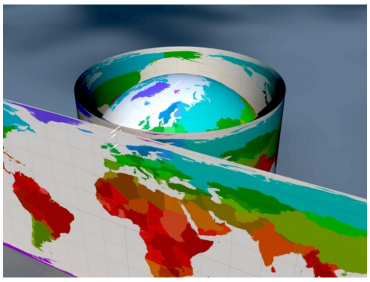
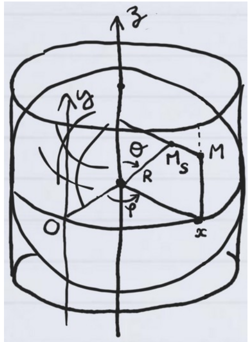
# Metric and Lagrangian

## Exercise 1 *Lambert cylindrical projection*

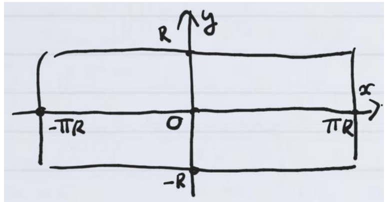


1) Find the induced metric on the cylinder *M(x,y)*.
   Points: A(0,R), B(-πR,R/2) and C(πR,R/2).
2) On the map find with the metric:
   - a- the distance OA along a longitude.
   - b- the distance BC along a latitude.
   - c- the surface of the map.

   Are the results consistent?

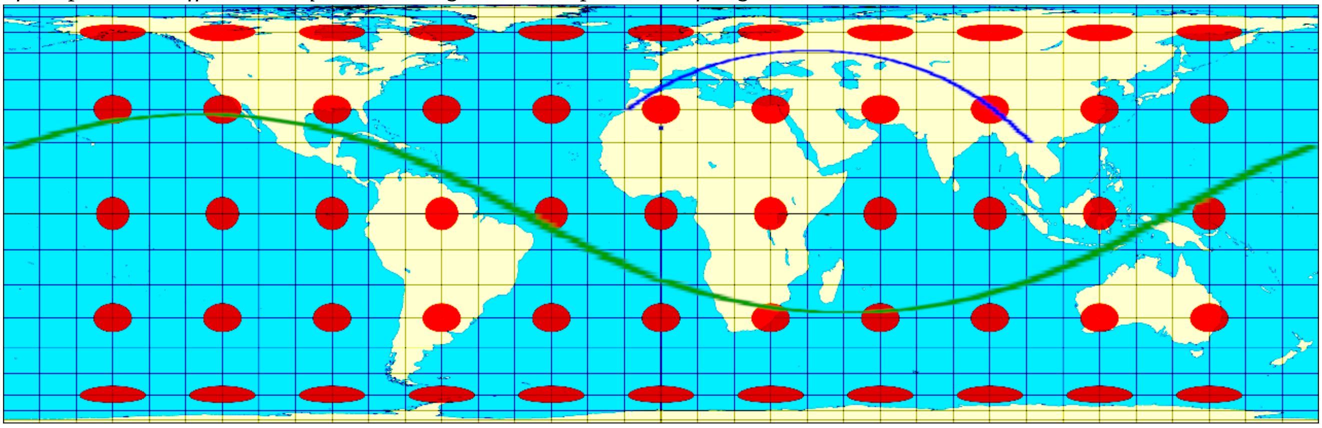



3) Is this map easy to use for navigation?
4) Show that this map preserves the surface areas.
5) Find the differential equations of the geodesics:




$$\begin{cases} \ddot{u}(1-v^2)-2\dot{u}\dot{v}v=0 \\ \ddot{v}(1-v^2)+\dot{v}^2 v+\dot{u}^2 v(1-v^2)^2=0 \end{cases} \quad \text{with} \quad \begin{cases} u=x/R \\ v=y/R \end{cases}$$

*Globe Lambert cylindrical map with Tissot's indicatrix of deformation, in blue a geodesic find from a numerical resolution of the previous differential equations, the green line represents a full great circle:*



## Exercise 2 *Lagrangian and Maxwell's equations*

For a continuous field $\phi_\nu(x^\mu)$ there is a lagrangian density $\mathscr{L}(\phi_\nu,\partial_\mu\phi_\nu)$.
The lagrangian $L$ is then expressed as a volume integral:

$$L=\int \mathscr{L}(\phi_\nu,\partial_\mu\phi_\nu)\,d^3x \quad \text{and the equations of motion are} \quad \frac{\partial \mathscr{L}}{\partial \phi_\nu}=\partial_\mu \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi_\nu)}.$$

In the case of the electromagnetic field $A_\nu$ interacting with charges:

the field is the four-potential $A^\mu=(V/c,\vec{A})$ and $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$

1) Find the known relations $\vec{E}=-\vec{\nabla}V-\partial\vec{A}/\partial t$ and $\vec{B}=\vec{\nabla}\wedge\vec{A}$ (see Ex.4, tensors sheet 1 for $F^{\mu\nu}$).

$$\text{now} \quad \phi_\nu=A_\nu, \quad \mathscr{L}=\mathscr{L}_{field}+\mathscr{L}_{interaction} \quad \text{and} \quad \mathscr{L}=k\,F_{\mu\nu}F^{\mu\nu}-j^\mu A_\mu$$

2) From the lagrangian density, find $k$ to obtain the Maxwell's equation with sources $\partial_\mu F^{\mu\nu}=\mu_0 j^\nu$.

3) Prove the homogeneous Maxwell's equation $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$.
   No need to use the lagrangian, it comes from the structure of $F^{\mu\nu}$.