context, but in the absence of gravity, it also applies to non-inertial reference frames. We study the cases of uniformly accelerated rectilinear motion and rotating frames. This is a very rich subject that allows for a smooth transition to general relativity, by familiarizing oneself with the concept of coordinate time, tensor calculus, and Lagrangian formalism.

3. In cartography, we use various metrics to represent the sphere on a plane. The shortest path is no longer necessarily a straight line. We use this analogy to understand geodesics that maximize proper time in general relativity.

4. Some chapters are based on a research paper and help to prepare to adopt the researcher's approach.

Good reading!

Always happy to receive mail from my readers: ecrire@incertitudes.fr

Recommended prerequisites:
Math and mechanics L1, special relativity, and electromagnetism (Maxwell's equations).

For special relativity, reference will regularly be made to the book by the same author:

[SR] *Special Relativity, A Geometric Approach, followed by the conference Interstellar travel and antimatter., 2020, 536 pages.*

Files on github.com/Mathieu-ROUAUD

First Lecture

Special Relativity

Mathieu : Hello. Let's begin.
Since birth, we have lived in the social fiction of absolute time. However, as we will explain today, time is relative. This new concept needs to be clearly explained, otherwise it can lead to a lot of confusion. For example, the relativity of time in no way calls causality into question. It is impossible to travel into the past, or for someone from the future to visit us. However, as we will see later, we can travel into the future of others.

Arkaitz : Sir, but why not start directly with general relativity? We have already had special relativity classes since the bachelor's degree.

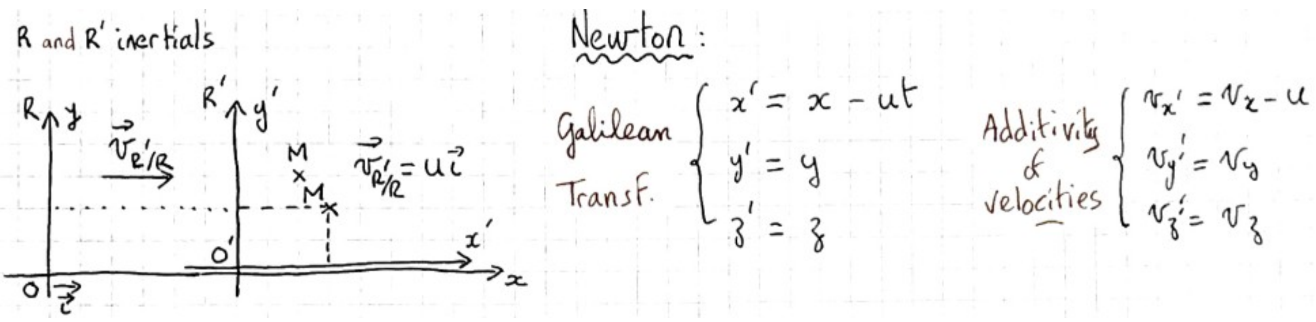
Mathieu : Indeed, it may seem unnecessary to repeat what we already know. But every teacher has a different approach to special relativity. Today, for example, we will look at a visual tool for determining time dilation that you have never seen before: the time triangle. In addition, this will be an opportunity for us to introduce notation, and in the third lecture we will introduce the tensor approach necessary for general relativity. With a new definition of special relativity that breaks away from the historical bias of light, and which, thanks to symmetries, has made it possible to construct the Standard Model.

Cristian : All right, and when will we begin general relativity itself?

Mathieu : We have a total of 18 lectures, only the last 6 of which will deal exclusively with general relativity. Before that, we need to complete your lectures on special relativity. You are not yet familiar with tensor calculus, and, for example, we are going to put Maxwell's equations into tensor form, which will then guide us by analogy for gravitational waves. We will discuss special relativity in accelerated reference frames, which will also be new to you. This will be an opportunity to introduce metric vector spaces and the Lagrangian.

Any other questions?

Okay, then... Historically, the concept of relative time is the result of a contradiction between two theories. On the one hand, we have Newton's theory from 1687, which describes the motion of objects and matter in general, and on the other, Maxwell's theory from 1864, which describes the behavior of light in terms of electromagnetic waves. These two theories, which have been thoroughly verified experimentally, state their laws in inertial reference frames. However, the first predicts the additivity of velocities, which contradicts the second theory's assertion that the speed of light in a vacuum is constant.



at t=0, O=O'

Maxwell (vacuum):

- Non invariance of eq. under Galileo

$R: \vec{\nabla}.\vec{E} = 0 \xrightarrow[\text{SR p433}]{\text{dem.}} R': \vec{\nabla}'.\vec{E}' \neq 0$

- $\vec{\nabla} \wedge \vec{E} = -\partial \vec{B}/\partial t$ and $\vec{\nabla} \wedge \vec{B} = \mu_0 \varepsilon_0 \, \partial \vec{E}/\partial t \Rightarrow \Box \vec{E} = \vec{0}$

$\frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0$ with $c = 1/\sqrt{\mu_0 \varepsilon_0}$

An idea, to put everyone in agreement and save the two theories, would be that light propagates in a particular referential named *ether*. However, the experiments of Michelson-Morley, carried out as early as 1887, prove that ether, supposed medium of propagation of light, does not exist.

Which leads Einstein to propose a new theory in 1905. This theory is much more satisfactory because it integrates matter and light in the same framework. Here are the two main **postulates** of Einstein's theory of special relativity:

1. *Principle of relativity*: The laws of physics are the same in all inertial frames.

   Note the word *physics* and not *mechanics* as with Newton:
   physics = mechanics + electromagnetism = matter + light

2. *Universality of the speed of light*: The speed of light in a vacuum is the same for all inertial observers.

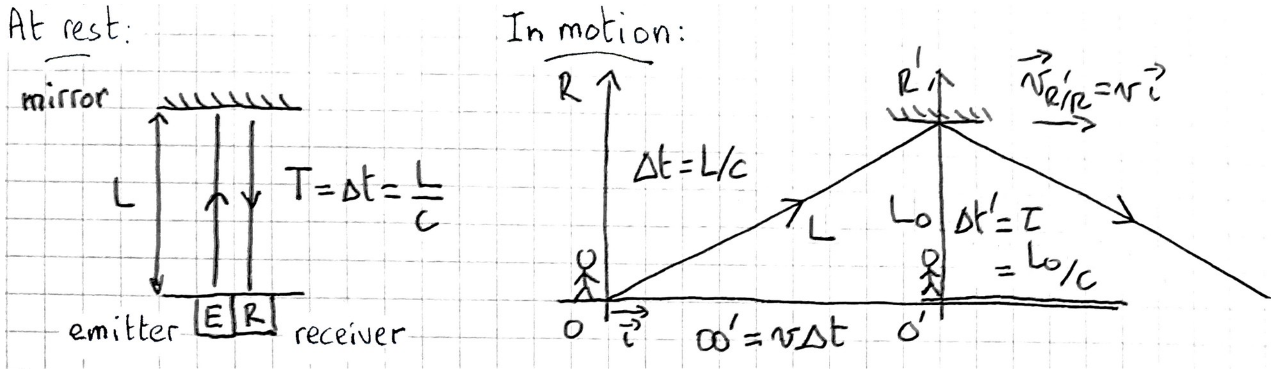
   c = 299 792 458 m/s (universal constant fixed by special relativity)

Arkaitz : It's surprising that light plays such a central role. Gravitational waves, as verified in 2017, also go to velocity c, and any particle of zero mass. I find it strange that the content and the container are linked.

Mathieu : Yes, the second postulate is actually useless. It is a consequence of simple assumptions about symmetries: 3 dimensions of space, one of time, homogeneity of time and space, isotropy of space, principles of relativity and causality. These assumptions seem natural and allow to find the Lorentz transform in a few pages of calculation[1]. The constant c then appears as a structural constant specific to space-time, unrelated to any physical object.
It is interesting to note that Newton had discovered the mathematical tools of differential calculus that could have allowed him to find this most general change of coordinates between two inertial frames of reference. Without realizing it, he restricted himself to the limit case where c is infinite, and space and time, absolute and disconnected.

Let's continue nevertheless with the 2 historical postulates of Einstein and describe the ***light clock***. Let's imagine a clock that uses light rays. A light ray is emitted, reflects on a mirror and returns to the same point after a *2T* duration:

1 Well explained and demonstrated by Jean-Marc Lévy-Leblond, « One more derivation of the Lorentz transformation », American Journal of Physics 44 (3) 271 - 277 (1976).



The clock is at rest in *R'* and moves at the velocity $\vec{v}$ with respect to *R*. The distance traveled by the ray in *R* is therefore larger, while keeping the speed constant:

$$\text{so} \quad L^2=L_0{}^2+OO'^2 \quad \Rightarrow \quad c^2\Delta t^2=c^2\Delta t'^2+v^2\Delta t^2 \quad \Rightarrow \quad \Delta t=\frac{1}{\sqrt{1-v^2/c^2}}\Delta t'$$

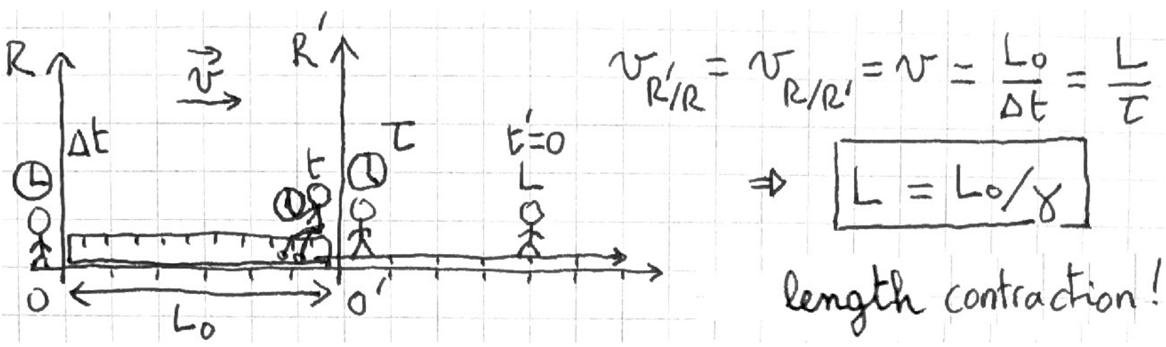
$$\gamma=\frac{1}{\sqrt{1-\beta^2}} \quad , \quad \beta=\frac{v}{c} \quad , \quad 0\leq\beta<1 \quad , \quad \gamma\geq 1 \quad \text{then} \quad \boxed{\Delta t=\gamma\tau} \Rightarrow \textbf{time dilation}!$$

Adriana : ... so an observer of *R* sees the clock of *R'* going more slowly, and it's relative, that of *R'* also sees a clock at rest in *R* going more slowly.

Mathieu : Yes, the situation is symmetrical. But, no, it's not what is seen but what is measured. We will see later the methods for measuring times and distances. Here the time dilation is the same for frames that move closer or further away. On the other hand, for what we see, it is different, because it is necessary to take into account, in addition, the propagation of light. We will see that in a next class, it's the Doppler effect.

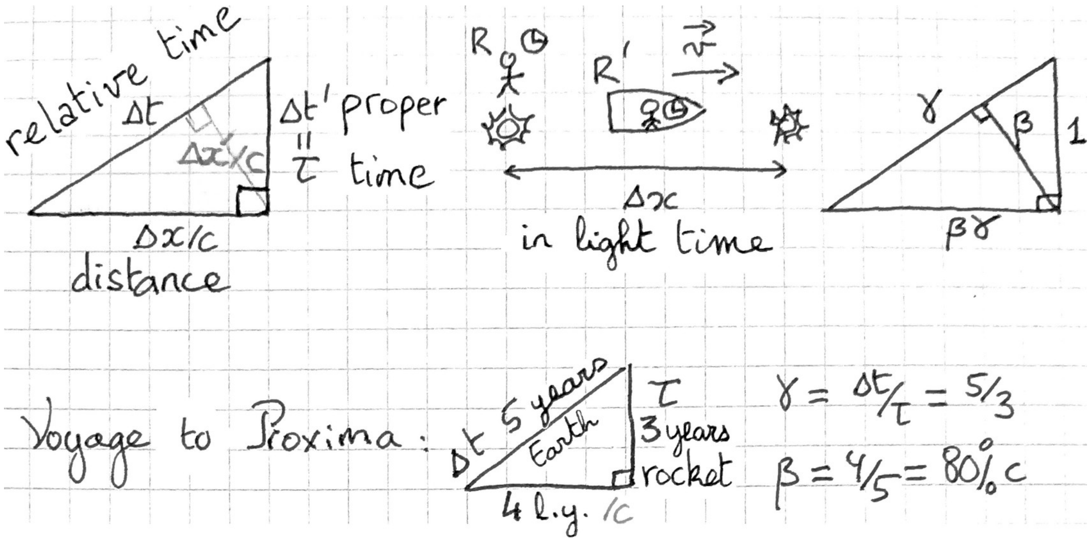
Time dilation can make one think of a spatio-temporal perspective effect. An analogy, from where I am, I can "hold" a student at the back of the room between my thumb and forefinger, he is very small, like a Smurf! But, he also sees me as tiny, it's mutual. In this case, we have a spatial perspective effect. And, of course, we actually both have the same size, it's just an illusion.
In the case of relativity it is more complex, because, unlike space, time is not isotropic.

Where were we? Ah yes! Moreover, time dilation is associated with a **length contraction**. Let's consider a ruler at rest in *R*, and, as before, a clock at rest in *R'*:



To measure a length in a given reference frame, it is necessary to identify the position of each end at the same moments.

Relying on the Einstein's light clock, let us now present the **time triangle** that summarizes these effects:



We are ready to address the twin paradox: imagine twins who are 20 years old, one stays on Earth, and the other goes in his spaceship at a speed of 80% of *c* for Proxima Centauri located 4 ly. The traveling twin arrives, on the exoplanet *Proxima b*, 3 years later, while it has been on Earth for 5 years. When he returns to Earth, it flows again 3 years in the ship and 5 years on Earth. So, when they meet again, the traveling twin will be 26, while the sedentary twin will be 30! If we imagine the situation symmetrically, it seems paradoxical. But one is undergoing acceleration to return, unlike the other. And ultimately, this is not just an illusion, a perspective effect, but a physical and measurable difference.

Luna : Here, the rocket always travels at the same speed, which is not realistic! You can't go from 0 to 80% of *c* instantly, and even less turn around suddenly by reversing speed. However, it is because of these accelerated phases that the traveler ages less than someone on Earth. Because the rest of the time, the situation is symmetrical. Do you think it is conceivable that such a rocket could be built one day?

Mathieu : Yes, excellent points. For greater realism, in the next lesson we will consider a uniformly accelerated rocket. The rocket's frame of reference is then non-inertial, and Einstein's postulates do not apply. The metric is no longer Minkowskian but of Rindler. Such a rocket is conceivable, but with many “ifs”. If we were able to collect, store, and use antimatter efficiently, a rocket the size of a *Saturn V* would enable a trip to *Proxima b* with an acceleration of 1 *g*, taking 3 years for the traveler and 5 Earth years.

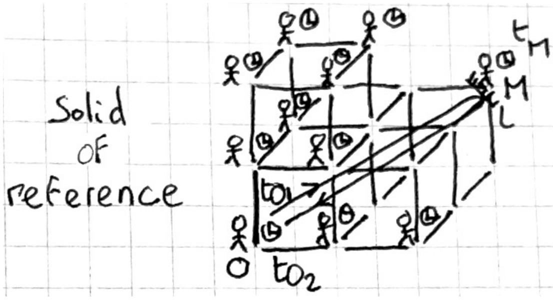
To conclude today, let's look at a fundamental point: the definition of an inertial reference frame in special relativity, the only reference frames in which the postulates are verified.

To locate ourselves in space, we proceed as in Newtonian mechanics. We place one-meter rulers perpendicular to each other to form a grid that covers the entire space. We also want to have clocks at each node of the grid, with all clocks constantly synchronized. In addition, we imagine an observer at each node that records events occurring at its level using labels (ct, x, y, z).
The difference with classical mechanics concerns the synchronization of clocks. If you place all the clocks at node *O*, set them to zero, and then move them to their respective nodes, they will become desynchronized during the motion, because they will then have a non-zero velocity and their time will dilate.

Hence ***Einstein's synchronization method***: the clocks are first placed at their nodes. Arbitrarily, the clock at O, H_O, is chosen as a reference. At time t_{O1} on H_O, the observer at O emits a light beam toward a clock at M to be synchronized. The ray is reflected by a mirror at M and returns to O. When the ray is reflected, the observer at M notes the time t_M indicated by his clock. When the ray returns, O notes t_{O2}. O communicates to M the values t_{O1} and t_{O2}, and M deduces the synchronized t_M.

By symmetry of the trajectories between the outward and return paths (isotropy of space):

$(t_M)_{sync} = \frac{t_{O1}+t_{O2}}{2}$, H_M to be advanced by $(t_M)_{sync} - t_M$.



Cristian : Okay, so the clocks are initially synchronized. But what guarantees that they will remain so?

Mathieu : The homogeneity of space and time required in an inertial reference frame. No point should be favored; due to symmetry, there is no reason why one clock should start behaving differently from the others.
A major advantage of special relativity is that we have an experimental method for determining whether a reference frame is inertial: once synchronized, clocks remain synchronized. If the reference frame is non-inertial, they become desynchronized over time.
In Newtonian mechanics, time is absolute, and a reference frame is Galilean (inertial) if Newton's laws are verified, but the laws are only verified in a Galilean reference frame, so the definition is circular. In classical mechanics, if an object has a curved trajectory, we do not know whether this is due to the action of a force or the non-inertial nature of the reference frame.
In relativity, we use the clock crystal, and this clarifies the issue.

