Tensor Training

Surfaces, Volumes, Covariant Derivative and Curvatures

Exercise 1 *Surfaces and volumes*

$I_n = \int \sqrt{|det\, g|}\, d^n x$ for n dimensions.
For a surface $S = \int \sqrt{|det\, g_{ij}|}\, dx^i dx^j$ with g_{ij} represented by a 2-dimension matrix.
For a volume $V = \int \sqrt{|det\, g_{ij}|}\, dx^1 dx^2 dx^3$ with g_{ij} represented by a 3-dimension matrix.
This formulas works for all kinds of metrics.

1) Let's begin with a known example, the euclidean sphere.
In spherical coordinates we have the metric $dl^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$.

a- Give the $g_{\theta\varphi}$ matrix and det($g_{\theta\varphi}$). With the previous S formula find the known surface of a sphere of radius R. Here, a view from outside the sphere surface is used.
b- Give the spatial g_{ij} matrix and det(g). With the previous V formula find the known volume of a sphere of radius R.

2) Let's find again the surface S of the sphere from inside, with the Riemann projection induced metric: $dl^2 = \dfrac{dx^2+dy^2}{\left(1+\dfrac{x^2+y^2}{4R^2}\right)^2}$

a- To simplify the calculations, give the metric in polar coordinates ρ and θ.
b- Although $\rho \in [0,+\infty[$, with the previous method, do you find the same surface S?

Exercise 2 *Covariant Derivative*

The formula is $D_\mu A^\nu = \partial_\mu A^\nu + \Gamma^\nu{}_{\lambda\mu} A^\lambda$ with $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$.

For a rank-2 tensor $\boldsymbol{T} = T^{\mu\nu}\, \tilde{e}_\mu \otimes \tilde{e}_\nu$ and $D_\lambda T^{\mu\nu} = \partial_\lambda T^{\mu\nu} + \Gamma^\mu{}_{\sigma\lambda} T^{\sigma\nu} + \Gamma^\nu{}_{\sigma\lambda} T^{\mu\sigma}$.

Some demonstrations to do: i) Prove that $d(\tilde{e}^\nu \cdot \tilde{e}_\mu) = 0$

ii) To find the expression of the covariant derivative on a covariant vector: $D_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\lambda{}_{\nu\mu} A_\lambda$,
just do the calculation on the contravariant base: $d\tilde{A} = d(A_\nu \tilde{e}^\nu) = d A_\nu \tilde{e}^\nu + A_\nu d\tilde{e}^\nu$.

iii) Have a guess for $D_\lambda T^\mu{}_\nu$ and $D_\lambda T_{\mu\nu}$ iv) Prove the Ricci Theorem: $D_\lambda g_{\mu\nu} = 0$

v) We have the derivative product formula $D_\lambda(A^{\cdots}_{\cdots} B^{\cdots}_{\cdots}) = B^{\cdots}_{\cdots} D_\lambda A^{\cdots}_{\cdots} + A^{\cdots}_{\cdots} D_\lambda B^{\cdots}_{\cdots}$,
use it to prove that $D_\lambda g^{\mu\nu} = 0$ and $D_\lambda(g_{\mu\nu} A^\mu) = g_{\mu\nu} D_\lambda A^\mu$,
the metric passes through the covariant derivative!

vi) Prove that the covariant derivatives do not commute: $D_\lambda D_\sigma V^\mu \neq D_\sigma D_\lambda V^\mu$,
the Riemann curvature tensor is the commutator of the covariant derivatives:

$$(D_\mu D_\nu - D_\nu D_\mu) V^\lambda = R^\lambda{}_{\sigma\mu\nu} V^\sigma$$

And numerous other formulas to discover!

Like the Bianchi identity: $D_\lambda R_{\rho\sigma\mu\nu} + D_\rho R_{\sigma\lambda\mu\nu} + D_\sigma R_{\lambda\rho\mu\nu} = 0$
(easier to demonstrate this identity in the local minkowskian frame)