Examination

Everything must be fully justified on the basis of the definitions or the appendix.

Problem *Milne Universe*

The Milne model is a cosmological model of the universe. This model is a particular case of the Friedmann–Lemaître–Robertson–Walker model when the energy density is very small compared to the critical density (flat space). The Milne metric gives a flat space-time.

Milne metric: $ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - c^2 t^2 (d\chi^2 + \sinh^2\chi \, d\Omega^2)$ with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

1) Using Einstein's equation and the information given in the introduction, detail the demonstration to find the expression for the matter density ρ_0 of the universe according to Milne's model. Isolated particles are considered: $T^{\mu\nu} = \rho_0 u^\mu u^\nu$.

For simplicity's sake, let's consider a 3D Milne spacetime toy model:

$$ds^2 = c^2 dt^2 - c^2 t^2 (d\chi^2 + \sinh^2\chi \, d\theta^2) \quad \text{with} \quad x^\mu (x^0 = ct, x^1 = \chi, x^2 = \theta).$$

2) Calculate all the components of the Christoffel symbols.

3) Express all the components of the Riemann curvature tensor for this metric.

4) Is spacetime flat?

5) Now, let's consider the spatial part dl^2 of the metric. Is space flat?
Find the Gauss curvature. What kind of geometry do we have?

6) Do you know another example in relativity where there is a curved space within a flat spacetime?
Explain also with a metric.

APPENDIX:

Connections (Christoffel symbols): $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$

Riemann tensor: $R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\sigma\gamma} \Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta} \Gamma^\sigma{}_{\beta\gamma}$

Ricci tensor: $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$ *Ricci scalar:* $R = g^{\mu\nu} R_{\mu\nu}$

$(\sinh x)' = \cosh x$

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$\cosh^2 x - \sinh^2 x = 1$

Gauss' curvature (2D space): $K = \frac{1}{R_1 R_2} = \frac{R_{1212}}{\gamma_{11}\gamma_{22} - \gamma_{12}^2}$ with $dl^2 = \gamma_{ij} dx^i dx^j$

K=0 : flat space
K>0 : elliptic curvature
K<0 : hyperbolic curvature

$$\gamma_{ij} = -g_{ij} + \frac{g_{0i} g_{0j}}{g_{00}}$$

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