When we talk about measuring time and length, we are referring to this reference solid and its associated clock crystal. Each reference frame has a set of observers, each with their own rulers and clocks. Following an experiment, the observers gather all the data on the various events they have collected to reconstruct what happened[2].

Consider a ruler of length L_0 and a clock with period τ, both at rest in R. When a frame R' moves with the velocity $\vec{v}$, an observer in R' will sometimes coincide with a given observer in R. For example, there are two observers A_1 and A_2 of R at rest at each end of the ruler, and in R' at the same time t' two observers A'_1 and A'_2 will coincide with them:

$$\text{for all } t,\ L_0 = A_1 A_2 \quad ; \quad \text{same } t',\ L = A'_1 A'_2 = \sqrt{(x'_{A'_2} - x'_{A'_1})^2 + (y'_{A'_2} - y'_{A'_1})^2 + (z'_{A'_2} - z'_{A'_1})^2}.$$

For the clock, an observer of R is constantly at rest, and two observers of R' will coincide, one at the beginning and the other at the end of the period:

$$\text{same } x,\ \tau = t_2 - t_1 \text{ (one clock)} \quad ; \quad \text{different } x',\ \Delta t' = t'_{A'_2} - t'_{A'_1} \text{ (two clocks)}.$$

And it is according to these protocols that we have $L = L_0/\gamma$ and $\Delta t' = \gamma\tau$.

Thank you for your attention.

2 It should also be noted that clocks at rest are thus synchronized in a given inertial reference frame, but *a priori* they do not appear to be synchronized with respect to another reference frame in which they are in motion.

Summary of the notions covered today:

Lecture 1 - SPECIAL RELATIVITY

- Einstein's Postulates
- Light clock: Time Dilation
- Length Contraction
- Time Triangle
- Twin Paradox
- Definition of an inertial frame: Einstein's synchronization method

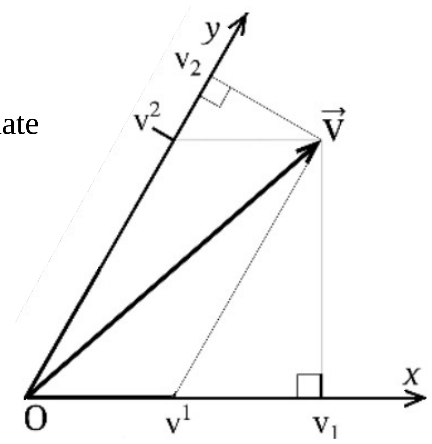
To review and practice:

- This lecture covers Chapter 1 of the SR book, which includes exercises 1 to 8 with their answers.

Minkowski diagrams

Mathieu : Hello.
Today we present a graphical tool proposed in 1908 by Hermann Minkowski.
In relativity, position in space-time is specified by four coordinates grouped together in the position four-vector $\tilde{x}$ with components x^{μ} where $\mu=0,1,2,3$. Here, the index is placed above, and it is said to be *contravariant* (it is not a power!).
In this course we will take: $\tilde{x} = (x^0=ct\,,\ \ x^1=x\,,\ \ x^2=y\,,\ \ x^3=z)$.
When a particle moves through space-time, the set of successive events constitutes a *worldline* in a 4-dimensional space of a reference frame *R*.

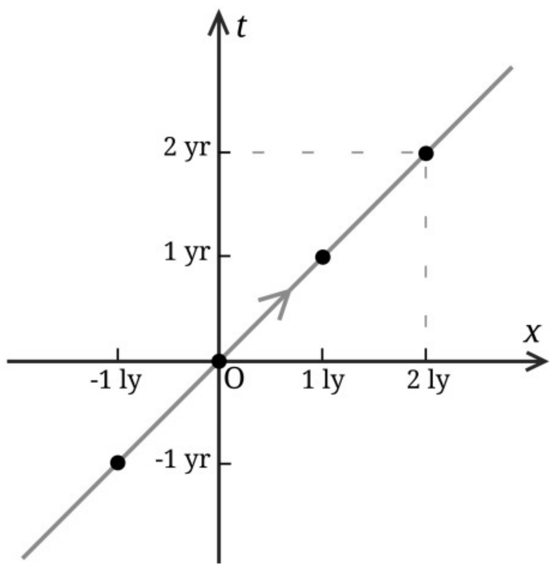
Adriana : ... what if we put the index at the bottom?
Would that change anything?



Mathieu : For a vector of $\mathbb{R}^3$ projected on an orthogonal coordinate system, this does not change anything, but in general it is different.
In a non-orthogonal coordinate system, there are two possibilities for projecting a vector: the contravariant components v^{μ} are projected parallel to the axes, and the *covariant* components v_{μ} are projected perpendicular to them:

More detailed explanations will be given in the next lesson on vector spaces.

So, an event is denoted *E(ct, x, y ,z)*, and in many cases it is possible to limit ourselves to a straight-line motion and a single spatial dimension: *E(ct, x)*.
In a Minkowski diagram, the time axis *t*, or *ct*, is always vertical. On the *x*-axis, the units of distance are light-times. As *c* is a universal constant, it can be used to unambiguously match a time to a length. For interstellar travel, we choose the year and the light-year.
An object at rest at *O* is represented by a vertical line, the time axis, an worldline pointing upwards, not only the event *O(0, 0)* at the origin for *t=0.*

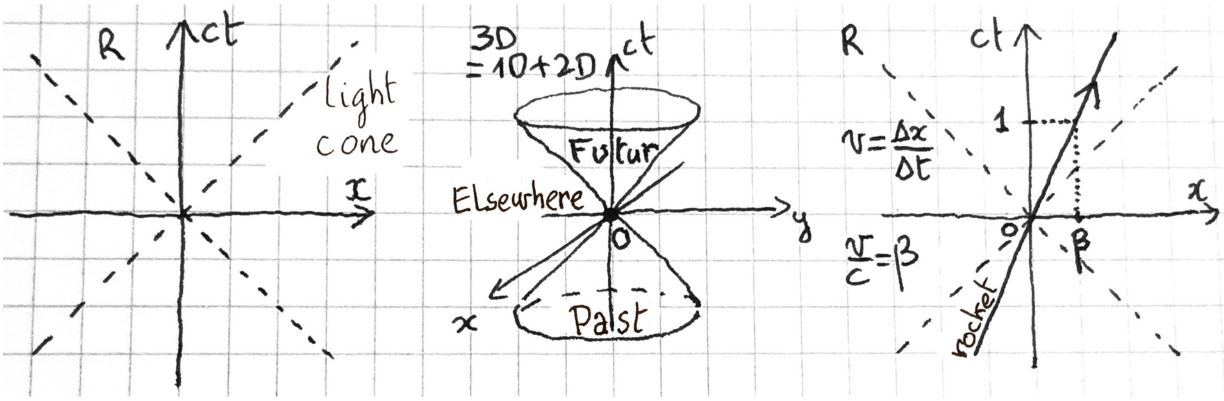


A light pulse traveling to the right and at *O* at *t=0*, will be at one l.y. in one year, and was at *x=-1 ly* one year ago, from which the light's worldline represented in gray according to the first bisector.

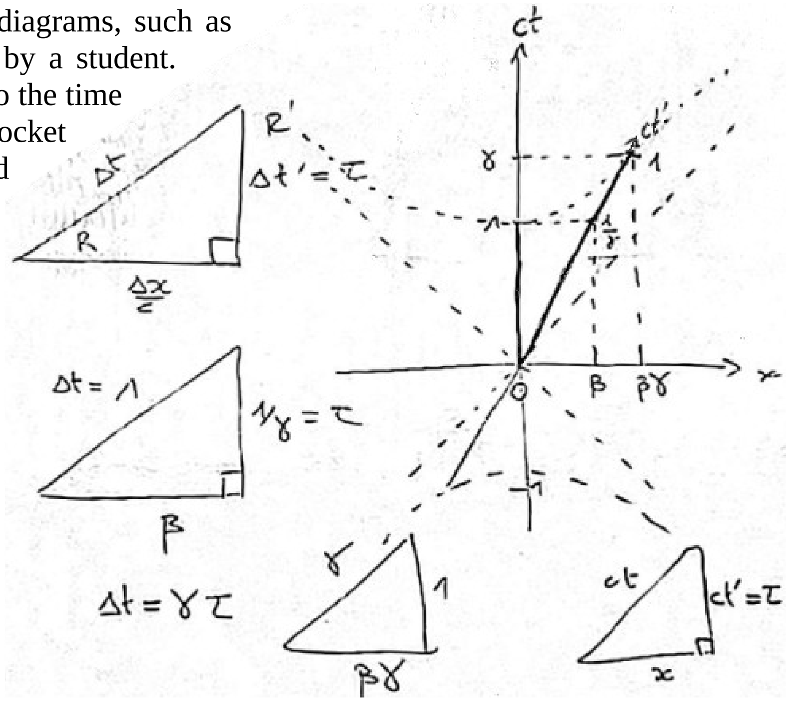
On a space-time diagram, the worldlines of light rays passing through *O* are always represented by dotted lines. They define the light cone (next page).

Arkaitz : Actually, the cone only appears in 3D!

Mathieu : Yes! And, in general, it's 4D with a hypercone!



On the drawing on the right, you see the worldline of an object moving at constant speed *v* along the *x*-axis. For *ct=1*, you can see its speed β, as a percentage of *c*. From *O*, all points inside the cone can be connected by a worldline, but there is no possible connection with those outside because the speed is less than *c*.



The time triangle can also be linked to diagrams, such as this picture of a board taken in Bolivia by a student. The rocket's worldline in *R* corresponds to the time axis *t'=τ* in the reference frame *R'* of the rocket at rest. And according to the second triangle, we read, in *R*, *t'=1/γ* when *t=1*. And in the third triangle, multiplied by a factor γ, time dilation appears explicitly. We have two segments of length 1, which appear to be of different lengths, along *ct* or *ct'* !
The reason is that we are not in a Euclidean space but in a hyperbolic space. Indeed, according to the last triangle:

$$c^2t^2 - x^2 = c^2\tau^2$$

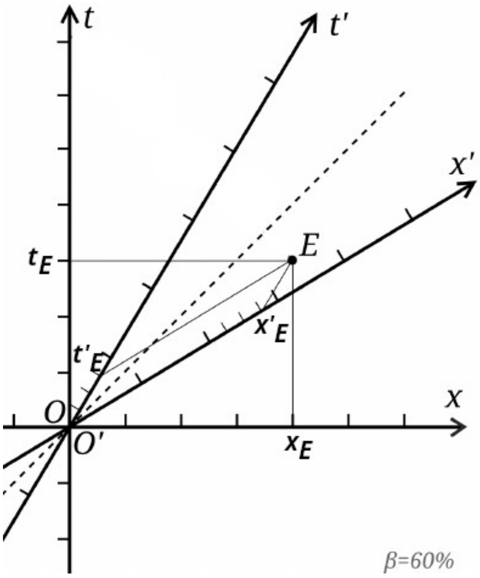
We recognize the equation of a hyperbola. Here, the invariant is the proper time τ, and the curves of constant *proper time* are hyperbolas. In Euclidean geometry, length is invariant, and we are at a constant distance on a circle: $x^2 + y^2 = L^2$. Representing Euclidean geometry on a sheet that is itself Euclidean is fairly simple, but hyperbolic geometry on a Euclidean sheet requires careful attention.

Adriana : But... that's very strange! The lengths are not what we think they are. And the angles?

Mathieu : Also... For example, for any point *M* on the hyperbola, the line *(OM)* is locally orthogonal to the hyperbola, and not only when *M* is on the *ct* axis. By analogy in Euclidean geometry, the radius of a circle is always perpendicular to the circle. We will see later how to perform a scalar product in Minkowski space.

Let us now represent the twins' experience in a diagram (next page). Our reasoning is based on the galactic reference frame, where the stars are assumed to be fixed. The worldline of our sun, and therefore of the twin on Earth, corresponds to the time axis. That of Proxima Centauri is parallel to it, such that *x=4 ly*. For the twin traveler's round trip, his broken world line clearly shows the evolution of his proper time; he is 4 years younger than his brother when he returns.

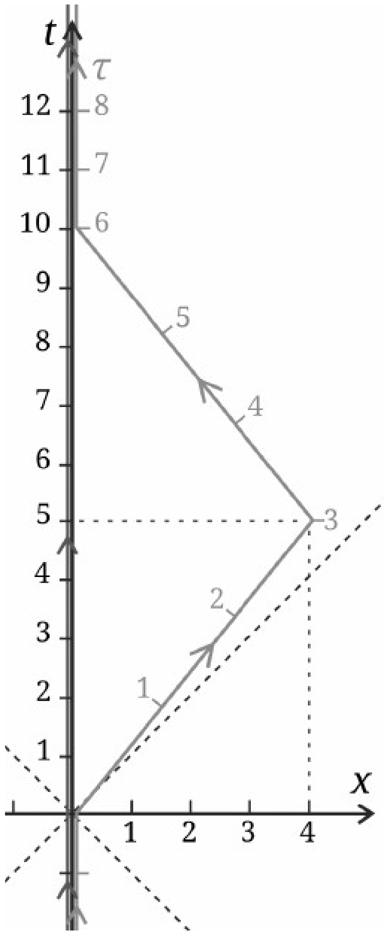
Let's see how to perform a ***change of reference frame***. For a frame *R′* moving at a velocity $\vec{v}=v\,\vec{i}$ with respect to *R*, the axis *ct′* corresponds to the world line of *O′*. For the axis *x′*, we use the invariance of the speed of light: the axes *x′* and *ct′* must be symmetrical with respect to the light cone.



$$E=(ct_E,\ x_E)=(ct'_E,\ x'_E)$$

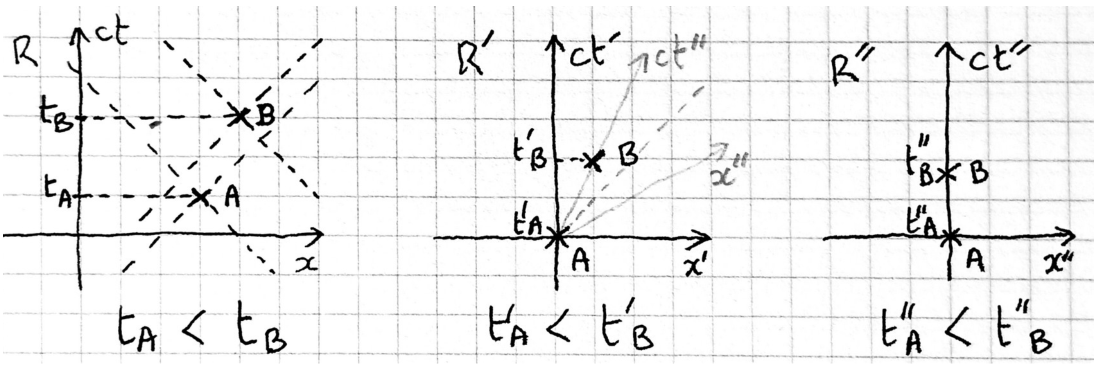
$$(3\ ;\ 4)_R=(0{,}75\ ;\ 2{,}75)_{R'}$$

The frame *R* is orthonormal. An inexperienced Euclidean mind might think that *R′* is not orthogonal. However, it is: two lines that are symmetrical with respect to the cones of light are orthogonal.
Furthermore, if we visualize the hyperbolas, we understand that the norm is the same.



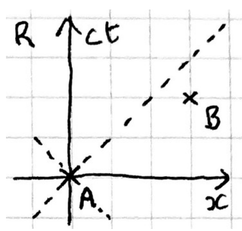
Now let's talk about ***causality***:
If event *A* precedes event *B* in *R*, is this true in all reference frames *R′*?

<u>First case</u>, *A* and *B* are in their respective light cones:



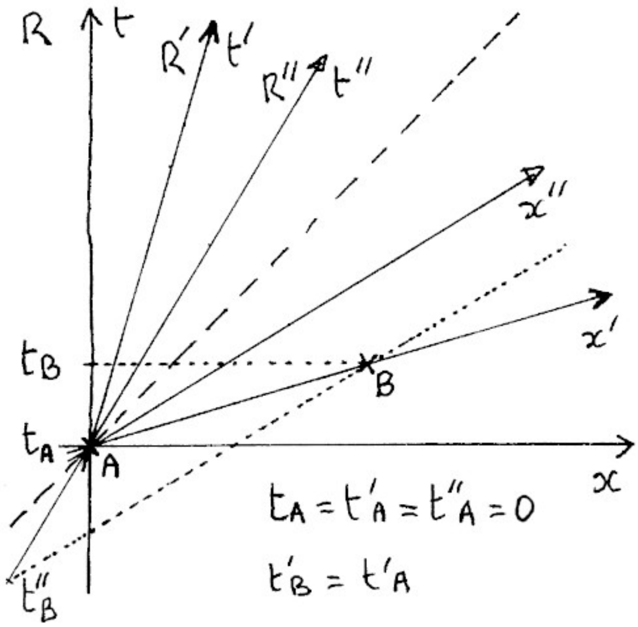
From *R* to *R′*, it is a simple translation. Then, we can always perform a *Lorentz transformation* from *R′* to *R″*, such as *A* and *B* are at rest. The chronology is respected, and we can return to a frame *R* at any velocity relative to *R″*. Also in *R″*, this is the simplest case for measuring dates: a single clock is needed, and the anteriority of *A* over *B* is measured directly. It follows that: $\Delta t''_{AB} = \tau < \Delta t'_{AB} = \Delta t_{AB}$, where the proper time τ is the minimum duration.

