ANSWERS

Everything must be fully justified on the basis of the definitions or the appendix.

Problem *Milne Universe*

The Milne model is a cosmological model of the universe. This model is a particular case of the Friedmann–Lemaître–Robertson–Walker model when the energy density is very small compared to the critical density (flat space). The Milne metric gives a flat space-time.

Milne metric: $ds^2 = c^2 dt^2 - dl^2 = c^2 dt^2 - c^2 t^2 (d\chi^2 + \sinh^2\chi \, d\Omega^2)$ with $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$.

1) Using Einstein's equation and the information given in the introduction, detail the demonstration to find the expression for the matter density ρ_0 of the universe according to Milne's model. Isolated particles are considered: $T^{\mu\nu} = \rho_0 u^\mu u^\nu$.

Einstein's Equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$.

The Milne metric gives a flat space-time, then all the components of the Riemann curvature tensor are null: $R^\alpha{}_{\beta\gamma\delta} = 0$. So, the Ricci tensor is identically null: $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha} = 0$. And the Ricci scalar is zero: $R = g^{\mu\nu} R_{\mu\nu} = 0$.
The Einstein's Equation gives then $T_{\mu\nu} = 0$, so according to the formula $T^{\mu\nu} = \rho_0 u^\mu u^\nu$, $\rho_0 = 0$.
The cosmological Milne model is a universe with no matter, the limit of the Friedmann–Lemaître–Robertson–Walker model when the density tends to zero.

For simplicity's sake, let's consider a 3D Milne spacetime toy model:

$$ds^2 = c^2 dt^2 - c^2 t^2 (d\chi^2 + \sinh^2\chi \, d\theta^2) \quad \text{with} \quad x^\mu (x^0 = ct, x^1 = \chi, x^2 = \theta).$$

2) Calculate all the components of the Christoffel symbols.

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$: $g_{00} = 1$, $g_{11} = -c^2 t^2$ and $g_{22} = -c^2 t^2 \sinh^2\chi$. Dependencies: $g_{11}(t)$ and $g_{22}(t, \chi)$, then only the $\partial_0 g_{11}$, $\partial_0 g_{22}$, and $\partial_1 g_{22}$ are different from zero. Metric diagonal: $g^{\mu\mu} = 1/g_{\mu\mu}$ and all the connections with 3 different indices are nulls. Connections symmetric on the covariant indices.
$\partial_0 g_{11} = -2ct$, $\partial_0 g_{22} = -2ct \sinh^2\chi$ and $\partial_1 g_{22} = -2c^2 t^2 \sinh\chi \cosh\chi$.

So: $\Gamma^0{}_{11} = \frac{1}{2} \frac{1}{g_{00}} (\partial_1 g_{01} + \partial_1 g_{01} - \partial_0 g_{11}) = -\frac{1}{2} \frac{1}{g_{00}} \partial_0 g_{11} = -\frac{1}{2}(1)(-2ct) = ct$

$$\Gamma^0{}_{22} = \frac{1}{2} \frac{1}{g_{00}} (-1) \partial_0 g_{22} = ct \sinh^2\chi \qquad \Gamma^1{}_{22} = \frac{1}{2} \frac{1}{g_{11}} (-1) \partial_1 g_{22} = -\sinh\chi \cosh\chi$$

$$\Gamma^1{}_{10} = \Gamma^1{}_{01} = \frac{1}{2} \frac{1}{g_{11}} \partial_0 g_{11} = \frac{1}{ct} \qquad \Gamma^2{}_{20} = \Gamma^2{}_{02} = \frac{1}{2} \frac{1}{g_{22}} \partial_0 g_{22} = \frac{1}{ct} \qquad \Gamma^2{}_{21} = \Gamma^2{}_{12} = \frac{1}{2} \frac{1}{g_{22}} \partial_1 g_{22} = \frac{\cosh\chi}{\sinh\chi}$$

$$\Gamma^0{}_{01} = \Gamma^0{}_{10} = 0 \quad \Gamma^1{}_{00} = 0 \quad \Gamma^1{}_{11} = 0 \quad \Gamma^0{}_{00} = 0 \quad \Gamma^1{}_{12} = \Gamma^1{}_{21} = 0 \quad \Gamma^2{}_{22} = 0 \quad \Gamma^0{}_{02} = \Gamma^0{}_{20} = 0 \quad \Gamma^2{}_{00} = 0 \quad \Gamma^2{}_{11} = 0$$

