Tensor Training

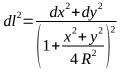
Surfaces, Volumes, Covariant Derivative and Curvatures

Exercise 1 Surfaces and volumes

 $I_n = \int \sqrt{|\det g|} d^n x \text{ for } n \text{ dimensions.}$ For a surface $S = \int \sqrt{|\det g_{ij}|} dx^i dx^j$ with g_{ij} represented by a 2-dimension matrix. For a volume $V = \int \sqrt{|\det g_{ij}|} dx^1 dx^2 dx^3$ with g_{ij} represented by a 3-dimension matrix. This formulas works for all kinds of metrics.

- 1) Let's begin with a known example, the euclidean sphere. In spherical coordinates we have the metric $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$.
 - a- Give the $g_{\theta\phi}$ matrix and det $(g_{\theta\phi})$. With the previous *S* formula find the known surface of a sphere of radius *R*. Here, a view from outside the sphere surface is used.
 - b- Give the spatial g_{ij} matrix and det(g). With the previous V formula find the known volume of a sphere of radius R.

2) Let's find again the surface *S* of the sphere from inside, with the Riemann projection induced metric:



a- To simplify the calculations, give the metric in polar coordinates ρ and θ .

b- Although $\rho \in [0, +\infty[$, with the previous method, do you find the same surface *S*?

Exercise 2 Covariant Derivative

The formula is $D_{\mu}A^{\nu} = \partial_{\mu}A^{\nu} + \Gamma^{\nu}{}_{\lambda\mu}A^{\lambda}$ with $\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_{\mu}g_{\beta\nu} + \partial_{\nu}g_{\beta\mu} - \partial_{\beta}g_{\mu\nu}).$

For a rank-2 tensor $T = T^{\mu\nu} \widetilde{e}_{\mu} \otimes \widetilde{e}_{\nu}$ and $D_{\lambda} T^{\mu\nu} = \partial_{\lambda} T^{\mu\nu} + \Gamma^{\mu}_{\sigma\lambda} T^{\sigma\nu} + \Gamma^{\nu}_{\sigma\lambda} T^{\mu\sigma}$.

Some demonstrations to do: i) Prove that $d(\tilde{e}^{\nu} \cdot \tilde{e}_{\mu}) = 0$

- ii) To find the expression of the covariant derivative on a covariant vector: $D_{\mu}A_{\nu} = \partial_{\mu}A_{\nu} \Gamma^{\lambda}_{\nu\mu}A_{\lambda}$, just do the calculation on the contravariant base: $d\widetilde{A} = d(A_{\nu}\widetilde{e}^{\nu}) = dA_{\nu}\widetilde{e}^{\nu} + A_{\nu}d\widetilde{e}^{\nu}$.
 - iii) Have a guess for $D_{\lambda}T^{\mu}_{\nu}$ and $D_{\lambda}T_{\mu\nu}$ iv) Prove the Ricci Theorem: $D_{\lambda}g_{\mu\nu}=0$

v) We have the derivative product formula $D_{\lambda}(A_{...}^{...}B_{...}^{...}) = B_{...}^{...}D_{\lambda}A_{...}^{...} + A_{...}^{...}D_{\lambda}B_{...}^{...}$, use it to prove that $D_{\lambda}g^{\mu\nu} = 0$ and $D_{\lambda}(g_{\mu\nu}A^{\mu}) = g_{\mu\nu}D_{\lambda}A^{\mu}$, the metric passes through the covariant derivative!

vi) Prove that the covariant derivatives do not commute: $D_{\lambda}D_{\sigma}V^{\mu} \neq D_{\sigma}D_{\lambda}V^{\mu}$, the Riemann curvature tensor is the commutator of the covariant derivatives: $(D_{\mu}D_{\nu}-D_{\nu}D_{\mu})V^{\lambda}=R^{\lambda}_{\sigma\mu\nu}V^{\sigma}$

And numerous other formulas to discover!

Like the Bianchi identity: $D_{\lambda}R_{\rho\sigma\mu\nu} + D_{\rho}R_{\sigma\lambda\mu\nu} + D_{\sigma}R_{\lambda\rho\mu\nu} = 0$ (easier to demonstrate this identity in the local minkowskian frame)