<u>Second case</u>, *A* and *B* are not in their cones:
In *R*, *A* comes before *B* because $t_B > t_A$.
In this case, it is impossible to find a frame of reference in which *A* and *B* are at rest, because there is no worldline connecting the two events.
In *R′*, such that *x′=(AB)*, see next page, *A* and *B* are simultaneous!
In *R″*, *A* comes after *B*!
But whatever the frame of reference, *B* is outside the cone of *A*, and has no causal link with it.



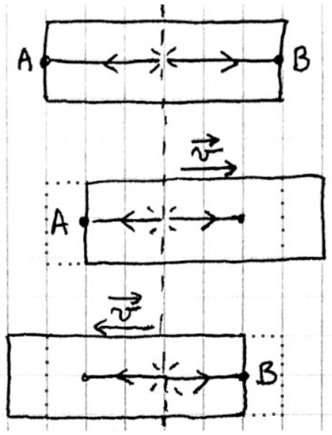
Conclusion, when two events have a possible cause-and-effect relationship, the chronology is respected for all observers, and the principle of causality is therefore verified.
When two events cannot influence each other, it does not matter whether one occurs before the other or whether they occur simultaneously. The chronology then depends on the observer and has no consequence on real life. They have no causal link, so the principle of causality (preservation of the direction of the causal link) does not need to be verified.

Thus, in relativity, causality is just as rigorously verified as in Newtonian mechanics. And the fact that time and simultaneity are relative does not change anything.



Luna : I remember the example where a flash of light is emitted in the middle of a train car, and the light rays reach both ends of the car at the same time. But if the car moves to the right, since the speed of light is constant, the left ray arrives first. And if it moves to the left, it's the right ray. The order of events changes!

Mathieu : Yes, good example. Whereas in classical mechanics simultaneity would be verified in all reference frames, due to the additivity of velocities, one ray would travel at *c*-*v* and the other at *c*+*v* compensating for the motion of the wagon.

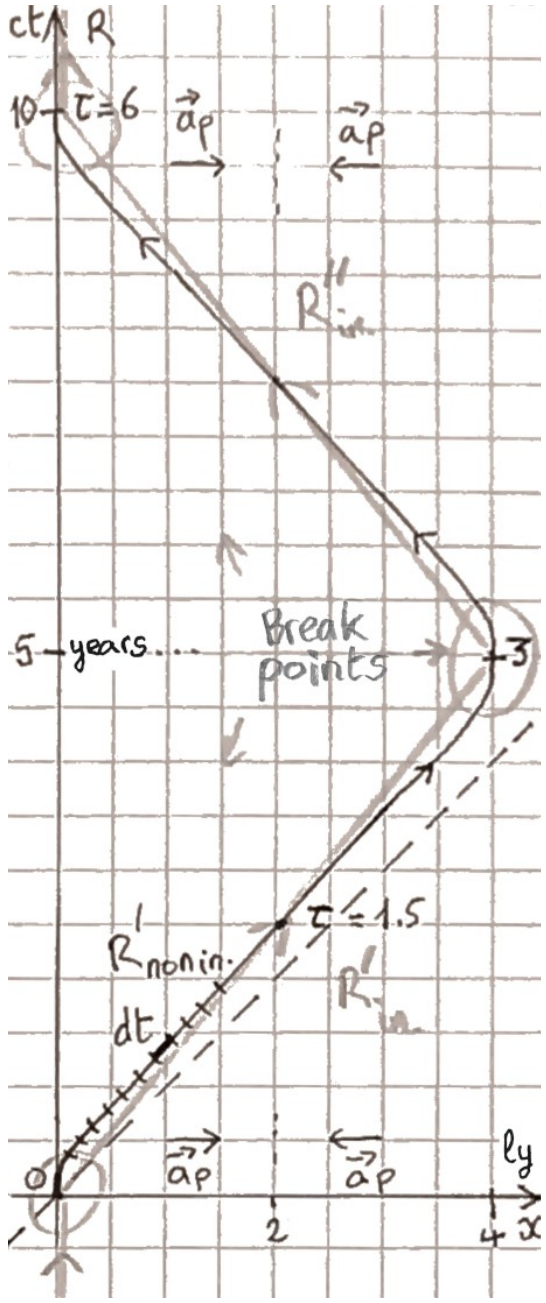


To conclude today, let's return to the twin paradox.
For greater realism, we replace the broken line with non-physical angular points, with a curve with smooth variations (next page). The rocket travels along a straight line at a constant proper acceleration $\vec{a}_p$, and turns around halfway to arrive at Proxima at zero speed. The passengers experience an artificial gravity $\vec{g}=-\vec{a}_p$. The rocket's reference frame *R*' is non-inertial, nevertheless at any moment we can make a second inertial rocket with the same velocity $\vec{v}$ and zero acceleration coincide. For a short duration, as long as the rockets are roughly immobile relative to each other, the clocks of the two rockets will keep the same pace. These ideas that make it possible to deal with non-inertial frames of reference in special relativity are summarized in the following two new postulates:

3. *Equivalence principle*: Locally, at any point in space-time, there is always a coincident inertial reference frame.

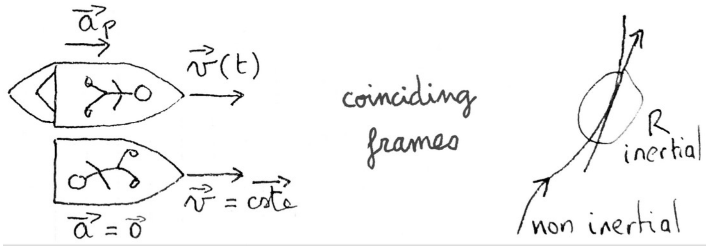
This frame is locally minkowskian.

4. *Clock hypothesis*: Two clocks that coincide with the same speed will have the same time dilation, regardless of their accelerations or rotations.



The time elapsed in the rocket can be calculated from the frame *R*. On a worldline:

$$dt=\gamma\, d\tau \qquad \text{and} \qquad \tau=\int d\tau=\int\sqrt{1-\frac{v^2}{c^2}}\,dt$$

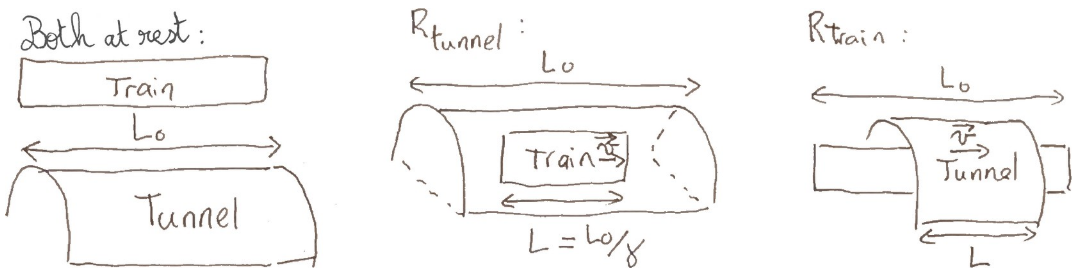


Cristian : Will these new postulates remain true in general relativity?

Mathieu : Yes. And for gravitation, we will complete these four postulates by adding *Einstein's equation*, which links the *metric* to the *energy-momentum tensor*.

The formula for *τ* shows us how, from the perspective of the observer on Earth, the traveler ages more slowly than he does. *dτ* is smaller in relation to *dt* as *v* increases. On the worldline of accelerated motion, it is mainly outside the breakpoints zones that the slowing of aging relative to Earth is most significant.

Ah, but since we still have a few minutes left, we can finish with a nice paradox, involving a relativistic train passing through a tunnel. Both the train and the tunnel have a rest length of $L_0 = 1000\ m$. Let's assume that the tunnel is at rest and that the train is arriving with a γ of 2. The train, contracted by a factor of 2, easily enters the tunnel. We can even close the doors at both ends of the tunnel at the same time to trap the train for a microsecond!
What about from the point of view of the train at rest? Now it is the tunnel's turn to be contracted by a factor of 2, and it is impossible to put the train inside...
How can we resolve this apparent contradiction?



Summary of the notions covered today:

Lecture 2 - MINKOWSKI DIAGRAMS

- Event and worldline
- Light cone and hyperbolic space
- Change of reference frame
- Principle of Causality
- Relativity of Simultaneity
- Back to the Twin Paradox
- Clock Hypothesis
- Equivalence Principle
- Train Paradox

To review and practice:

- Chapter 2 of the SR book, exercises 1 to 4 with their answers, and Chapter 3.
- A worksheet follows.
- Represent the paradox of the flash of light emitted in the middle of a train car, then that of the train and the tunnel, using Minkowski diagrams.

Worksheet No. 1

Time dilation and spacetime diagrams

Exercise 1

In the galactic year 2125, you undertake the Sun-Sirius voyage to study this binary stellar system.
After 8 years in your spaceship, you land on an exoplanet around Sirius A.
In what galactic year are we then, and what was the speed of your ship?
In 2128, three years after you left, the earthlings send you a radio message.
When do you receive this message?
When you arrive on the planet, you send a picture of your new home. When will they receive it?
Draw a Minkowski diagram that shows the various worldlines.

Sun-Sirius distance in the galactic frame of reference: 9 ly.

Exercise 2

Carlos, 40, has to travel, for a secret mission, to the star *Kappa Ceti*, 30 ly away in the equatorial constellation of *Cetus*. The Carlos' vessel is a β95 model. When he left, is daughter Maria, is 15 years old. Maria sends a letter each month to her father with an electromagnetic wave. How often does Carlos receive letters during his trip to *Kappa Ceti*?
Just arrived at *Kappa Ceti*, Carlos came back to the Sun. Show the Minkowski diagram.
When they meet again, how old are the daughter and the father?

β95 model: $\beta=3/\sqrt{10}$

Exercise 3

In Japan, the Shinkansen train takes 3 hours to travel from Tokyo to Shin-Aomori, 675 km away. Initially, two atomic clocks are synchronized at Tokyo. One remains on the station, and the other makes a round trip to Shin-Aomori on the train. Back to Tokyo, what is the desynchronization between the two clocks? Is it measurable? The atomic clocks accuracy is 0.5 ns/day.

The terrestrial reference frame is supposed inertial.

Exercise 4

A plane flies around the Earth straight on the equator toward the east, covering 40,000 km at 1000 km/h ground speed. At the beginning, two atomic clocks at rest are synchronized. One remains on the ground, and the other makes the round trip on the plane around the world. Back to the initial point, what is the desynchronization between the two clocks? What if the plane travels toward west?

Altitude variations are ignored. Geocentric frame considered inertial.

Exercise 5

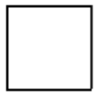
What is the probability for a muon created at 15 km with a vertical velocity of 0.995c to reach the sea level? Chacaltaya, 5240m?

Muon mean life at rest: 2.2 µs

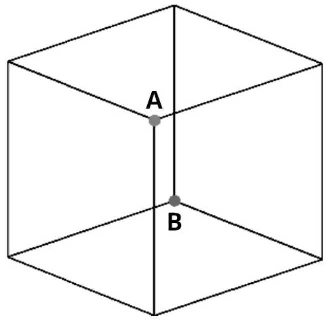
Lecture 3

Vector Space

Mathieu : Hello.
Let's talk about dimensions first. On a board, a 2D object, it's easy to draw a square:



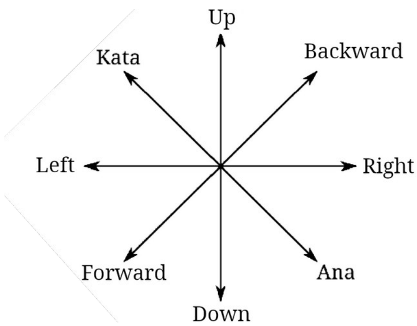
However, *a priori*, it is impossible to draw a 3D object, such as a cube.
Moreover, our eyes only see in 2D – two 2D images displaced on each of our retinas are interpreted as a volume. This is a very useful conceptualization, developed since childhood. For example, we can represent a cube on a sheet of 2D paper as follows:

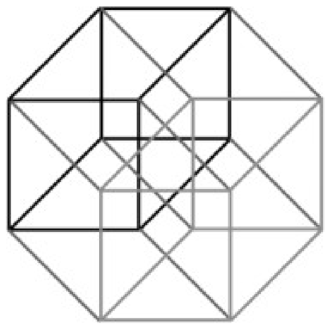


It's a trick, because the object is fundamentally flat, it's our brain that builds a three-dimensional object. With some ambiguities, for example, which point is in front? The *A* or the *B* ?

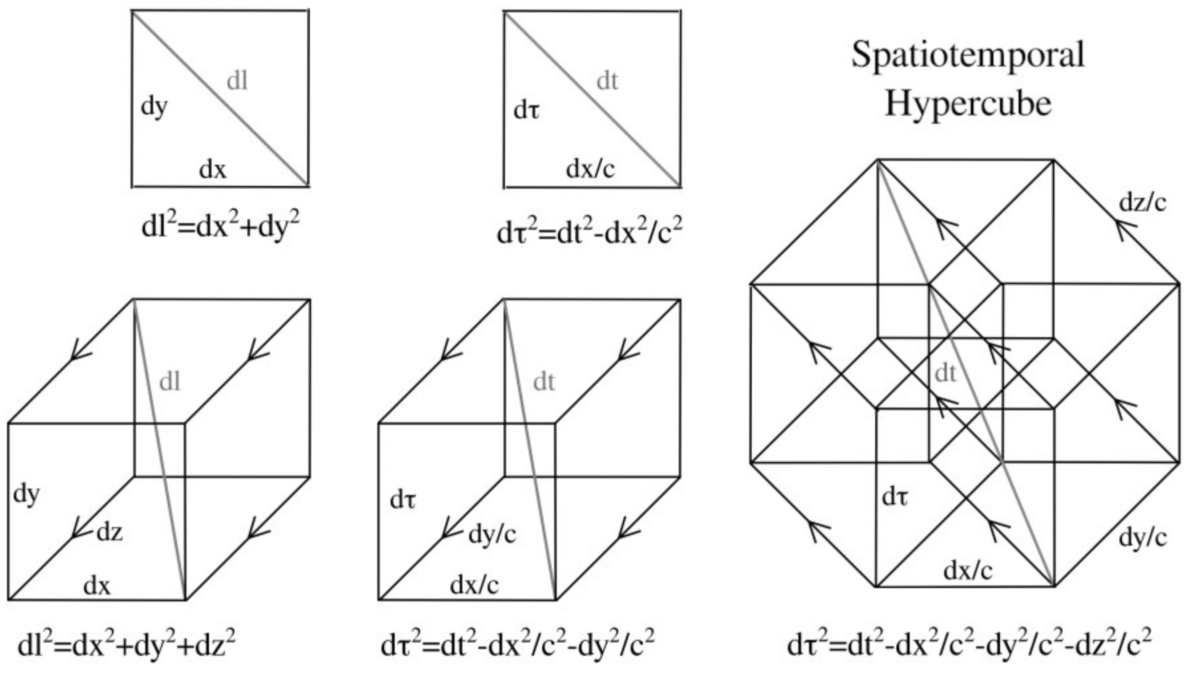
But if our brain builds 3D, why not also practice "seeing" a 4D object?

In the same way that a third dimension is coded, we can add a fourth. This gives us a hypercube, a 4-dimensional object, on a 2D sheet.

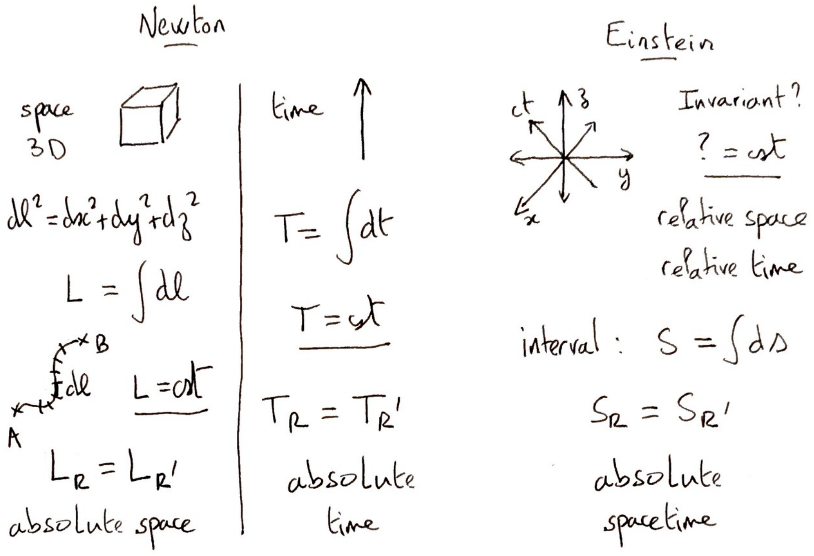


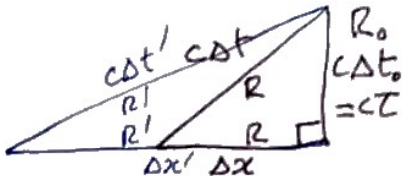


Below, we compare Euclid's 3D space with Minkowski's 4D space-time:



In Newton's theory, time and space are separate, and independently, lengths and durations are invariant when we change from one frame of reference to another.
In relativity, these quantities change from one reference frame to another.
What is the invariant in absolute spacetime?