3) Express all the components of the Riemann curvature tensor for this metric.
6 independent components in 3D spacetime: $R^0{}_{101}$, $R^0{}_{202}$, $R^1{}_{212}$, $R^0{}_{102}$, $R^1{}_{012}$ and $R^0{}_{212}$.
Most of the terms are null or cancel:
$R^0{}_{101} = \Gamma^0{}_{11,0} - \Gamma^0{}_{10,1} + \Gamma^0{}_{00}\Gamma^0{}_{11} - \Gamma^0{}_{01}\Gamma^0{}_{10} + \Gamma^0{}_{10}\Gamma^1{}_{11} - \Gamma^0{}_{11}\Gamma^1{}_{10} + \Gamma^0{}_{20}\Gamma^2{}_{11} - \Gamma^0{}_{21}\Gamma^2{}_{10}$
and $R^0{}_{101} = 1 - 0 + 0 - 0 + 0 - ct/ct + 0 - 0 = 0$

$R^0{}_{202} = \Gamma^0{}_{22,0} - \Gamma^0{}_{20,2} + \Gamma^0{}_{00}\Gamma^0{}_{22} - \Gamma^0{}_{02}\Gamma^0{}_{20} + \Gamma^0{}_{10}\Gamma^1{}_{22} - \Gamma^0{}_{12}\Gamma^1{}_{20} + \Gamma^0{}_{20}\Gamma^2{}_{22} - \Gamma^0{}_{22}\Gamma^2{}_{20}$
and $R^0{}_{202} = \sinh^2\chi - 0 + 0 - 0 0 - 0 + 0 - ct \sinh^2\chi / ct = 0$

$R^1{}_{212} = \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{01}\Gamma^0{}_{22} - \Gamma^1{}_{02}\Gamma^0{}_{21} + \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{22}\Gamma^2{}_{21}$

and $R^1{}_{212} = -d/d\chi(\sinh\chi\cosh\chi) - 0 + 1/ct \times ct\sinh^2\chi - 00 - 0 + 0 - (-\sinh\chi\cosh\chi)\cosh\chi/\sinh\chi$
eventually $R^1{}_{212} = -\cosh^2\chi - \sinh^2\chi + \sinh^2\chi + \cosh^2\chi = 0$

All terms null:

$R^0{}_{102} = \Gamma^0{}_{12,0} - \Gamma^0{}_{10,2} + \Gamma^0{}_{00}\Gamma^0{}_{12} - \Gamma^0{}_{02}\Gamma^0{}_{10} + \Gamma^0{}_{10}\Gamma^1{}_{12} - \Gamma^0{}_{12}\Gamma^1{}_{10} + \Gamma^0{}_{20}\Gamma^2{}_{12} - \Gamma^0{}_{22}\Gamma^2{}_{10} = 0$
$R^1{}_{012} = \Gamma^1{}_{02,1} - \Gamma^1{}_{01,2} + \Gamma^1{}_{01}\Gamma^0{}_{02} - \Gamma^1{}_{02}\Gamma^0{}_{01} + \Gamma^1{}_{11}\Gamma^1{}_{02} - \Gamma^1{}_{12}\Gamma^1{}_{01} + \Gamma^1{}_{21}\Gamma^2{}_{02} - \Gamma^1{}_{22}\Gamma^2{}_{01} = 0$

$R^0{}_{212} = \Gamma^0{}_{22,1} - \Gamma^0{}_{21,2} + \Gamma^0{}_{01}\Gamma^0{}_{22} - \Gamma^0{}_{02}\Gamma^0{}_{21} + \Gamma^0{}_{11}\Gamma^1{}_{22} - \Gamma^0{}_{12}\Gamma^1{}_{21} + \Gamma^0{}_{21}\Gamma^2{}_{22} - \Gamma^0{}_{22}\Gamma^2{}_{21}$
and $R^0{}_{212} = ct\, d/d\chi(\sinh^2\chi) - 0 + 0 - 0 + ct(-\sinh\chi\cosh\chi) - 0 + 0 - ct\sinh^2\chi\cosh\chi/\sinh\chi = 0$

4) Is spacetime flat?
Spacetime is flat because all the components of the curvature tensor are null.

5) Now, let's consider the spatial part dl^2 of the metric. Is space flat?
Find the Gauss curvature. What kind of geometry do we have?

