Gravitational Waves

Exercise 1 Lecture: considering the weak gravitational field: $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$

1) Show at order 1 in *h*: $E_{\mu\nu}=\left(-\partial_\mu\partial_\alpha h^\alpha_\nu-\partial_\nu\partial_\alpha h^\alpha_\mu+\partial_\mu\partial_\nu h+\Box h_{\mu\nu}+\eta_{\mu\nu}\partial^\beta\partial_\alpha h^\alpha_\beta-\eta_{\mu\nu}\Box h\right)/2$.

2) With the gauge conditions $\partial_\mu\bar{h}^\mu_\nu=0$, $\bar{h}^\mu_\nu=h^\mu_\nu-\delta^\mu_\nu h/2$ and a traceless metric perturbation,
find $\Box h_{\mu\nu}=\frac{16\pi G}{c^4}T_{\mu\nu}$.

Exercise 2 Lorenz gauge for the gravitational waves (harmonic gauge)

On the local tangent minkowskian plane, the first derivatives of the metric are null.
This condition gives the Lorentz gauge.

1) Show that $h_{\mu\nu}$ transforms as a covariant tensor in the Minkowski space. The flat metric can be used.

Let's consider an infinitesimal change of coordinates in the coinciding minkowskian plane:

$$x'^\mu=x^\mu+\xi^\mu\left(x^\nu\right)$$

2) Perform an infinitesimal local change of coordinates of $g_{\mu\nu}$ to find $h'_{\mu\nu}=h_{\mu\nu}-\partial_\mu\xi_\nu-\partial_\nu\xi_\mu$.
So, as the first derivatives of the metric are zero on the local tangent plane, $h'_{\mu\nu}$ and $h_{\mu\nu}$ have the same physical meaning for all ξ. Like gauge invariance $A'_\mu=A^{new}_\mu=A_\mu-\partial_\mu f$ in electromagnetism.

3) Find for the trace: $h'=h-2\partial_\mu\xi^\mu$.

4) And demonstrate for $\bar{h}_{\mu\nu}=h_{\mu\nu}-\eta_{\mu\nu}h/2$ that $\bar{h}'_{\mu\nu}=\bar{h}^{new}_{\mu\nu}=\bar{h}_{\mu\nu}-\partial_\mu\xi_\nu-\partial_\nu\xi_\mu+\eta_{\mu\nu}\partial_\alpha\xi^\alpha$

5) Prove that if ξ is chosen as $\Box\xi_\mu=\partial^\nu\bar{h}_{\mu\nu}$ then for the new field the Lorenz gauge $\partial^\nu\bar{h}_{\mu\nu}=0$ is satisfied.

The Lorenz gauge does not fix everything. From the Einstein equation, the divergence of $T_{\mu\nu}$ and $E_{\mu\nu}$ are null, then $\partial^\nu\bar{h}_{\mu\nu}=0$. That is, we have the additional condition $\Box\xi_\mu=0$ witch leaves only two degrees of freedom for the plane wave polarization (Hobson p498 / Sénéchal p145 / Blanchet p53).