Similar to Euclidean space, there is a quantity that does not change when the inertial reference frame is changed: $c^2\tau^2 = c^2\Delta t^2 - \Delta x^2 = c^2\Delta t'^2 - \Delta x'^2$. Proper time is an invariant, or the space-time interval *s*, with units of length:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = c^2 d\tau^2 = ds'^2.$$

Arkaitz : Excuse me, but in our bachelor's degree course, I believe we had $ds^2 = -c^2 d\tau^2$.

Mathieu : It depends on the signature chosen, according to the teacher or the book. Here, we have taken the signs (+ - - -). Other authors take the signature (+ + + -), with a minus sign for time.
Also, for the position of an event, we will choose the indices 0, 1, 2 and 3

$$x^\mu = (x^0 = ct\,,\ x^1 = x\,,\ x^2 = y\,,\ x^3 = z).$$

Others use the indices 1, 2, 3, 4, with 4 for time. In addition, we choose Greek indices to designate the four indices that run over space-time, and Latin indices, such as *i, j, k*... for the three spatial indices.

With these notations, we have: $ds^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$,

so: $ds^2 = \eta_{00} dx^0 dx^0 + \sum_{i=1}^{3} \eta_{ii} dx^i dx^i = \sum_{\mu\nu} \eta_{\mu\nu} dx^\mu dx^\nu$ and eventually $ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$.

For the last expression, we used **Einstein's convention**: the same index, repeated above and below, implies a summation. Four-vectors, rank-1 tensors, x^μ and dx^μ are contravariants, and the rank-2 tensor, $\eta_{\mu\nu}$, is twice covariant.
This type of tensorial relationship is very useful in non-inertial relativity. In the particular case of inertial relativity, in an orthonormal Cartesian coordinate system, we simply have $\eta_{00} = 1$ and $\eta_{ii} = -1$, or the metric in matrix form:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$ *called the Minkowski metric* (denoted by the letter η).

Tensor calculus: let's start with the case of Euclidean space $\mathbb{R}^3$, $x^i = (x^1, x^2, x^3) = (x, y, z)$.

Vector projected onto a covariant basis: $\vec{x} = x^1 \vec{e}_1 + x^2 \vec{e}_2 + x^3 \vec{e}_3 = \sum_{i=1}^{3} x^i \vec{e}_i = x^i \vec{e}_i = x^j \vec{e}_j$ (dummy index).

Scalar product: $\vec{x}\cdot\vec{y}=x^i\vec{e}_i\cdot y^j\vec{e}_j=g_{ij}x^i y^j=x^i y_i=x_j y^j$, where we have defined the **metric**:

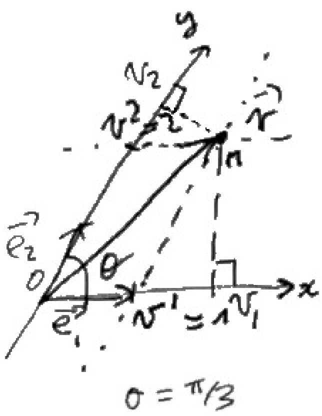
$$g_{ij}=\vec{e}_i\cdot\vec{e}_j=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix},\ \textit{symmetric tensor}:\ g_{ij}=g_{ji}.$$

In an orthonormal basis, you get an identity matrix.

The covariant metric allows you to lower the indices: $x_i=g_{ij}x^j$, here $x_i=x^i$, but in general $x_i\neq x^i$.

For example: $dl^2=\vec{dl}\cdot\vec{dl}=dx^i\vec{e}_i\cdot dx^j\vec{e}_j=g_{ij}dx^i dx^j=dx_k dx^k=dx^2+dy^2+dz^2$, Euclidean distance.

Vector Space : Case of a non-orthogonal Euclidean basis. Study based on an example.



Let us consider: $\overrightarrow{OM}=\vec{v}=v^1\vec{e}_1+v^2\vec{e}_2=v^i\vec{e}_i=\vec{e}_1+2\vec{e}_2$

(contravariant components by projection onto a covariant basis).

Here we obtain the metric: $g_{ij}=\vec{e}_i\cdot\vec{e}_j=\begin{pmatrix}1&\cos\theta\\\cos\theta&1\end{pmatrix}$

but, $v_i=g_{ij}v^j$ so $v_1=g_{11}v^1+g_{12}v^2=1+\frac{1}{2}.2=2$ and $v_2=g_{21}v^1+g_{22}v^2=\frac{5}{2}$.

The ***contravariant metric*** is defined by the relation: $g_{ik}g^{kj}=\delta_i^j$.

That makes four equations: $g_{11}g^{11}+g_{12}g^{21}=1$, $g_{11}g^{12}+g_{12}g^{22}=0$, etc.

The system is solved: $g^{ij}=\frac{1}{\sin^2\theta}\begin{pmatrix}1&-\cos\theta\\-\cos\theta&1\end{pmatrix}$.

To raise the indices: $v^i=g^{ij}v_j$ et $\vec{e}^{\,i}=g^{ij}\vec{e}_j$.

A vector can also be expressed in its contravariant basis:

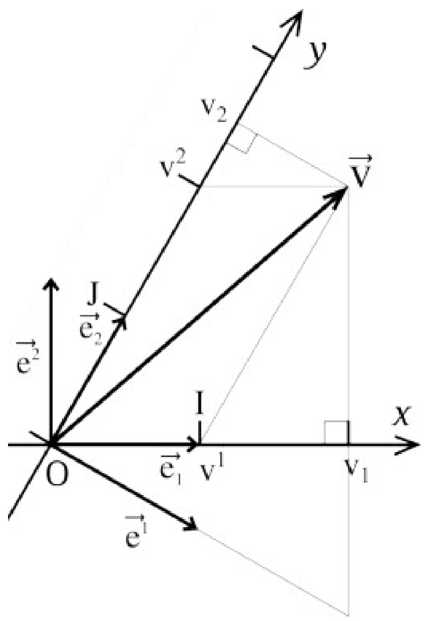
$$\vec{v}=v^i\vec{e}_i=v_i\vec{e}^{\,i}=2\vec{e}^{\,1}+\frac{5}{2}\vec{e}^{\,2}$$

with $\vec{e}^{\,1}=g^{11}\vec{e}_1+g^{12}\vec{e}_2=\frac{2}{3}(2\vec{e}_1-\vec{e}_2)$ and $\vec{e}^{\,2}=\frac{2}{3}(-\vec{e}_1+2\vec{e}_2)$.

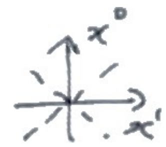
We can also use the metric to calculate the norm of the vector $\vec{v}$:

$$\vec{v}\cdot\vec{v}=\|\vec{v}\|^2=g_{ij}v^i v^j=(v^1)^2+(v^2)^2+\cos\theta\, v^1 v^2+\cos\theta\, v^2 v^1$$

and $v=\sqrt{5+4\cos\theta}$.



Minkowski Space: case of an orthonormal basis in an inertial frame *R*. *E(ct, x)* :



Orthogonal basis: $g_{01}=\eta_{01}=\tilde{e}_0\cdot\tilde{e}_1=0$,

A tilde ~ is chosen for a vector in Minkowski space, an arrow → in Euclidean space.

Normal basis: $\eta_{00}=\tilde{e}_0\cdot\tilde{e}_0=1$ et $\eta_{11}=\tilde{e}_1\cdot\tilde{e}_1=-1$

Consider two 4-vectors $\tilde{a}$ and $\tilde{b}$: $\tilde{a}=(a_0, a_1, a_2, a_3)=(a_0, \vec{a})$ $\tilde{b}=(b_0, \vec{b})$

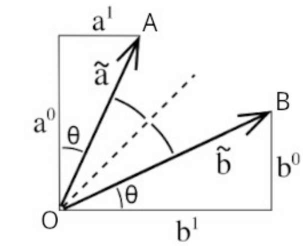
In 2D : $\tilde{a}.\tilde{b}=g_{\mu\nu}a^\mu b^\nu=\eta_{00}a^0b^0+\eta_{11}a^1b^1$

$\tilde{a}.\tilde{b}=a^0b^0-a^1b^1$

Two orthogonal 4-vectors are symmetric with respect to the photon worldlines:

$$\tilde{a}\cdot\tilde{b}=0 \quad\Rightarrow\quad a^0b^0=a^1b^1 \qquad \tan\theta=\frac{a^1}{a^0}=\frac{b^0}{b^1}$$

In the context of Minkowski space, OAB is a right triangle at O.



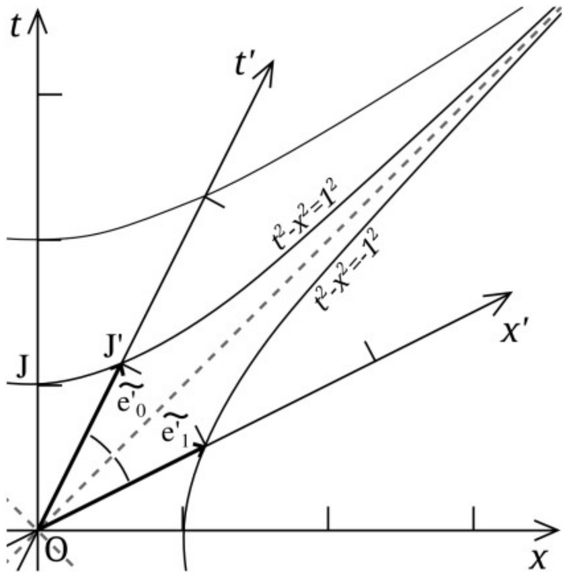
Now let us consider another reference frame *R′* moving relative to *R* :

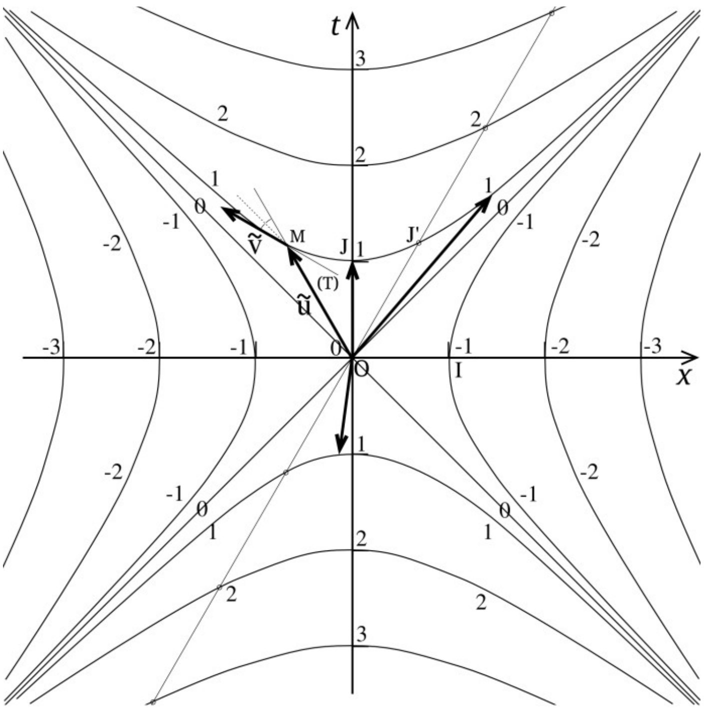
$\eta'_{01}=\eta_{01}=\tilde{e'}_0\cdot\tilde{e'}_1=0$, the basis of *R′* is orthogonal,

$\eta'_{00}=\eta_{00}=\tilde{e'}_0\cdot\tilde{e'}_0=1$ et $\eta'_{11}=\eta_{11}=\tilde{e'}_1\cdot\tilde{e'}_1=-1$,

the *R′* basis is normal.

From Minkowski's point of view: *OJ = OJ′*.





With or without primes :

$$\eta'_{\mu\nu}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}=\eta'^{\mu\nu}$$

$$x'_0=x'^0,\quad x'_1=-x'^1,$$

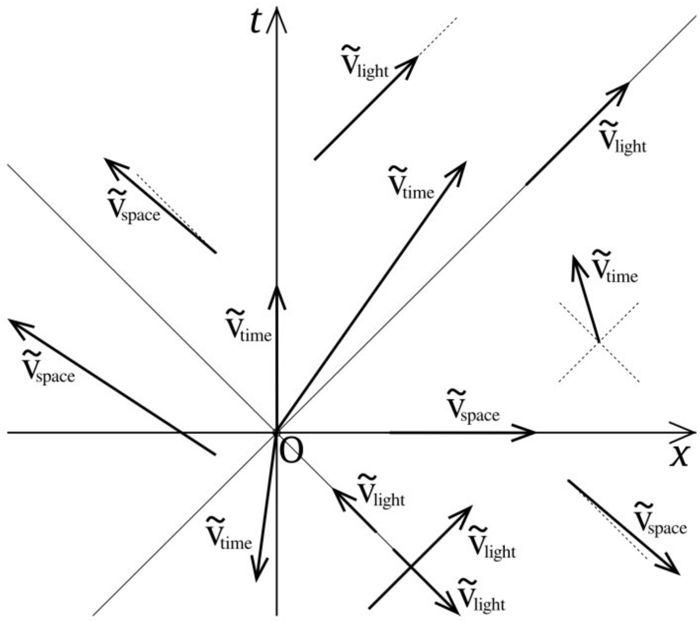
$$\tilde{e}'^0=\tilde{e}'_0 \quad \text{et} \quad \tilde{e}'^1=-\tilde{e}'_1.$$

In the figure on the left: $\tilde{u}\cdot\tilde{v}=0$.

The ***pseudo-norm***, $\tilde{v}\cdot\tilde{v}=(v^0)^2-(v^1)^2$, is positive, zero, or negative.

We have three kinds of 4-vectors $\tilde{v}$:

- timelike : $\tilde{v}\cdot\tilde{v}>0$
- lightlike : $\tilde{v}\cdot\tilde{v}=0$
- spacelike : $\tilde{v}\cdot\tilde{v}<0$



Depending on the sign of the time component, a four-vector can point toward the future or the past.

Proper time, scalar product, ortho-gonality, pseudonorm, kind, and sign of the temporal component are quantities or properties that remain unchanged when the reference frame changes.

To summarize, in special or general relativity,
a four-vector can be written using two sets of components, contravariant or covariant:

$$\tilde{v}=v^\mu\tilde{e}_\mu=v_\mu\tilde{e}^\mu.$$

The metric tensor is defined by $g_{\mu\nu}=\tilde{e}_\mu\cdot\tilde{e}_\nu$.

For the scalar product, you have: $\tilde{a}\cdot\tilde{b}=g_{\mu\nu}a^\mu b^\nu=a^\mu b_\mu=a_\mu b^\mu=g^{\mu\nu}a_\mu b_\nu$.

The metric tensor helps to move indices, for example: $\tilde{e}_\mu\cdot\tilde{e}^\nu=\tilde{e}_\mu\cdot g^{\nu\lambda}\tilde{e}_\lambda=g_{\mu\lambda}g^{\lambda\nu}=\delta^\nu_\mu$.

Adriana : Are Minkowski diagrams also used in general relativity?

Mathieu : Yes, but the interpretation is less immediate. First of all, for the simple reason that clocks cannot be synchronized in a non-inertial reference frame. It is therefore impossible, *a priori*, to define a time t ! Since we cannot use proper clocks, we use coordinated clocks. In fact, we can continue to synchronize clocks, but they do not run at the same rate as proper clocks... Locally, you now have two clocks, a proper clock and a coordinate clock. This may seem artificial, because depending on the point in space considered, a coordinate clock may run faster or slower than the proper clock. That said, we can therefore label all events in our non-inertial reference frame, define a “time,” and ultimately deduce proper time. From this *coordinate time*, we also have a *coordinate velocity*. And although locally, in our Minkowski reference frame, relative to the proper clock, the speed of light is always c, the coordinate speed of light may therefore be less than or greater than c. And the light cones will be more or less flattened or elongated at different points on the diagram.

Summary of the notions covered today:

Lecture 3 – VECTOR SPACE

- Space-time dimensions
- Proper time as an invariant
- Einstein's convention
- Covariant metric: $ds^2 = c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$
- Components and bases, contravariant and covariant
- Contravariant metric: $g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu$
- lowering and raising indices: $v_\nu = g_{\nu\mu} v^\mu$ and $v^\nu = g^{\nu\mu} v_\mu$
- Orthogonality in Minkowski space
- Pseudonorm and kind

To review and practice:

- Chapter 7 of SR pages 173 to 198.
- Worksheet on tensor calculus page Erreur : source de la référence non trouvée: exercise 1

Lecture 4

Change of Coordinates

Mathieu : Hello.

Question: What would be the most beautiful physics theory?

An idea: a theory that is true in all reference frames, invariant under coordinate changes, true for all observers, regardless of their situation, the laws of physics would be expressed in the same way.

As we will see today, this is precisely what can be achieved with 4-dimensional tensors.

First of all, what about the theories you already know?

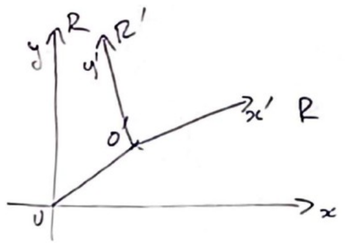
In Newtonian mechanics and quantum physics, if time is not taken into account, the laws are invariant. By translation, which simply means changing the origin, and by rotation of the axes. This is normal because in $\mathbb{R}^3$ there are vector or scalar relations, such as $\vec{F}=m\vec{a}$ or $d(1/2\, m\vec{v}\cdot\vec{v})/dt=\vec{F}\cdot\vec{v}$, the components of the vectors change, but the relationships between the vectors, or their scalar products, remain the same. By now adding time, we have invariance under time translation, the origin of time can be changed, and a Galilean transformation can be performed. However, all other transformations do not leave the laws invariant: $\vec{v}_{R'/R}=\vec{v}(t)$, $\vec{\Omega}_{R'/R}(t)$ and $\vec{a}_{R'/R}$.
In special relativity, and therefore also in electromagnetism, the Lorentz transformation is used, but the same observation applies.

