ANSWERS

Accelerating Frame and Rotating Disk

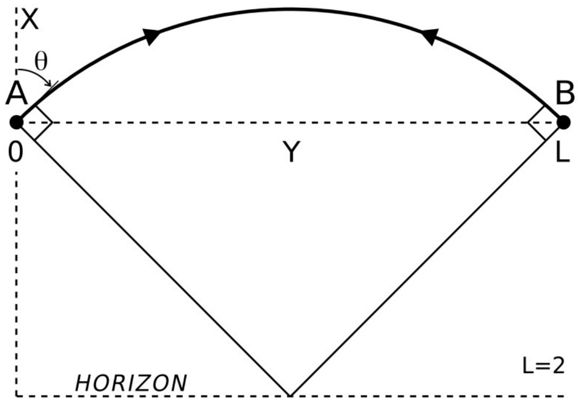
Exercise 1 *Two rockets with the same proper acceleration synchronized*

Equations of the trajectory of an objet in an accelerating frame, at $T=0$, $X=0$ and $Y=0$:

$$X = \frac{1}{\cosh T - \beta_0 \sinh T \cos\theta} - 1 \qquad \text{and} \qquad Y = \frac{\beta_0 \sinh T \sin\theta}{\cosh T - \beta_0 \sinh T \cos\theta},$$

β_0c is the initial speed and θ the angle with the vertical direction. $T=t/t_H$, $X=x/d_H$ and $Y=y/d_H$.

1) Two rockets in the accelerating frame, at rest at A and B, exchange photons to verify they are well synchronized (see the figure, $X_A=X_B=0$, $Y_A=0$ and $Y_B=2$).



a- How long it takes for a photon to go from A to B?

The arc AB is a quarter circle ⇒ $\theta=\pi/4$.
Light speed for X=0: $\beta_0=1$.
so $X=0 \Rightarrow\ ch T - sh T/\sqrt{2}=1$ and $Y=2 \Rightarrow sh T=2\sqrt{2}$
and $T=argsh(2\sqrt{2})\simeq 1.76$

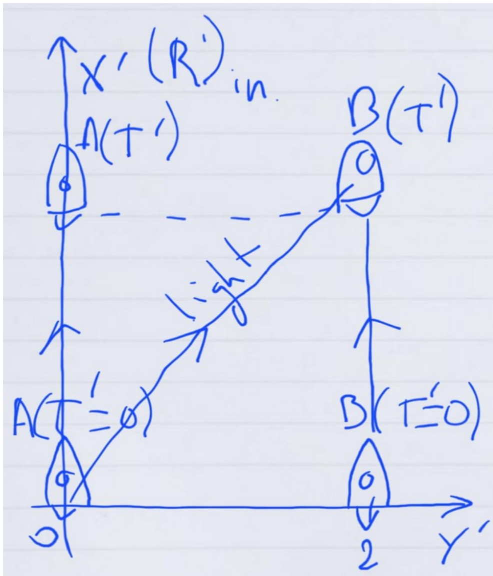
b- What distance did the photon travel?

Circle radius: $R=\sqrt{2}$. Arc $AB=\pi R/2=\pi/\sqrt{2}\simeq 2.22$

c- What is its average speed?

$v/c=\pi/\sqrt{2}/\text{argsh } 2\sqrt{2}\simeq 1.26$

2) Let's now consider the point of view of an inertial observer in R'. At $T'=0$, $T=0$ and $X'=X=0$.
Change of coordinates:
$T'=(X+1)\, shT$, $X'=(X+1)\, chT-1$ and $Y'=Y$.



a- Drawn the two rockets and the ray of light in R'.
b- Give the coordinates of the events: the photon starts from A, and, the photon arrives at B.

A: T=0, T'=0, X'=0, Y'=0.
B: $T=\text{argsh}2\sqrt{2}$, $T'=(0+1)\, 2\sqrt{2}=2\sqrt{2}$,
$X'=(0+1)\text{ch}(\text{argsh}2\sqrt{2})-1=2$ and $Y'=Y=2$.

c- How long it takes for a photon to go from A to B?
$\Delta T'=T'_B-T'_A=2\sqrt{2}$

d- What distance did the photon travel?
$AB=\sqrt{(2^2+2^2)}=2\sqrt{2}$

e- What is its average speed? $D'/T'=1=v/c \Rightarrow v=c$

3) Comments.
In the non-inertial frame of the rocket, as seen with the metric with $ds^2=0$, the coordinate speed of light is greater than c when $X>1$ then the average speed v/c is greater than 1.
In an inertial frame of reference we always have the speed of light equal c, and the geodesic is rectilinear.

Exercise 2 *Ehrenfest Paradox*

Metric on the rotating disk: $ds^2=\left(1-\frac{\rho^2\omega^2}{c^2}\right)c^2dt^2-2\rho^2\omega\, dt\, d\theta-d\rho^2-\rho^2 d\theta^2-dz^2$

1) Find the expression of the connections $\Gamma^{\lambda}{}_{\mu\nu}$.

Done on page 412 of the book SRfriends.pdf :

$$g_{tt}=g_{00}=1-\frac{\omega^2\rho^2}{c^2},\ g_{t\theta}=g_{02}=g_{20}=-2\omega\frac{\rho^2}{c},\ g_{11}=-1,\ g_{22}=-\rho^2 \text{ and } g_{33}=-1.$$

The metric is not diagonal. Only the $\partial_1 g_{00}$, $\partial_1 g_{22}$ and $\partial_1 g_{02}$ are different from zero.