That is a 2D space and there is only one independent component: $(R^1{}_{212})_{2D}$.
$(R^1{}_{212})_{2D} = R^1{}_{212} - (\Gamma^1{}_{01}\Gamma^0{}_{22} - \Gamma^1{}_{02}\Gamma^0{}_{21}) = 0 - (1/ct \times ct\sinh^2\chi - 0) = -\sinh^2\chi \neq 0$
then space is not flat but curved.

$K = \dfrac{(-g_{11})R^1{}_{212}}{(-g_{11})(-g_{22})} = \dfrac{-\sinh^2\chi}{c^2t^2\sinh^2\chi} = -\dfrac{1}{c^2t^2} < 0$, hyperbolic curvature.

6) Do you know another example in relativity where there is a curved space within a flat spacetime?
Explain also with a metric.

The Ehrenfest Paradox on the rotating disk: radially there is no length contraction because the relative velocity with respect to the laboratory is always perpendicular to the proper line disk element, on the contrary, orthoradially there is a length contraction factor $\gamma = 1/\sqrt{(1-\beta^2)}$ with $\beta = \rho\omega/c$, and then *Perimeter/Diameter* $= \gamma\pi > \pi$. According to the Euclid's Postulates space is curved.

From the inertial laboratory metric $ds'^2 = c^2dt'^2 - d\rho'^2 - \rho'^2 d\theta'^2$, we find the metric on the disk with the change of coordinates $\begin{cases} \rho = \rho' \\ \theta = \theta' - \omega t' \\ t = t' \end{cases}$ and $ds^2 = \left(1 - \dfrac{\rho^2\omega^2}{c^2}\right)c^2dt^2 - 2\rho^2\omega\, dt\, d\theta - d\rho^2 - \rho^2 d\theta^2$. There is a global change of coordinates from the spacetime of Minkowski, then spacetime is flat on the disk.

Space metric: $dl^2 = \gamma_{ij}dx^i dx^j = d\rho^2 + \gamma^2\rho^2 d\theta^2$, indeed $\gamma_{11} = -g_{11} + \dfrac{g_{01}g_{01}}{g_{00}} = 1$, $\gamma_{12} = \gamma_{21} = -g_{12} + \dfrac{g_{01}g_{02}}{g_{00}} = 0$ and $\gamma_{22} = -g_{22} + \dfrac{g_{02}g_{02}}{g_{00}} = \gamma^2\rho^2$, then $R^1{}_{212} = \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{11}\Gamma^1{}_{22} - \Gamma^1{}_{12}\Gamma^1{}_{21} + \Gamma^1{}_{21}\Gamma^2{}_{22} - \Gamma^1{}_{22}\Gamma^2{}_{21}$
$\gamma' = \beta^2\gamma^3/\rho$. Two non null connections: $\Gamma^1{}_{22} = -1/2\,\gamma^{11}\partial_1\gamma_{22} = -\rho\gamma^4$ and $\Gamma^2{}_{21} = 1/2\,\gamma^{22}\partial_1\gamma_{22} = \gamma^2/\rho$,
$R^1{}_{212} = -\gamma^4 - 4\rho\gamma^3\beta^2\gamma^3/\rho + \gamma^6 = -\gamma^4(1 + 4\beta^2\gamma^2 - \gamma^2)$ hence $R^1{}_{212} = -3\beta^2\gamma^6 \neq 0$ and space is curve.

APPENDIX:

Connections (Christoffel symbols): $\Gamma^\alpha{}_{\mu\nu} = \dfrac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$

Riemann tensor: $R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}$

Ricci tensor: $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$ *Ricci scalar:* $R = g^{\mu\nu}R_{\mu\nu}$

$(\sinh x)' = \cosh x$
$(\cosh x)' = \sinh x$
$\cosh^2 x - \sinh^2 x = 1$

Gauss' curvature (2D space): $K = \dfrac{1}{R_1 R_2} = \dfrac{R_{1212}}{\gamma_{11}\gamma_{22} - \gamma_{12}^2}$ with $dl^2 = \gamma_{ij}dx^i dx^j$

K=0 : flat space
K>0 : elliptic curvature
K<0 : hyperbolic curvature

$\gamma_{ij} = -g_{ij} + \dfrac{g_{0i}g_{0j}}{g_{00}}$

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