The idea is actually quite simple: if time becomes a coordinate alongside the three spatial coordinates, it is possible to construct 4-dimensional scalars, vectors, and tensors, leading to the invariance of the laws that link these quantities. This is similar to classical mechanics in 3D. However, there was a limitation, as the laws were established using 3D objects in a 4D world.

Luna : But in that case, shouldn't it already work in special relativity?

Mathieu : Yes, this is indeed the case when replacing derivatives with *covariant derivatives*.
They will be studied later.

Let's start by looking at what happens in 2D when changing from one coordinate system to another:



R (x,y)
R' (x',y')

$$\begin{cases} x'(x,y) \\ y'(x,y) \end{cases}$$

A little math allows us to obtain the change for an infinitesimal element:

Maths $f(x,y)$

$$df = f(r') - f(r) = \left(\frac{\partial f}{\partial x}\right)_y dx + \frac{\partial f}{\partial y} dy$$

$f = x'$: $$dx' = \frac{\partial x'}{\partial x} dx + \frac{\partial x'}{\partial y} dy$$

Then let's generalize the formula for $x'^{\mu}(x^{\nu})$:

$x'^1 = x'$ $\quad x^1 = x$

$x'^2 = y'$ $\quad x^2 = y$

$$dx'^1 = \frac{\partial x'^1}{\partial x^i} dx^i \quad i = 1,2$$

Change of basis matrices, *lambda matrices*:

$$\Lambda^{\mu}{}_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} \qquad \Lambda_{\mu}{}^{\nu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}}$$

$$\boxed{dx'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} dx^{\nu}}$$

For an infinitesimal vector:

$$d\tilde{x} = dx^{\mu}\,\tilde{e}_{\mu} = dx'^{\mu}\,\tilde{e}'_{\mu}$$

with $dx'^{\mu} = \Lambda^{\mu}{}_{\nu} dx^{\nu}$ and $\tilde{e}'_{\mu} = \Lambda_{\mu}{}^{\nu}\tilde{e}_{\nu}$.

The same applies to any vector:

$$v'^{\mu} = \Lambda^{\mu}{}_{\nu} v^{\nu}$$

Before continuing, we have a useful formula to demonstrate:

$$\begin{cases} x(x',y') \\ y(x',y') \end{cases} \qquad dx = \frac{\partial x}{\partial x'}dx' + \frac{\partial x}{\partial y'}dy' \qquad \frac{\partial x'}{\partial x^i}\frac{\partial x^i}{\partial x'} = 1 \qquad \frac{\partial x'^1}{\partial x^i}\frac{\partial x^i}{\partial x'^2} = 0$$

$$dy = \frac{\partial y}{\partial x'}dx' + \frac{\partial y}{\partial y'}dy'$$

$$dx' = \underbrace{\left[\frac{\partial x'}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial x'}{\partial y}\frac{\partial y}{\partial x'}\right]}_{1}dx' + \underbrace{\left[\frac{\partial x'}{\partial x}\frac{\partial x}{\partial y'} + \frac{\partial x'}{\partial y}\frac{\partial y}{\partial y'}\right]}_{0}dy' \qquad \boxed{\frac{\partial x'^\alpha}{\partial x^\nu}\frac{\partial x^\nu}{\partial x'^\beta} = \delta^\alpha_\beta}$$

To recap, here are the changes of coordinates:

4-vectors : $\quad v'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)v^\nu \qquad v'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)v_\nu$

Rank 2 tensors: $\quad T'^{\mu\nu} = \left(\frac{\partial x'^\mu}{\partial x^\alpha}\right)\left(\frac{\partial x'^\nu}{\partial x^\beta}\right)T^{\alpha\beta} \qquad T'_{\mu\nu} = \left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right)T_{\alpha\beta}$

Corollary: *A tensor that is identically null in one frame is also identically null in any other frame.*

Components of a tensor: $\boldsymbol{T} = T^{\mu\nu}\tilde{e}_\mu \otimes \tilde{e}_\nu = T_{\mu\nu}\tilde{e}^\mu \otimes \tilde{e}^\nu = T^\mu{}_\nu \tilde{e}_\mu \otimes \tilde{e}^\nu = T_\mu{}^\nu \tilde{e}^\mu \otimes \tilde{e}_\nu$ ($\otimes$: tensor product)

This can be generalized for a tensor of any rank with any type of covariant or contravariant indices. If we have a relationship between physical quantities in one reference frame, it is also true in any other reference frame, for example:

$$a^\mu{}_\nu = b^\mu c_\nu \quad \Rightarrow \quad \left(\frac{\partial x'^\mu}{\partial x^\alpha}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right)a^\alpha{}_\beta = \left(\frac{\partial x'^\mu}{\partial x^\alpha}\right)b^\alpha\left(\frac{\partial x^\beta}{\partial x'^\nu}\right)c_\beta \quad \Rightarrow \quad a'^\mu{}_\nu = b'^\mu c'_\nu$$

Scalars: $\quad T_\mu{}^{\mu} = T'_\mu{}^{\mu} = T = T' \qquad T_{\mu\nu}T^{\mu\nu} = T'_{\mu\nu}T'^{\mu\nu} = S = S'$

$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = S$ (Kreschmann scalar constructed with Riemann's curvature tensor)

Let's take the case of the scalar of two four-vectors for the demonstration:

$$\tilde{u}\cdot\tilde{v} = u^\mu v_\mu = u'^\mu v'_\mu = invariant, \quad \text{indeed} \quad u'^\mu v'_\mu = \left(\frac{\partial x'^\mu}{\partial x^\lambda}\right)u^\lambda\left(\frac{\partial x^\sigma}{\partial x'^\mu}\right)v_\sigma = \delta^\sigma_\lambda u^\lambda v_\sigma = u^\mu v_\mu.$$

For inertial special relativity, only changes in reference frames that leave the Minkowski metric invariant between two orthonormal Cartesian coordinate systems *(O, ct, x, y, z)* and *(O', ct', x', y', z')* are considered:

$$\eta'_{\mu\nu} = \left(\frac{\partial x^\alpha}{\partial x'^\mu}\right)\left(\frac{\partial x^\beta}{\partial x'^\nu}\right)\eta_{\alpha\beta} = \eta_{\mu\nu} = diag(1,-1,-1,-1)$$

It is then shown that only a Lorentz Transform (LT) satisfies this condition:

Special Lorentz Transform: $\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{cases}$, $\vec{v} = v\,\vec{i}$, *(Ox)=(O'x')* and *O(t=0)=O'(t'=0)*.

Special Inverse Lorentz Transformation: $\begin{cases} ct=\gamma(ct'+\beta x') \\ x=\gamma(x'+\beta ct') \\ y=y' \\ z=z' \end{cases}$, $v\rightarrow -v$ and $\beta\rightarrow-\beta$.

Lambda matrix for the special Lorentz transformation:

$$\eta_{\mu\nu}=\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta \eta_{\alpha\beta},\quad \Lambda_\nu{}^\mu=\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},\ \text{for } \Lambda^\mu{}_\nu\ \beta\rightarrow-\beta,\quad \Lambda^\mu{}_\lambda \Lambda_\nu{}^\lambda=\delta^\mu_\nu \ \text{ and }\ \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1}=\boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda}=\boldsymbol{I}.$$

Cristian : Performing a change of reference frame for a rank 2 tensor, is equivalent to performing a matrix product with 3 matrices?

Mathieu : I don't think so... Let's take a look.
For the product of two matrices $M=A\times B$, we have $m^i{}_j=a^i{}_k b^k{}_j$, the index k runs through the columns of A and the rows of B at fixed i and j. For 3 matrices $N=A\times B\times C$ and $n^i{}_j=a^i{}_k b^k{}_l c^l{}_j$, it is associative, so it is possible to do $A\times B$ or $B\times C$ first.
However, for a tensor $T'_{\mu\nu}=\Lambda_\mu{}^\alpha \Lambda_\nu{}^\beta T_{\alpha\beta}$, the sum of products is done in a single step.
For example, for the previous case of $\eta'_{\mu\nu}$, if you do the product $\Lambda\times\Lambda\times\eta$, it does not work.
However, $\eta'_{00}=\Lambda_0{}^\alpha \Lambda_0{}^\beta \eta_{\alpha\beta}=\Lambda_0{}^0 \Lambda_0{}^0 \eta_{00}+\Lambda_0{}^1 \Lambda_0{}^1 \eta_{11}+0\ldots=\gamma^2\times1+\gamma^2\beta^2\times(-1)=1$, as expected.

In inertial special relativity, every 4-vector follows the Lorentz transformation: $\boxed{\begin{cases} v'^0=\gamma(v^0-\beta v^1) \\ v'^1=\gamma(v^1-\beta v^0) \\ v'^2=v^2 \\ v'^3=v^3 \end{cases}}$.

With these new tools, let's revisit some properties and discover new ones.

► Using the Lorentz transform for x^μ :

Time Dilation : Using the transformation for x^μ, in the reference frame R, where a clock is at rest, we measure the proper time $\tau = t_{E2} - t_{E1}$ between two events E_1 and E_2, at the same $x=0$, at the beginning and end of the period. Thus, according to LT, $t'_E=\gamma(t_E-\beta x)$, $\Delta t'=t'_{E2}-t'_{E1}=\gamma t_{E2}-\gamma t_{E1}$, and $\Delta t'=\gamma\tau$.

Length Contraction : In the reference frame R where the ruler is at rest, $\Delta x=x_{E2}-x_{E1}=L_0-0=L_0$.
In the moving reference frame, we measure the ends at the same instants: $t'_{E2}=t'_{E1}$.
Therefor $ct_{E2}-\beta x_{E2}=ct_{E1}-\beta x_{E1}$ and $c\Delta t=\beta\Delta x$. So, $\Delta x'=L=x'_{E2}-x'_{E1}=\gamma(\Delta x-\beta c\Delta t)=\gamma L_0(1-\beta^2)=L_0/\gamma$.

► Using the Lorentz transform for dx^μ: $\begin{cases} cdt'=\gamma(cdt-\beta dx) & (1) \\ dx'=\gamma(dx-\beta cdt) & (2) \\ dy'=dy \\ dz'=dz \end{cases}$, $\vec{v}_{R'/R}=v\,\vec{i}=\beta c\,\vec{i}$

Law of velocity composition: if the velocity is defined as in classical mechanics $v_x'=\frac{dx'}{dt'}$, $v_x=\frac{dx}{dt}$, and $\frac{(2)}{(1)}$ gives $v_x'=\frac{v_x-v}{1-v\,v_x/c^2}$. You already know this formula, but what intrigues us today is that this formula does not correspond to a Lorentz transform!

If it did, the formula would be:: $v'^1=\gamma(v^1-\beta v^0)$ and $v'^x=\frac{v^x-v}{\sqrt{1-v^2/c^2}}$, a different formula.

The vector $\vec{v}=(v_x,v_y,v_z)$, built according to the Newtonian approach, does not correspond to the spatial part of a four-vector. Therefore, with this definition of velocity, it is not possible to construct laws that are invariant under a change of reference frame In the next lesson, we will build the four-velocity u^μ, a quantity that transforms in a *covariant* manner and which, for example in the context of inertial SR, has components that transform according to LT. If we want to interpret relativity correctly, we must use the concepts specific to it.

►Using the Lorentz transform for the wave vector k^μ :

Maxwell's equations give the electromagnetic wave equation. And the solutions for the plane monochromatic wave with rectilinear polarization are of the form $\vec{E}=\vec{E}_0\cos(\omega t-k\,x)$.
The phase is an invariant: $\omega t-k\,x=\omega t-\vec{k}\cdot\vec{r}=k^\mu x_\mu$, with $k^\mu=(\omega/c\,,\vec{k})$, *wave four-vector*.

Doppler Effect:
An electromagnetic wave in a vacuum has a pulsation $\omega=2\pi f$ in a reference frame R.
What is the pulsation ω' perceived in R'?

$$\begin{cases} k'^0=\gamma(k^0-\beta k^1)\\ k'^1=\gamma(k^1-\beta k^0)\\ k'^2=k^2\\ k'^3=k^3\end{cases},$$

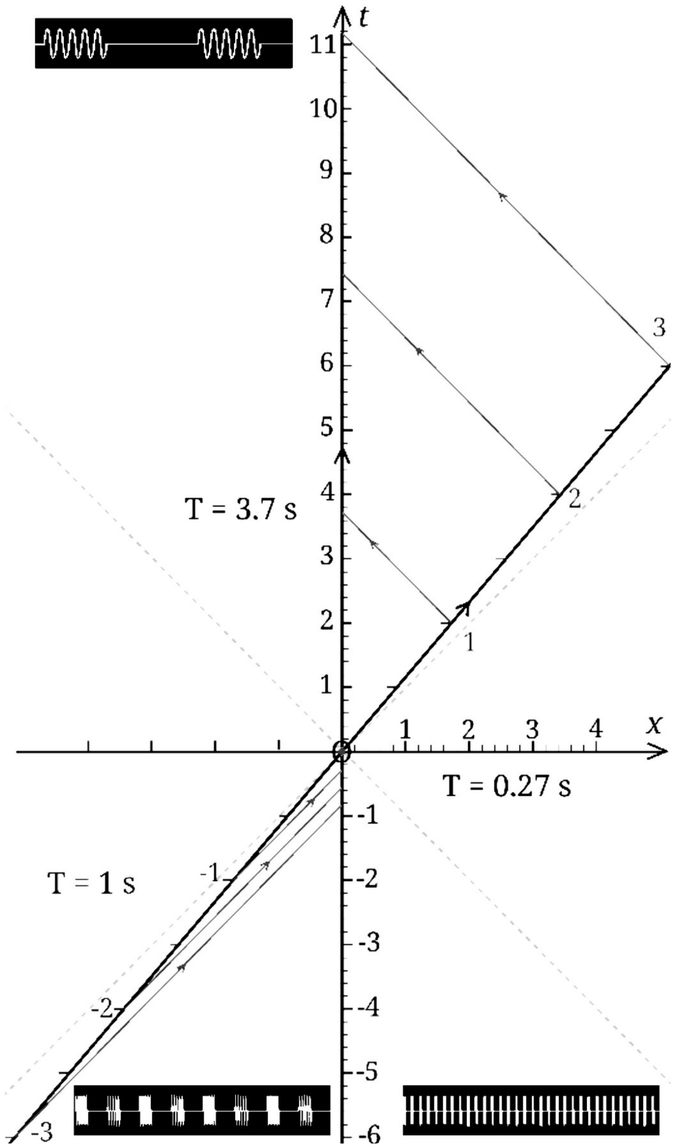
for the *longitudinal* Doppler Effect,

$\vec{k}=k^x\vec{i}=k\,\vec{i}$, $k=\omega/c$, and $\omega'=\gamma(\omega-\beta\omega)$,

eventually: $f'=f\sqrt{\frac{1\mp\beta}{1\pm\beta}}$

depending on whether one moves away from or closer to the source.

On a Minkowski diagram, you can visualize the Doppler effect. A spacecraft is traveling at γ=2, approximately 87 % of c. A light source at rest inside the spacecraft emits a yellow light pulse every second.
Tracing the worldlines of light rays, for the approach, a higher frequency is perceived in the ultraviolet. Conversely, light is perceived in the infrared when the spacecraft is moving away.



► Using the Lorentz transform for the electromagnetic tensor $F^{\mu\nu}$:

In Maxwell's four equations, electric and magnetic fields appear as trivectors. To ensure invariance, we could try to rewrite Maxwell's equations using four-vectors. The fields would then be the spatial parts of these 4-vectors. But that doesn't work, instead, an antisymmetric rank-2 tensor must be considered. For $\vec{E}$ and $\vec{B}$, there are 3+3= 6 degrees of freedom. For a rank 2 tensor, there are $4\times4=16$ independent components. Due to antisymmetry, $F^{\mu\mu}=0$ and $F^{\mu\nu}=-F^{\nu\mu}$, we also have $16-4-6=6$ independent components. It will be shown later that to satisfy Maxwell's equations, the tensor is written as: $\boldsymbol{F}=F^{\mu\nu}\tilde{e}_\mu\otimes\tilde{e}_\nu$ with $F^{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$ (to simplify, $E_i/c\rightarrow E_i$, $c\rightarrow1$).

$$F'^{\mu\nu}=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta F^{\alpha\beta} \text{ then } F'^{\mu\mu}=\Lambda^\mu{}_\nu\Lambda^\mu{}_\nu F^{\nu\nu}+\Lambda^\mu{}_\alpha\Lambda^\mu{}_\beta\left(F^{\alpha\beta}+F^{\beta\alpha}\right)=0 \quad (\alpha\neq\beta)$$

$$E'_x=F'^{10}=\Lambda^1{}_0\Lambda^0{}_1F^{01}+\Lambda^1{}_1\Lambda^0{}_0F^{10}=\beta^2\gamma^2\times(-E_x)+\gamma^2\times E_x=E_x \quad \text{(10 terms, including 8 null terms)}$$

$$E'_y=F'^{20}=\Lambda^2{}_2\Lambda^0{}_0F^{20}+\Lambda^2{}_2\Lambda^0{}_1F^{21}=\gamma\times E_y-\beta\gamma\times B_z=\gamma\left(E_y-\beta B_z\right)$$

and so on, to be completed as an exercise.