Also $g^{\mu\nu}g_{\nu\sigma}=\delta^{\mu}_{\sigma}\Rightarrow\ g^{33}=1/g_{33},\ g^{11}=1/g_{11}$, but, $g^{02}g_{20}+g^{00}g_{00}=1,\ g^{02}g_{22}+g^{00}g_{02}=0,$ $g^{22}g_{22}+g^{20}g_{02}=1$ and $g^{22}g_{20}+g^{20}g_{00}=0$

then: $g^{\mu\nu}=\begin{pmatrix}1 & 0 & -\frac{\omega}{c} & 0\\ 0 & -1 & 0 & 0\\ -\frac{\omega}{c} & 0 & \frac{\omega^2}{c^2}-\frac{1}{\rho^2} & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$. $\quad\partial_1 g_{00}=-2\frac{\omega^2\rho}{c^2},\ \partial_1 g_{22}=-2\rho,\ \partial_1 g_{02}=-4\frac{\omega\rho}{c}.$

Connections symmetric on the covariant indices.

$$\Gamma^2{}_{12}=\frac{1}{2}g^{22}(\partial_1 g_{22}+\partial_2 g_{21}-\partial_2 g_{12})+\frac{1}{2}g^{20}(\partial_1 g_{02}+\partial_2 g_{01}-\partial_0 g_{12})$$

$\Gamma^1{}_{00}=\frac{1}{2}g^{11}(\partial_0 g_{10}+\partial_0 g_{10}-\partial_1 g_{00})$ and after calculations only 5 non null connections:

$$\Gamma^1{}_{00}=-\frac{\rho\omega^2}{c^2}\qquad \Gamma^1{}_{02}=-\frac{\rho\omega}{c}\qquad \Gamma^1{}_{22}=-\rho\qquad \Gamma^2{}_{10}=\frac{\omega}{\rho c}\qquad \Gamma^2{}_{12}=\frac{1}{\rho}$$

2) Find the components of the Riemann tensor $R^{\alpha}{}_{\beta\mu\nu}$. Comments.

20 independent components:

$R^1{}_{001}=0-\Gamma^1{}_{00,1}+\Gamma^1{}_{\sigma 0}\Gamma^{\sigma}{}_{01}-0=\frac{\omega^2}{c^2}-\rho\frac{\omega}{c}\frac{\omega}{\rho c}=0$

... etc, done page 415 ... all the components are null, spacetime is flat.

Normal, because the metric is obtain from the minkowskian metric with a global change of coordinates: $R^{\alpha}{}_{\beta\mu\nu}=\frac{\partial x^{\alpha}}{\partial x'^{\alpha'}}\frac{\partial x'^{\beta'}}{\partial x^{\beta}}\frac{\partial x'^{\mu'}}{\partial x^{\mu}}\frac{\partial x'^{\nu'}}{\partial x^{\nu}}(R'^{\alpha'}{}_{\beta'\mu'\nu'})_{Mink}=0$

3) Now, let's find the space curvature.

a- When the disk is motionless, give the metric and the surface of the disk of radius R.

$ds^2=c^2dt^2-d\rho^2-\rho^2 d\theta^2-dz^2$ (Minkowski), Euclidean space: $S=\pi R^2$.

b- When the disk rotates with an angular speed ω the reference system, has said by *Landau*, is not "synchronous", $g_{0i}\neq 0$, the temporal coordinate is not directly separated from the spatial coordinates. Then, it is shown that the spatial metric is:

$$dl^2=\gamma_{ij}dx^i dx^j\qquad \text{with}\qquad \gamma_{ij}=-g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}}$$

(Landau/Lifchitz, *The Classical Theory of Field*, § Distances and time intervals)

The lengths and the surface $S=\int\sqrt{|det\,\gamma_{ij}|}\,dx^i\,dx^j$ can be calculate:

i) Express the three-dimensional metric tensor γ_{ij}.

Page 419: ... $\gamma_{ij}=\begin{pmatrix}1 & 0 & 0\\ 0 & \gamma^2\rho^2 & 0\\ 0 & 0 & 1\end{pmatrix}$

ii) Determine the diameter D and the perimeter P of the disk.
What is the value of P/D? Comments.

Page 420: ... $\frac{P}{D}=\frac{\int_{\theta=0}^{\theta=2\pi}\sqrt{\gamma_{22}}\,d\theta}{2\int_{\rho=0}^{\rho}\sqrt{\gamma_{11}}\,d\rho}=\frac{\gamma\rho\int_{\theta=0}^{\theta=2\pi}d\theta}{2\int_{\rho=0}^{\rho}d\rho}=\gamma\pi>\pi$, space is curve (even if spacetime is flat).

After, an other way to prove that the space of the disk is curve,
also in the book, the calculation of $R^1{}_{212}$ is done: $R^1{}_{212}=-3\beta^2\gamma^6\neq 0$, and $K<0$.

iii) Find S and compare with the euclidean surface when $\omega\ll c/R$.

$det\,\gamma_{ij}=1\times\gamma^2\rho^2\times 1$ and $\sqrt{|det\,\gamma_{ij}|}=\frac{\rho}{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}}$ then $S=\int_{\theta=0}^{2\pi}\int_{\rho=0}^{R}\frac{\rho\,d\rho\,d\theta}{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}}=2\pi\frac{c^2}{\omega^2}\int_{u=0}^{R\frac{\omega}{c}}\frac{u\,du}{\sqrt{1-u^2}}$

$$\dots\ S=2\pi\frac{c^2}{\omega^2}\left[1-\sqrt{1-\left(R\frac{\omega}{c}\right)^2}\right]\simeq\pi R^2+\frac{1}{8}\left(\frac{R\omega}{c}\right)^4>S_{Euclid}$$