Connections and Riemann Curvature Tensor

For the following exercises the connection and the Riemann tensor:

$$\Gamma^{\alpha}{}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}(\partial_{\mu}g_{\beta\nu}+\partial_{\nu}g_{\beta\mu}-\partial_{\beta}g_{\mu\nu}) \qquad \text{and} \qquad R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}{}_{\beta\delta,\gamma}-\Gamma^{\alpha}{}_{\beta\gamma,\delta}+\Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta}-\Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}$$

2D, Gauss' Curvature: $$K=\frac{1}{R_1 R_2}=\frac{R_{1212}}{g_{11}g_{22}-g_{12}^2}$$

Exercise 1 *Riemann Tensor Symmetries*

1) Show the following anti-symmetry: $R^{\alpha}{}_{\beta\gamma\delta}=-R^{\alpha}{}_{\beta\delta\gamma}$.

2) Give the expression of $\Gamma_{\lambda\mu\nu}$. Is there any symmetry between the first two indices?

3) Give the expression of the purely covariant version of the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$.

To see the symmetries of $R_{\alpha\beta\gamma\delta}$, one can develop all the terms as second derivatives of the metric, and products of the first derivatives. The calculations are quite long. But, there is a trick, you can do the calculations in the local inertial frame, where it is much simpler (link).

The following properties are obtained:

- Anti-symmetries: $R_{\alpha\beta\gamma\delta}=-R_{\beta\alpha\gamma\delta}=-R_{\alpha\beta\delta\gamma}$
- Symmetry under interchange of the first two indices with the last two: $R_{\alpha\beta\gamma\delta}=R_{\gamma\delta\alpha\beta}$
- Cyclic permutation of the three last indices: $R_{\alpha\beta\gamma\delta}+R_{\alpha\delta\beta\gamma}+R_{\alpha\gamma\delta\beta}=0$

For a rank-4 tensor in four dimensions there are $4^4=256$ components. Here, from the symmetries, only 20 components are independents (link).

Exercise 2 *Riemann Stereographic Sphere Projection / induced metric:* $dl^2=\dfrac{dx^2+dy^2}{\left(1+\dfrac{x^2+y^2}{4R^2}\right)^2}=\dfrac{dx^2+dy^2}{h^2}$

1) Give the expression of the connection components $\Gamma^{i}{}_{jk}$.

2) Find the Riemann tensor components $R^{1}{}_{212}$.

3) Determine the Gauss' curvatures R_1 and R_2.

Exercise 3 *Schwarzschild Metric:* $ds^2=g(r)c^2dt^2-\dfrac{dr^2}{g(r)}-r^2d\theta^2-r^2\sin^2\theta\, d\phi^2$ & $g(r)=1-\dfrac{r_S}{r}$

1) Prove the following expressions of the connection components: $\Gamma^{0}{}_{10}=g'/2g$ and $\Gamma^{2}{}_{21}=\Gamma^{3}{}_{31}=1/r$.

Given expressions: $\Gamma^{1}{}_{11}=-g'/2g$, $\Gamma^{1}{}_{00}=g\,g'/2$, $\Gamma^{1}{}_{22}=-rg$, $\Gamma^{1}{}_{33}=-rg\sin^2\theta$, $\Gamma^{2}{}_{33}=-\sin\theta\cos\theta$, $\Gamma^{3}{}_{32}=1/\tan\theta$, all the others connections are nulls.

2) Show that: $R_{0101}=\dfrac{r_S}{r^3}$.