The components of $\vec{E}$ and $\vec{B}$ mix together. The mathematical entity that has physical meaning and represents electromagnetic phenomena is not a four-vector but a rank-2 tensor.

► In general relativity, it is the Riemann curvature tensor $R^{\alpha\beta\mu\nu}$ that plays a central role. This rank-4 tensor is covariant for all coordinate changes. It encodes all the information about curvatures at all points in space-time. It has 4^4= 256 components, but due to symmetries, only 20 of them are independent. To compare in two dimensions of space, this same tensor has 2^4= 16 components, and only 2 independent ones that correspond to the Gaussian curvatures R_1 and R_2 that are observed on two main perpendicular planes at each point on a surface.

Thank you for your attention.

Summary of the notions covered today:

Lecture 4 – CHANGE OF COORDINATES

- Lambda matrix – change of basis
- Vectors, tensors and scalars
- Building covariant laws
- Particular case of Inertial Special Relativity
- Lorentz Transform
- Time Dilation
- Length Contraction
- Composition of velocities
- Doppler Effect

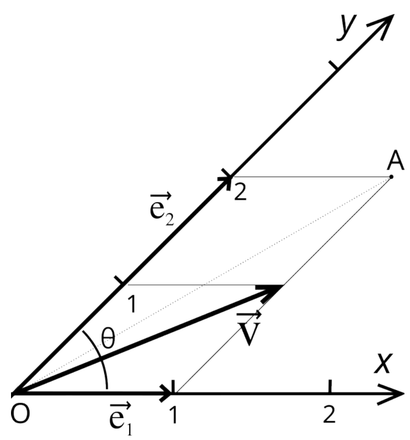
To review and practice:

- Exercise sheet no. 2 follows.
- Chapter 7 of the SR book, pages 199 to 205, exercise 1, chapter 3, pages 59 to 69, exercises 1 to 3, (chapter 4 and exercises).
- Worksheet on tensor calculus: scalar, vector, and matrix products, operators, and conversion of Maxwell's equations into tensor form.

Worksheet No. 2

Vector Space, Change of coordinates and Composition of velocities

Exercise 1 Let's consider an euclidean space with a non -normal and non-orthogonal Cartesian basis, with θ=π/4:



a) Give the covariant components g_{ij} of the metric.

b) Write the metric and determine the distance OA.

c) Express the vector $\vec{v}$ with the contravariant components and the covariant basis. Represent the components on the graph.

d) Find the contravariant components g^{ij} of the metric.

e) Determine the contravariant basis and the covariant components of $\vec{v}$. Draw them on the figure.

Exercise 2 Minkowski Diagram

a) On a spacetime diagram, represent the events $E_0(ct=1, x=0)$ and $E_1(ct=\gamma, x=\beta\gamma)$.
E_1 represented for $\beta=1/2$.

b) Find the pseudonorm of $\widetilde{OE_0}$, $\widetilde{OE_1}$ and $\widetilde{E_O E_1}$. Are they spacelike or timelike vectors?

c) Represent the curve(s) for $OE=\tau=1$.

d) Find an event E_2 as $\widetilde{E_O E_2}$ and $\widetilde{E_O E_1}$ orthogonal and $E_0E_2=1$.

e) From a minkowskian point of view, what kind of triangles are OE_0E_1 and $E_0E_1E_2$?

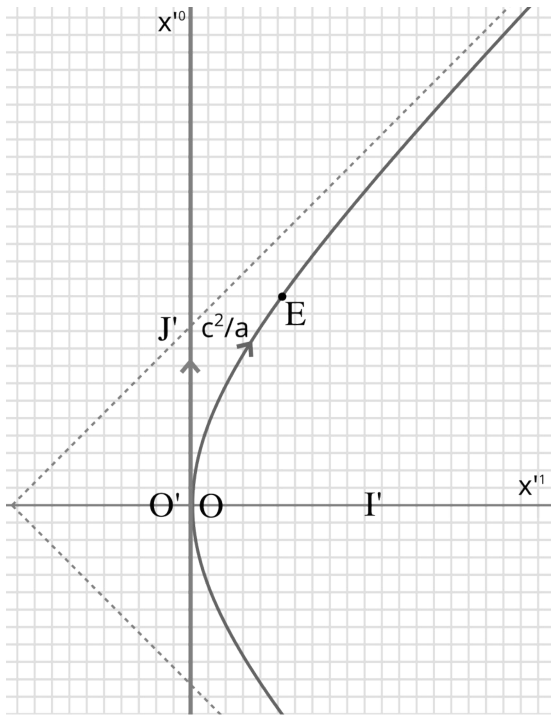
f) Do we have Minkowskian equilateral triangles? It depends on the definition. Prove there are no triangles with the three sides the same pseudonorm (and then the same kinds, timelike, lightlike or spacelike). However, it is possible if the kinds are mixed.
Example: find a value of β for OE_0E_1 to be an equilateral triangle.

Exercise 3 Metric in an accelerated reference frame. On the Minkowski diagram of the inertial frame R', two worldlines are represented, vertically the one of the observer at rest at O', and on the right the one of an uniformly accelerated rocket in the x direction. R denote the accelerating reference frame. At $t'=0$, $t=0$, and $O'(t'=0)=O(t=0)$. The change of coordinates in given:

$$ct'=\left(x+\frac{c^2}{a}\right)\sinh\left(\frac{at}{c}\right) \quad \text{and} \quad x'=\left(x+\frac{c^2}{a}\right)\cosh\left(\frac{at}{c}\right)-\frac{c^2}{a}$$

Find the metric in the accelerated rocket.

Represent R with the axis (ct, x) at $E=O(t>0)$.



Exercise 4

From Earth, we measure with our inertial frame of reference, 5 years after the departure of a spacecraft, a huge stellar eruption produced by the star Alpha Centauri B located 4 light-years away. The ship is moving from Earth to Alpha Centauri with $\gamma = \sqrt{2}$. In the reference frame of the spaceship where and when does the eruption occur?
When the light flare will be seen in a telescope from Earth? From the vessel?
Drawn a space-time diagram to illustrate all the exercise results.

Composition of velocities formula:

Exercise 5

1) Two rockets come face to face with a speed c/2. What is their relative speed?

2) Two rockets travels in the same direction with the speeds 50%c and 75%c.
What is their relative speed?

Exercise 6

Two rockets race in pursuit. Ahead, the pirates' rocket flees at speed 3/4 c. Behind the police rocket speed at c/2 and can't catch up with the pirates. So, from their rocket, the police use a proton cannon, the muzzle velocity of the protons is c/3. Does the protons reach the pirate rocket? Draw also a space diagram to illustrate the situation.

Exercise 7

A laser emits a 650 nm monochromatic red light. An atom absorbs 2.9 eV photons.

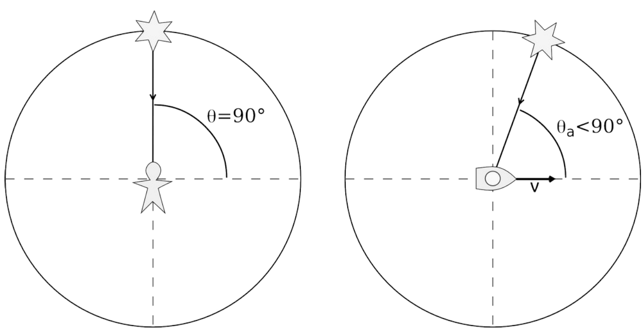
1) As the laser, the atom is at rest and receives the laser light, does it absorb the photons?

2) How the atom has to move to absorb a photon?

Data: $h=6.62\times10^{-34}\,J.s$, $c=3\times10^{8}\,m/s$ *and* $e=1.6\times10^{-19}\,C$.

Exercise 8

Let's determine the change in the perception of the starry sky depending of the speed of the ship. In addition to the change in the perceived color of the stars by *Doppler effect*, their position in the sky is modified, this is called the *aberration of light*. When we are at rest in the galactic frame of reference *R*, the stars are, as a whole, motionless. Let's focus on the case $\theta=\pi/2$, $\beta=v/c=0.8$ and $\lambda=580$ nm (yellow).



1) Drawn a Minkowski diagram (ct, x, y) with $\vec{v} = v\,\vec{i}$, $v>0$, the vessel velocity. Express the equation of the light ray worldline in *R*, and with the help of the Lorentz boost give it in *R'*, the frame of the ship. What's the apparent angle θ_a of the star from the vessel?

2) What's the color of the star from the vessel?

Tensor Training: Introduction

Exercise 1

1) In a metric vector space $\tilde{a}\cdot\tilde{b}=g_{ij}a^i b^j$ with the symmetric metric **g**. With this formula show the two following properties:

i) $\tilde{a}\cdot\tilde{b}=\tilde{b}\cdot\tilde{a}$ ii) $\tilde{a}\cdot(\tilde{b}+\tilde{c})=\tilde{a}\cdot\tilde{b}+\tilde{a}\cdot\tilde{c}$.

Let's consider the completely antisymmetric unit rank 3 tensor ϵ_{ijk} with $\epsilon_{123}=1$, if you exchange two indices the sign change, for example $\epsilon_{213}=-1$, if two indices are equal the component is null, for example $\epsilon_{iki}=0$. If you keep the cycling order the sign stay the same: $\epsilon_{ijk}=\epsilon_{jki}=\epsilon_{kij}=-\epsilon_{kji}$.

2) We are in three dimensions, in an euclidean space and an rectangular system of coordinates.
We continue to use the contravariant and covariant notations as: $\vec{a}\cdot\vec{b}=a^i b_i=g_{ij}a^i b^j=a_i b^i=g^{ij}a_i b_j$.
Find the values or prove the following expressions with tensor calculus:

i) ϵ_{321} ii) ϵ^{123} iii) $\epsilon_{12k}\epsilon^{12k}$ iv) $\epsilon_{ijk}\epsilon^{ijk}$ v) $\epsilon_{imn}\epsilon^{jmn}=2\delta_i^j$ vi) $\epsilon_{ijk}\epsilon^{mnk}=\delta_i^m\delta_j^n-\delta_i^n\delta_j^m$, the following property can be used: $\epsilon_{ij}\epsilon^{mn}=\begin{vmatrix}\delta_i^m & \delta_i^n\\ \delta_j^m & \delta_j^n\end{vmatrix}$, which can be generalized to higher dimensions.

vii) $(\vec{a}\wedge\vec{b})_i=\epsilon_{ijk}a^j b^k$ viii) $\vec{a}\wedge\vec{a}=\vec{0}$ ix) $\vec{a}\wedge\vec{b}=-\vec{b}\wedge\vec{a}$ x) $(\vec{a}\wedge\vec{b})^i$

xi) $(\vec{u}\wedge\vec{v})\cdot\vec{w}=\vec{u}\cdot(\vec{v}\wedge\vec{w})$ xii) $\vec{a}\wedge(\vec{b}\wedge\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$

Nabla operator vector definition: $(\vec{\nabla})_i=\dfrac{\partial}{\partial x^i}=\partial_i$, $\vec{\nabla}=\vec{e}^i\partial_i$ and $\partial^i=g^{ij}\partial_j$.

Write the following expressions or prove them with tensor calculus:

i) $(\vec{\nabla}f)_i$ ii) $\vec{\nabla}f$ iii) $\vec{\nabla}.\vec{V}$ iv) $\vec{\nabla}.\vec{\nabla}f=\Delta f=\partial_i\partial^i f$ v) $(\vec{\nabla}\wedge\vec{V})^i$ vi) $(\vec{\nabla}\wedge\vec{V})_i$

vii) $(\vec{\nabla}\wedge\vec{\nabla})_i=0$ viii) $\vec{\nabla}\cdot(\vec{\nabla}\wedge\vec{V})=0$ ix) $\vec{\nabla}\wedge(\vec{\nabla}\wedge\vec{V})=\vec{\nabla}(\vec{\nabla}\cdot\vec{V})-\Delta\vec{V}$

There are a lot of operator formulas. Some are obvious, like $\vec{\nabla}(fg)=f\vec{\nabla}g+g\vec{\nabla}f$. Other do not turn out to be simple, as illustrated by: $\vec{\nabla}(\vec{u}\cdot\vec{v})=(\vec{u}\cdot\vec{\nabla})\vec{v}+(\vec{v}\cdot\vec{\nabla})\vec{u}+\vec{u}\wedge(\vec{\nabla}\wedge\vec{v})+\vec{v}\wedge(\vec{\nabla}\wedge\vec{u})$. Indeed $(\vec{\nabla}(\vec{u}\cdot\vec{v}))_i=\partial_i(u^j v_j)=u^j\partial_i v_j+v_j\partial_i u^j$ but, for example, $((\vec{u}\cdot\vec{\nabla})\vec{v})_i=u^j\partial_j v_i$. So to verify the formula one could develop on *x*, *y* and *z*...

Exercise 2

In Special Relativity with the rectangular metric $\eta_{\mu\nu}$, let's consider the completely antisymmetric unit rank-4 tensor $\varepsilon_{\alpha\beta\gamma\delta}$, with $\varepsilon_{0123}=1$. Find the following expressions with tensor calculus:

i) ϵ_{3210} ii) ϵ^{0123} iii) $\epsilon_{\alpha\beta\gamma\delta}\epsilon^{\alpha\beta\gamma\delta}$ iv) $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\rho\beta\gamma\delta}$

Apply the special Lorentz transform to show that the completely antisymmetric unit rank-4 tensor is invariant in SR: $\varepsilon'^{\alpha\beta\gamma\delta}=\varepsilon^{\alpha\beta\gamma\delta}$.

Hint: the definition of the determinant can be recognized, det $A=\epsilon^{\alpha\beta\gamma\delta}a^0{}_\alpha a^1{}_\beta a^2{}_\gamma a^3{}_\delta$.

Exercise 3 Some demonstrations to do: i) If $T^{\mu\nu}$ is symmetric then $T_{\mu\nu}$ is also symmetric.

ii) If $T^{\mu\nu}$ symmetric then $T_{\mu}{}^{\nu} = T^{\nu}{}_{\mu}$. iii) Show that every tensors $T_{\mu\nu}$ can be considered as the sum of an antisymmetric $A_{\mu\nu}$ and symmetric $S_{\mu\nu}$ tensors ($S_{\mu\nu}=S_{\nu\mu}$ and $A_{\mu\nu}=-A_{\nu\mu}$).

iv) $S_{\mu\nu}A^{\mu\nu}=0$ v) Change of coordinates: show that $T_{\mu\nu}T^{\mu\nu}= T'_{\mu\nu}T'^{\mu\nu}$ (invariant scalar)

Exercise 4 Electromagnetism and dynamics

Electromagnetic tensor: $\boldsymbol{F}=F^{\mu\nu}=\begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$, 4-vector current: $\tilde{j}=q\,\tilde{u}$

In Special Relativity:

1) Show that the tensor $F^{\mu\nu}$ is antisymmetric.

2) Determine the components of the tensor $F_{\mu\nu}$.

3) Find the transform laws of $\vec{E}$ and $\vec{B}$ for a special Lorentz boost:

Application: In the reference frame of the laboratory *R*, we have a homokinetic beam of protons of velocity $\vec{v}$, radius *r* and density *n*. We call *R'* the proper frame of the protons at rest. With the Gauss' and Ampère's circuital laws, determine the electric and magnetic fields outside the beam in *R'*. Do the same in *R*. Can you find the same results with the Lorentz transformation shown on the right?

$$\begin{cases} E'_x=E_x \\ E'_y=\gamma(E_y-\beta c B_z) \\ E'_z=\gamma(E_z+\beta c B_y) \\ B'_x=B_x \\ B'_y=\gamma(B_y+\beta E_z/c) \\ B'_z=\gamma(B_z-\beta E_y/c) \end{cases}$$

In Special Relativity ∂_μ is a covariant 4-vector.

4) Why the following equation $\partial_\mu F^{\mu\nu}=\mu_0 j^\nu$ is covariant? An equation is said covariant, if the equation keep the same form under a transformation group, here the Lorentz group, so in all inertial frames of reference *R'*, $\partial'_\mu F'^{\mu\nu}=\mu_0 j'^\nu$. Show that this equation gives back the Maxwell equations with sources.

5) Show that the equation $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$ is covariant, and gives back the other two Maxwell equations $\vec{\nabla}\wedge\vec{E}=-\partial\vec{B}/\partial t$ and $\vec{\nabla}\cdot\vec{B}=0$.

6) Why $F^{\mu\nu}F_{\mu\nu}$ and $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ are Lorentz invariants? Then, show that $\vec{B}^2-\vec{E}^2/c^2$ and $\vec{E}\cdot\vec{B}$ are invariants. Use this Lorentz invariants to find again the fields in *R* from then in *R'* in the application in question 3).

7) Show that $\frac{d\vec{p}}{dt}=q(\vec{E}+\vec{v}\wedge\vec{B})$ and $\frac{dE}{dt}=q\vec{E}\cdot\vec{v}$ from the covariant equation $\frac{d\tilde{p}}{d\tau}=\boldsymbol{F}\tilde{j}$, that is, for example, $\frac{dp^\mu}{d\tau}=F^{\mu\nu}j_\nu$.

Lecture 5

4-vectors

Mathieu : Hello.
Today we will study the mathematical objects that allow us to formulate the laws of physics in special relativity (these objects and laws will later be adapted for general relativity).
These objects will, of course, be tensors ─ vectors and scalars.
Furthermore, these laws must give Newton's laws in the limit of low speeds.
So, tell me, what four-vectors and scalars do we already have at our disposal for doing physics?

Arkaitz : The position four-vector!

Luna : The proper time...

Mathieu : That's right. We have $\tilde{x}$, its infinitesimal $d\tilde{x}$ and $d\tau$, from its scalar product (pseudonorm). These three objects are *covariants*[3] under the Lorentz transformation.
We are now ready to construct a covariant object homogeneous with a speed:

$$\textbf{4-velocity}: \boxed{\tilde{u}=\frac{d\tilde{x}}{d\tau}} \quad \text{so} \quad u^{\mu}=\frac{dx^{\mu}}{d\tau} \quad \text{and} \quad \tilde{u}=\left(c\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right).$$

$$\text{Now}: \; c^2 d\tau^2=c^2 dt^2-dx^2-dy^2-dz^2 \;\Rightarrow\; c^2 d\tau^2=c^2 dt^2\left(1-v_x^2-v_y^2-v_z^2\right) \;\Rightarrow\; d\tau=dt/\gamma$$

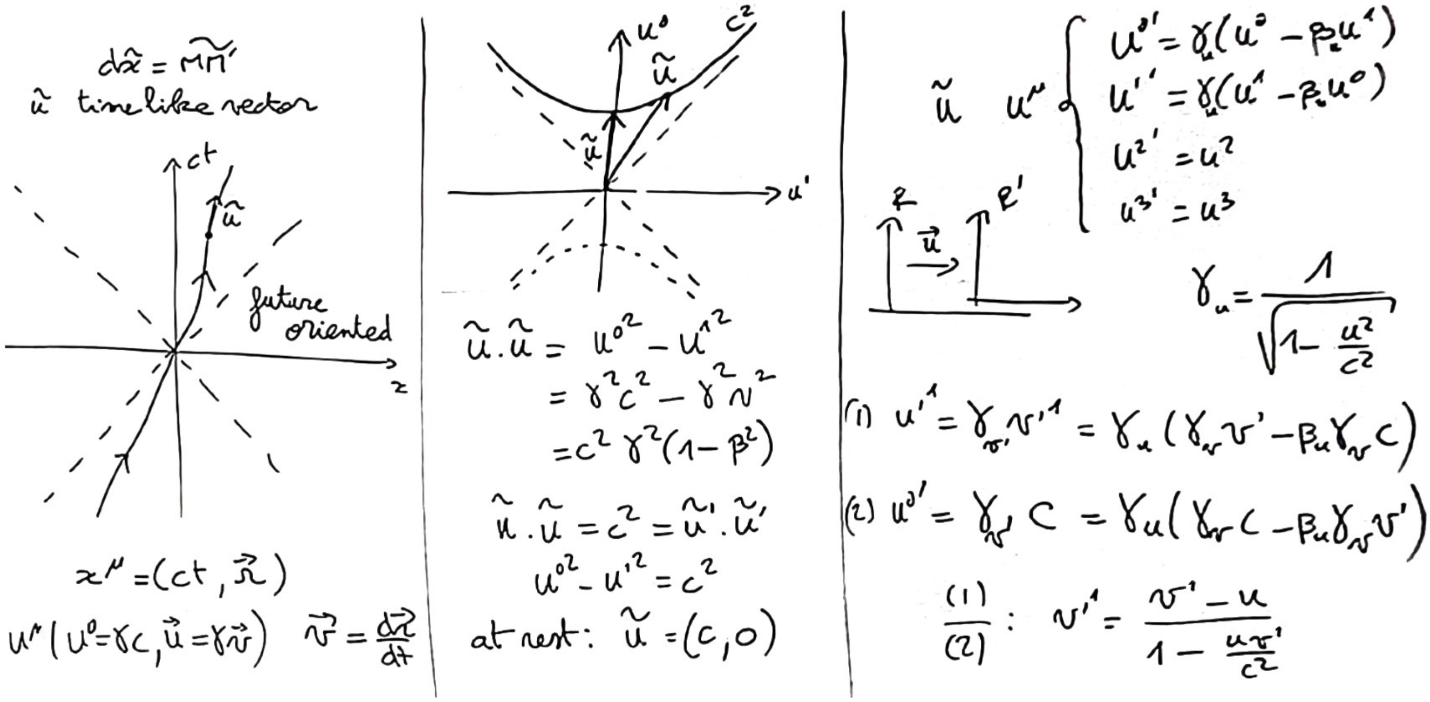
where v is Newton's velocity $v^i=\dfrac{dx^i}{dt}$ (coordinate velocity), not Einstein's velocity $u^i=\dfrac{dx^i}{d\tau}$!

$$\text{Thus}: \quad \tilde{u}=(\gamma c,\gamma\vec{v}) \quad \text{with} \quad \gamma=\gamma_v=\frac{1}{\sqrt{1-\beta_v^2}} \quad \text{and} \quad \beta_v=\beta=\frac{v}{c}$$

$\tilde{u}$ is covariant because $d\tilde{x}$ and $d\tau$ both are. And indeed, for the spatial component, Newton's velocity holds for $v \ll c$ and $\gamma \to 1$. Therefore, the maximum velocity u_{max} for a particle in the framework of relativity is infinite: γ tends to infinity as the velocity v in Newtonian theory tends to a constant c.
To understand relativity, we must apply its new concepts. The theory of relativity is so groundbreaking that we tend to fall back on old formulations derived from classical mechanics. Even in academic contexts, pre-relativistic concepts sometimes persist in relativity courses. At the undergraduate level, ambiguous terms may remain; for example, the law of composition of velocities in relativity is often presented using Newtonian velocity rather than the spatial component of the covariant velocity. So let’s set the record straight: in Einstein’s theory, the speed of light is the infinite speed. Therefore, we do not have to wonder if we will one day be able to travel faster than light. For example, an electron traveling at a speed close to that of light can cross our galaxy, which is 100,000 ly across, in just a few seconds, measured in its own frame of reference. For this electron $u \to \infty$. Going faster than c in the Newtonian sense is therefore simply impossible, because at $v \simeq c$ the electron would have already traversed the galaxy instantaneously! When I was an undergraduate, we used to talk with fascination about *tachions* : « *True, we cannot cross the insurmountable barrier of light speed, since that would require infinite energy, but there might be particles that have always been faster than light...* ». No, tachyons do not exist within the framework of the theory of relativity. Let’s reserve faster-than-light speeds and time travel to the realm of fantasy, even in science fiction it's better to avoid it!

3 Note that there is a potential source of confusion: the term “covariant” does not have the same meaning here as it does when referring to so-called covariant or contravariant indices. Generally speaking, we use covariant to describe a tensor and invariant to describe a scalar associated with that tensor.

Below is the space-time diagram for four-velocity, and on the right is a demonstration of the law of composition of velocities ─ in the classical sense ─ based on the covariance of four-velocity.



Now let's construct a covariant acceleration.

Cristian : All we have to do is differentiate the four-velocity again with respect to τ ?

Mathieu : Yes, we'll get: $\boxed{\tilde{w} = \frac{d\tilde{u}}{d\tau}}$ for the **4-acceleration**.

The letter w is chosen to denote the four-acceleration. Indeed, as we will see in the following calculation, the relationship between the spatial component of the relativistic acceleration $\tilde{w}$ and the Newtonian acceleration $\vec{a}$ is far from trivial.

For an invariant, using the scalar product of the 4-velocity, we obtained the structure constant c. In the case of the 4-acceleration, we obtain the proper acceleration a_p (shown on the next page).

The relation $w^\mu = du^\mu / d\tau$ holds true in inertial special relativity and rectangular coordinates. In the general case, there will be additional terms, so that we always have $\tilde{w} = \tilde{0}$ for a free particle.

<u>Comment</u> : for $f(x^\mu)$ we have $df = \frac{\partial f}{\partial x^\mu} dx^\mu = dx^\mu \frac{\partial f}{\partial x^\mu} = dx^\mu \partial_\mu f$

hence the operator $d\square = dx^\mu \partial_\mu \square$ so $d = dx^\mu \partial_\mu$. We then have: $\frac{d}{d\tau} = u^\mu \partial_\mu$ and $w^\mu = \frac{du^\mu}{d\tau} = u^\nu \partial_\nu u^\mu$

Then w^μ is not covariant: the 4-velocity is covariant, but ∂_μ is not ─ except in inertial relativity with cartesian coordinates. This definition of acceleration therefore works in this chapter, but ∂_μ will subsequently be replaced by a covariant derivative.

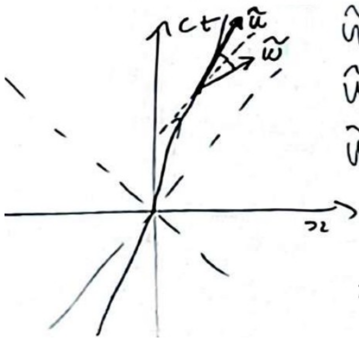
Properties :

$$\frac{d}{d\tau}\big(\underbrace{\tilde{u}\cdot\tilde{u}}_{c^2}\big)=0$$

$$=\frac{d\tilde{u}}{d\tau}\cdot\tilde{u}+\tilde{u}\cdot\frac{d\tilde{u}}{d\tau}=2\,\tilde{w}\cdot\tilde{u}$$

$$\Rightarrow \boxed{\tilde{w}\cdot\tilde{u}=0}$$

$\tilde{w}$ and $\tilde{u}$ orthogonal



$\tilde{w}$ space like

$\tilde{w}=(w^0,\vec{w})$

$\tilde{w}=(w^0,w^1)$

future oriented

$$w^0=\frac{du^0}{d\tau}=\frac{d}{d\tau}(\gamma c)=\gamma c\,\frac{d\gamma}{dt}$$

$$dt=\gamma\, d\tau$$

$$\frac{d\gamma}{dt}=\frac{d}{dt}\left[(1-\beta^2)^{-1/2}\right]=-\frac{1}{2}(-2\beta\dot{\beta})(1-\beta^2)^{-3/2}$$

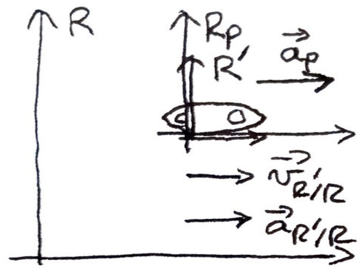
$\dot{\beta}=\frac{d\beta}{dt}$ $\qquad w^0=\gamma^4 c\,\beta\dot{\beta}$

$$\dots\quad \tilde{w}=\left(\gamma^4\,\vec{a}\cdot\vec{\beta}\,;\,\gamma^4(\vec{a}\cdot\vec{\beta})\vec{\beta}+\gamma^2\vec{a}\right)$$

R at rest: $\tilde{w}=(0\,;\vec{a})$

$$\boxed{\tilde{w}\cdot\tilde{w}=-a_p^2}=\tilde{w}'\cdot\tilde{w}'$$

Application : motion of a rocket with a constant proper acceleration.



R_p : proper rocket frame, non-inertial, $\vec{a}_p=\vec{g}$.

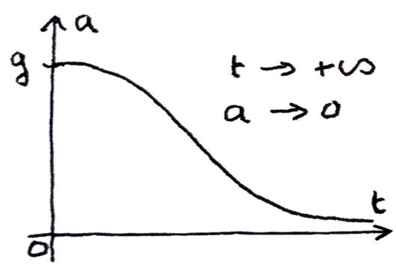
R' : coinciding reference frame, $\tilde{w}'=(0,\vec{a}_p)$, inertial.

R : galactic frame where the rocket moves, inertial, $\vec{v}=\vec{v}_{R'/R}$ and $\vec{a}=\vec{a}_{R'/R}$.

Lorentz transform: $\begin{cases} w^0=\gamma(w'^0+\beta w'^1) \\ w^1=\gamma(w'^1+\beta w'^0)\end{cases}$

then $\begin{cases} w^0=\gamma\beta a_p \\ w^1=\gamma a_p\end{cases}$ and $w^0=\gamma\beta a_p=\gamma^4\beta a$

so $a(t)=\dfrac{dv(t)}{dt}=\dfrac{a_p}{\gamma(t)^3}\neq cst$ and $\boxed{a=\dfrac{a_p}{\gamma^3}}$



Curve easy to plot using a spreadsheet (step-by-step method).

Adriana : But R_p, the rocket's frame, is not inertial, and the Lorentz transform is not valid. Why can we consider the coinciding frame?

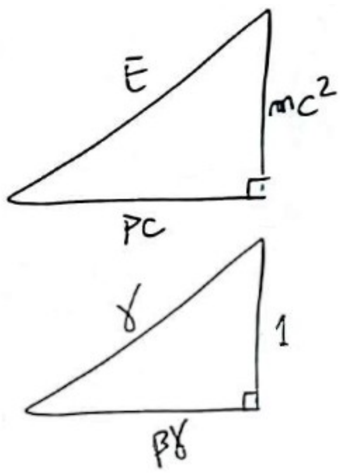
Mathieu : A tricky point. Instantly and locally, it is always possible to define a coinciding inertial frame; this is the *equivalence principle*. We then study the motion from this frame of reference R'. At low speeds, we must recover Newton's laws; when R_p and R' coincide, $\vec{v}_{R_p/R'}$ is close to $\vec{0}$, and classical mechanics applies: the law of composition of accelerations between R' and R_p, $\vec{a}_a=\vec{a}_r+\vec{a}_e+\vec{a}_c$ then $\vec{a}_{R'}(M)=\vec{0}+\vec{a}_p+\vec{0}=\vec{a}_p$. What needed to be demonstrated.

After introducing the two fundamental kinematic quantities $\tilde{u}$ and $\tilde{w}$, let us look for a kinetic quantity, which involves the mass *m* of the particle, and which corresponds to the momentum.

In Newtonian theory $\vec{p}=m\vec{v}$, it therefore seems natural to consider $\boxed{\tilde{p}=m\tilde{u}}$, the **four-momentum**, where *m* is an invariant characteristic of a particle. Then $\tilde{p}=(m\gamma c, m\gamma\vec{v})$. The term *momentum* is used for the 4-vector $\tilde{p}$ and its spatial component $\vec{p}=m\gamma\vec{v}$. You will notice that the letter *p* has been retained here, since the concept of momentum, as defined in relativity theory, has been fully adopted by the scientific community.
In $\vec{p}=m\gamma\vec{v}$, some associate γ with mass, and this is referred to as the relativistic mass γ*m*. As you have understood, in this course we prefer to associate γ with $\vec{v}$ to form the covariant velocity. Both interpretations are interesting.
For the spatial part of $\tilde{p}$, the classical momentum is found again when $v\to 0$: $\vec{p}=m\gamma\vec{v} \rightarrow m\vec{v}$.
On the other hand, what does the time component represent? It can be expressed as a quantity homogeneous to an energy: $p^0=m\gamma c=E/c$, $\tilde{p}=(E/c, \vec{p})$ and $E=m\gamma c^2$.
Could we give a physical meaning to this energy? (You know the answer!)
Classical limit: $E=m\gamma c^2=mc^2(1-(v/c)^2)^{-1/2}\simeq mc^2+\frac{1}{2}mv^2+...$, at order 1 in *v/c*, the classical kinetic energy appears! And a completely new zero-order term: an energy associated with the particle’s mass!! Thus, even at rest, a particle would have an energy… Well yes! And we call it *mass energy*: $E_0=mc^2$. A concept discovered by Einstein in his second 1905 paper ─ a new idea with applications that have profoundly change our society. All of this from a little game with covariant tensors…

We adopt the following definition: $E=E_0+T$, where *E* is the relativistic energy, divided into, E_0 energy at rest, and *T* the relativistic kinetic energy.

In the coinciding frame the particle is at rest: $\tilde{p}'=(mc, \vec{0})$
then $\tilde{p}'\cdot\tilde{p}'=m^2c^2=\tilde{p}\cdot\tilde{p}=\eta_{\mu\nu}p^\mu p^\nu=E^2/c^2-p^2$ and $E^2=m^2c^4+p^2c^2$.



The equation $E=mc^2$ implies a mass-energy equivalence: l’énergie peut se transformer en masse, et inversement, la masse en énergie. energy can be converted into mass, and conversely, mass into energy. For example, when two particles with high kinetic energy collide, particles can appear “out of nowhere.” This is how the antiproton was discovered in 1955 using a particle accelerator.

And of course, to conclude, a dynamic quantity, the **four-force**: $\frac{d\vec{p}}{dt}=\vec{F} \Rightarrow \boxed{\frac{d\tilde{p}}{d\tau}=\tilde{g}}$

so $\frac{dm\tilde{u}}{d\tau}=m\frac{d\tilde{u}}{d\tau}=m\tilde{w}=\tilde{g}$ and thus with the definition of classical force $\vec{F}=m\vec{a}$:

$$\tilde{g}=(g^0, \vec{g})=(\gamma^4\vec{F}\cdot\vec{\beta}, \gamma^4(\vec{F}\cdot\vec{\beta})\vec{\beta}+\gamma^2\vec{F})$$

In the next class, we will focus on collisions and the Lorentz force, examining the motion of a charged particle in an electromagnetic field.

Thank you.

Summary of the notions covered today:

Lecture 5 – 4-VECTORS

- 4-velocity / Diagram / Covariant velocity and maximum speed
- 4-acceleration / Orthogonality / Invariant
- Motion with constant proper acceleration
- Momentum and mass-energy equivalence
- Kinetic Energy / Energy-Mass-Momentum Triangle
- 4-force

To review and practice:

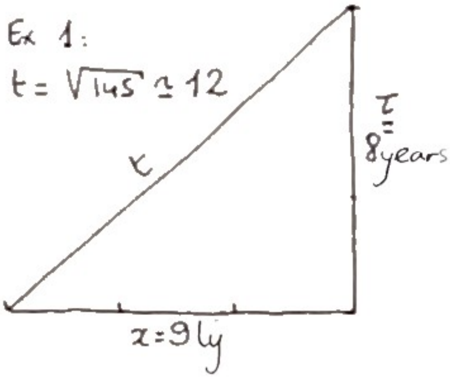
- Worksheet n°3 : exercises 1, 2 and 3.
- Chapter 7 of SR book pages 206 to 224, chapter 3, page 69.

Answers

Lectures 1 & 2

Ex 1:

$t=\sqrt{145}\simeq 12$

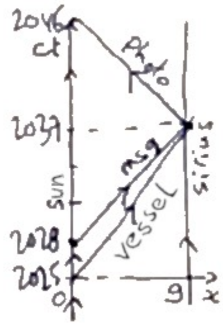


Arrival in 2125+12 = 2137

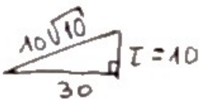
speed ≃ 9/12 c ≃ 75% of c

Message received the same year than the arrival : 3+9 = 12

Pic received in 2046 (+9)

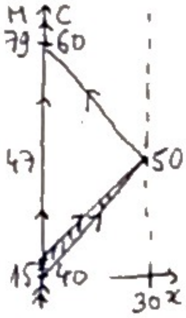


Ex 2:



$\gamma=\sqrt{10}\simeq 3.16$

Back, Maria has 79 and Carlos 60.



The letters sent between the ages of 15 and 16,6 by Maria will arrive before Carlos' arrival on Sirius : 19 Letters

Ex 3: $\Delta t=\gamma\tau$

$\Delta t-\tau$

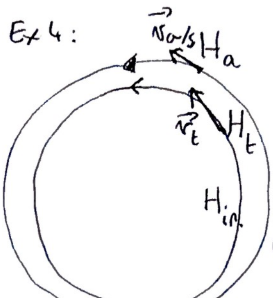
$=\Delta t\left(1-\sqrt{1-\beta^2}\right)$

$=\frac{1}{2}\Delta t\,\beta^2$

$\simeq 0{,}47\ ns$

drift 0,125 ns OK

Ex 4:



$(\Delta t)_t=\gamma_t(\Delta t)_{in.}$

$(\Delta t)_a=\gamma_a(\Delta t)_{in}$

$=\gamma_a/\gamma_t\,(\Delta t)_t$

$(\Delta t)_t-(\Delta t)_a=(1-\gamma_a/\gamma_t)(\Delta t)_t$

$\simeq\frac{1}{2}\cdot\frac{v_a^2-v_t^2}{c^2}(\Delta t)_t$

$v_t=4.10^7/(24\times 3600)\simeq 463\ m/s$

$v_a=\underbrace{v_{a/sol}}_{(-)}+v_t\simeq 741\ m/s$ $\quad\Delta t\simeq 161\ ms$

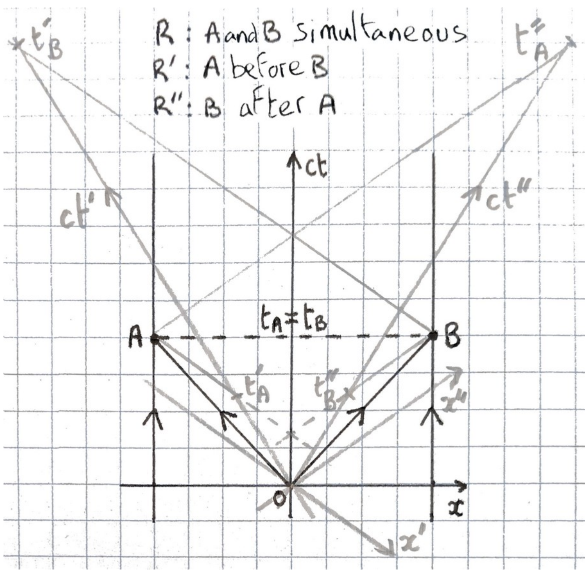
Terrestrial Clock in advance

$\simeq 278\ m/s$ Toward the west: $\Delta t\simeq -86\ ms$

Terrestrial Clock late

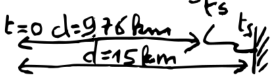
East	Ha/Hin	264ns	+161ns
	Ht/Hin	103ms	
West	Ha/Hin	16 ms	−86ns

Diagrams for the flashlight
in the middle of the wagon:



Exo 5: $P(t)=\lambda e^{-\lambda t}$; $p=P(T>t_{sol})=\int_{t_s}^{+\infty}\lambda e^{-\lambda t}dt=e^{-d/H}$ $\quad\lambda=1/\gamma\tau_0$ $\quad\tau=\gamma\tau_0$

$H=\gamma\tau\beta c\simeq 6575\ m$



15 km : $p\simeq 10{,}2\%$ $\quad$ 9760 m : $p\simeq 22{,}7\%$

(The memoryless law, on page 112 of the book by the same author : Probability, Stat. and Estimation)

For the train and tunnel paradox, see pages 75 and 76 of the book by the same author (fr):
Voyage pour Proxima, Comprendre Einstein plus vite que la lumière !

Lectures 3 & 4 **Exercise 1**

a) $g_{ij} = \vec{e}_i \cdot \vec{e}_j \qquad g_{ij} = \begin{pmatrix} 1 & 2\cos\theta \\ 2\cos\theta & 4 \end{pmatrix} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 4 \end{pmatrix} \qquad \|\vec{e}_1\| = 1 \quad \|\vec{e}_2\| = 2$

b) $dl^2 = d\vec{OM} \cdot d\vec{OM} = g_{ij}\, dx^i dx^j = g_{11}\, dx^1 dx^1 + g_{12}\, dx^1 dx^2 + g_{21}\, dx^2 dx^1 + g_{22}\, dx^2 dx^2$

$\Rightarrow dl^2 = dx^2 + 2\sqrt{2}\, dx\, dy + 4 dy^2$ (metric) $\qquad OA = \int_{\mathcal{C}} dl = \sqrt{\vec{OA} \cdot \vec{OA}} = \sqrt{g_{ij} x^i_A x^j_A} = \sqrt{5 + 2\sqrt{2}}$

with $\vec{OA} = \vec{e}_1 + \vec{e}_2 \qquad A(x^1_A = 1\ ;\ x^2_A = 1)$

c) $\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 = \vec{e}_1 + \frac{1}{2}\vec{e}_2 \qquad v^1 = 1 \qquad v^2 = 1/2$

d) $g^{ik} g_{kj} = \delta^i_j$ (System with 4 equations : $g^{11} g_{11} + g^{12} g_{21} = \delta^1_1 = 1$ ...)

or using matrices : $g\, g^{-1} = Id \xRightarrow{\text{maxima}}$ $\qquad g^{-1} = (g^{ij}) = \begin{pmatrix} 2 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 1/2 \end{pmatrix}$

e) $\vec{e}^{\,i} = g^{ij} \vec{e}_j$ or $\vec{e}^{\,i} \cdot \vec{e}_j = \delta^i_j \Rightarrow \vec{e}^{\,1}$ and $\vec{e}_2$ orthogonal as for $\vec{e}^{\,2}$ and $\vec{e}_1$.

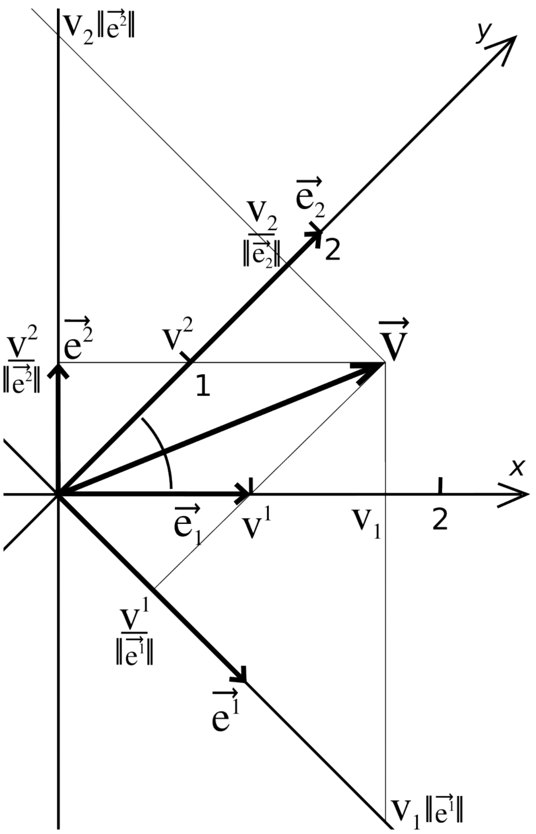
$\vec{e}^{\,1} = g^{11}\vec{e}_1 + g^{12}\vec{e}_2 = 2\vec{e}_1 - \frac{1}{\sqrt{2}}\vec{e}_2 \qquad \vec{e}^{\,2} = -\frac{1}{\sqrt{2}}\vec{e}_1 + \frac{1}{2}\vec{e}_2 \qquad \|\vec{e}^{\,1}\| = \sqrt{2} \qquad \|\vec{e}^{\,2}\| = \frac{1}{\sqrt{2}}$

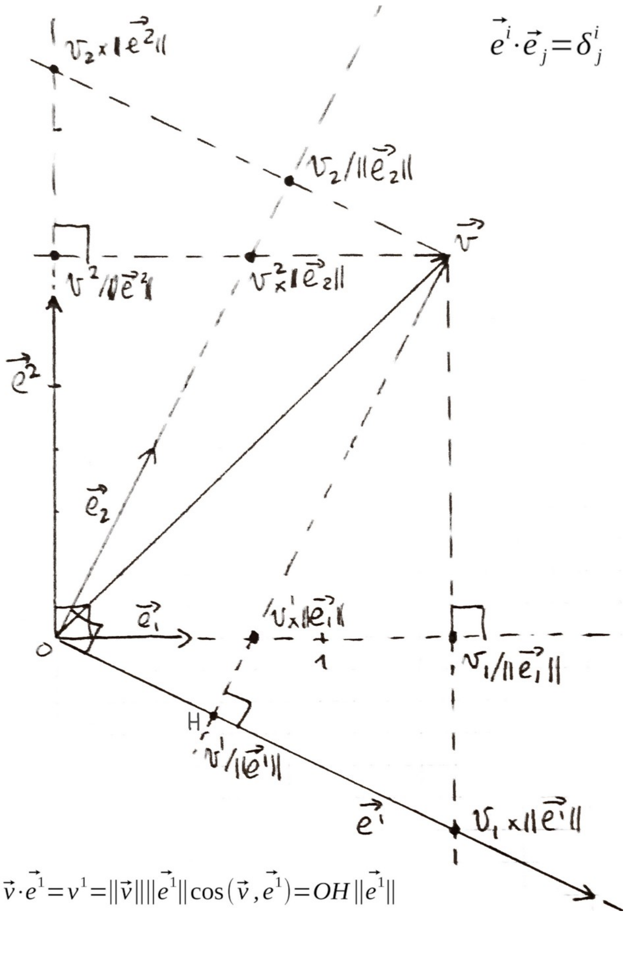
$v_i = g_{ij} v^j \qquad v_1 = g_{11} v^1 + g_{12} v^2 = 1 \times 1 + \sqrt{2} \times \frac{1}{2} \Rightarrow v_1 = 1 + \frac{1}{\sqrt{2}}$

$v_2 = g_{21} v^1 + g_{22} v^2 = \sqrt{2} + 4 \times \frac{1}{2} \Rightarrow v_2 = \sqrt{2} + 2$

$\vec{v} = v_1 \vec{e}^{\,1} + v_2 \vec{e}^{\,2} = (1 + 1/\sqrt{2})\vec{e}^{\,1} + (\sqrt{2} + 2)\vec{e}^{\,2}$

On the left is the figure for Exercise 1. On the right is the general case, showing the norms of the basis vectors to clearly illustrate the symmetry between the covariant and contravariant bases. The contravariant components of a vector are obtained by projecting parallel to the covariant basis. The covariant components of a vector are obtained by projecting orthogonally onto the covariant basis. The statements remain true if the words “covariant” and “contravariant” are interchanged. The two bases have interchangeable roles.





1) i) $\tilde{a}.\tilde{b} = g_{ij}a^ib^j = g_{ij}b^ja^i = g_{ji}b^ja^i = g_{ij}b^ia^j = \tilde{b}.\tilde{a}$ (g symetric / dummy indexes)

ii) $\tilde{a}.(\tilde{b}+\tilde{c}) = g_{ij}a^i(b^j+c^j) = g_{ij}a^ib^j + g_{ij}a^ic^j = \tilde{a}.\tilde{b} + \hat{a}.\tilde{c}$

2) i) $\varepsilon_{321} = -\varepsilon_{123}$ $(3 \to 1 \,/\, 1 \to 3)$ $\Rightarrow$ $\varepsilon_{321} = -1$ ii) $\varepsilon^{123} = g^{1i}g^{2j}g^{3k}\varepsilon_{ijk}$ or $g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \varepsilon^{123} = g^{11}g^{22}g^{33}\varepsilon_{123}$ $\Rightarrow$ $\varepsilon^{123} = 1$ (only one non-nul term out of g)

iii) $\varepsilon_{12k}\varepsilon^{12k} = \varepsilon_{123}\varepsilon^{123} + 0 + 0 = 1$ iv) $\varepsilon_{ijk}\varepsilon^{ijk} = 3\times2\times1\times(\pm1)^2 = 3!$

v) if $i \neq j$, $\varepsilon_{imn} = 0$ or $\varepsilon^{jmn} = 0$; if $i = j$, $\varepsilon_{imn}\varepsilon^{imn} = 3! = 2\delta_i^i = 2(\delta_1^1 + \delta_2^2 + \delta_3^3) = 6$ #

vi) $\varepsilon_{ijk}\varepsilon^{mnl} = \begin{vmatrix} \delta_i^m & \delta_i^n & \delta_i^l \\ \delta_j^m & \delta_j^n & \delta_j^l \\ \delta_k^m & \delta_k^n & \delta_k^l \end{vmatrix} = \delta_i^m(\delta_j^n\delta_k^l - \delta_k^n\delta_j^l) - \delta_i^n(\delta_j^m\delta_k^l - \delta_k^m\delta_j^l) + \delta_i^l(\delta_j^m\delta_k^n - \delta_k^m\delta_j^n)$

$\Rightarrow \varepsilon_{ijk}\varepsilon^{mnk} = 3\delta_i^m\delta_j^n - \delta_i^m\delta_j^n - 3\delta_j^m\delta_i^n + \delta_i^n\delta_j^m + \delta_i^n\delta_j^m - \delta_i^m\delta_j^n$

$\Rightarrow \varepsilon_{ijk}\varepsilon^{mnk} = \delta_i^m\delta_j^n - \delta_i^n\delta_j^m$ ((v): $\varepsilon_{ijk}\varepsilon^{mjk} = \delta_i^m\delta_j^j - \delta_i^j\delta_j^m = \delta_i^m\times3 - \delta_i^m = 2\delta_i^m$)

vii)

$\vec{a}\wedge\vec{b} = \begin{pmatrix} a^1 \\ a^2 \\ a^3 \end{pmatrix} \wedge \begin{pmatrix} b^1 \\ b^2 \\ b^3 \end{pmatrix} = \begin{pmatrix} a^2b^3 - a^3b^2 \\ a^3b^1 - a^1b^3 \\ a^1b^2 - a^2b^1 \end{pmatrix} = \begin{pmatrix} \varepsilon_{1jk}a^jb^k \\ \varepsilon_{2jk}a^jb^k \\ \varepsilon_{3jk}a^jb^k \end{pmatrix}$ $\Rightarrow$ $(\vec{a}\wedge\vec{b})_i = \varepsilon_{ijk}a^jb^k$

viii) $(\vec{a}\wedge\vec{a})_i = \varepsilon_{ijk}a^ja^k = \varepsilon_{ijk}a^ka^j = -\varepsilon_{ikj}a^ka^j = -\varepsilon_{ijk}a^ja^k = 0$

ix) $(\vec{a}\wedge\vec{b})_i = \varepsilon_{ijk}a^jb^k \underset{viii)}{\overset{idem}{=}} -\varepsilon_{ijk}b^ja^k = -(\vec{b}\wedge\vec{a})_i$

x) $(\vec{a}\wedge\vec{b})^l = g^{li}(\varepsilon_{ijk}a^jb^k) = g^{li}\varepsilon_{ijk}g^{jm}a_mg^{kn}b_n = \varepsilon^{lmn}a_mb_n$ $\Rightarrow$ $(\vec{a}\wedge\vec{b})^i = \varepsilon^{ijk}a_jb_k$

xi) $(\vec{u}\wedge\vec{v}).\vec{w} = \varepsilon_{ijk}u^jv^kw^i = u^j\varepsilon_{jki}v^kw^i = \vec{u}.(\vec{v}\wedge\vec{w})$

xii) $(\vec{a}\wedge(\vec{b}\wedge\vec{c}))_i = \varepsilon_{ijk}a^j\varepsilon^{kmn}b_mc_n = (\delta_i^m\delta_j^n - \delta_i^n\delta_j^m)a^jb_mc_n = b_ia^jc_j - c_ia^jb_j$