Examination

TRAINING - ANSWERS

Everything must be fully justified on the basis of the definitions or the appendix.

Problem *Cosmological constant*

Part 1: Einstein's equation

Let's consider the Einstein's equation with the cosmological constant Λ: $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R+\Lambda g_{\mu\nu}=\kappa T_{\mu\nu}$.

1) Demonstrate that the new term satisfy the properties of symmetry and null divergence.
New term: $\Lambda g_{\mu\nu}$. Λ is a constant, so let's verify if $g_{\mu\nu}$ symmetric with a null divergence.
Symmetric: using the interval definition $ds^2=g_{\mu\nu}dx^\mu dx^\nu=g_{\nu\mu}dx^\nu dx^\mu=g_{\nu\mu}dx^\mu dx^\nu$. Then the tensor **g** is symmetric: $g_{\mu\nu}=g_{\nu\mu}$. Another possible demonstration: $g_{\mu\nu}=\tilde{e}_\mu.\tilde{e}_\nu=\tilde{e}_\nu.\tilde{e}_\mu=g_{\nu\mu}$.
Null divergence: $D_\sigma g_{\mu\nu}=\partial_\sigma g_{\mu\nu}-\Gamma^\lambda{}_{\sigma\mu}g_{\lambda\nu}-\Gamma^\lambda{}_{\sigma\nu}g_{\mu\lambda}$

$$D_\sigma g_{\mu\nu}=\partial_\sigma g_{\mu\nu}-\frac{1}{2}g_{\lambda\nu}g^{\lambda\beta}(\partial_\sigma g_{\beta\mu}+\partial_\mu g_{\beta\sigma}-\partial_\beta g_{\sigma\mu})-\frac{1}{2}g_{\mu\lambda}g^{\lambda\beta}(\partial_\sigma g_{\beta\nu}+\partial_\nu g_{\beta\sigma}-\partial_\beta g_{\sigma\nu})$$

$$D_\sigma g_{\mu\nu}=\partial_\sigma g_{\mu\nu}-\frac{1}{2}\delta^\beta_\nu(\partial_\sigma g_{\beta\mu}+\partial_\mu g_{\beta\sigma}-\partial_\beta g_{\sigma\mu})-\frac{1}{2}\delta^\beta_\mu(\partial_\sigma g_{\beta\nu}+\partial_\nu g_{\beta\sigma}-\partial_\beta g_{\sigma\nu})$$

$$D_\sigma g_{\mu\nu}=\partial_\sigma g_{\mu\nu}-\frac{1}{2}(\partial_\sigma g_{\nu\mu}+\partial_\mu g_{\nu\sigma}-\partial_\nu g_{\sigma\mu})-\frac{1}{2}(\partial_\sigma g_{\mu\nu}+\partial_\nu g_{\mu\sigma}-\partial_\mu g_{\sigma\nu})=0$$

2) Express R as a function of T and the constants Λ and κ.
$g^{\mu\nu}R_{\mu\nu}-\frac{1}{2}g^{\mu\nu}g_{\mu\nu}R+\Lambda g^{\mu\nu}g_{\mu\nu}=\kappa g^{\mu\nu}T_{\mu\nu}$ and because a tensor scalar is invariant, let's consider the local minkowskian reference frame $g^{\mu\nu}g_{\mu\nu}=g'^{\mu\nu}g'_{\mu\nu}=\eta^{\mu\nu}\eta_{\mu\nu}=\eta^{00}\eta_{00}+\eta^{ii}\eta_{ii}=1^2+3\times(-1)^2=4$ indeed $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu_\sigma$ so $\eta^{\mu\mu}=1/\eta_{\mu\mu}$ and $\boldsymbol{\eta}=diag(1,-1,-1,-1)$ in rectangular coordinates. Also $R=g^{\mu\nu}R_{\mu\nu}$ then $R-\frac{1}{2}\times 4R+4\Lambda=\kappa T$ and $R=-\kappa T+4\Lambda$.

3) Express $R_{\mu\nu}$ as a function of $T_{\mu\nu}$, T, $g_{\mu\nu}$ and the constants Λ and κ.
$R_{\mu\nu}=\kappa T_{\mu\nu}+\frac{1}{2}g_{\mu\nu}(-\kappa T+4\Lambda)-\Lambda g_{\mu\nu}=\kappa(T_{\mu\nu}-\frac{1}{2}g_{\mu\nu}T)+\Lambda g_{\mu\nu}$

4) Give $R_{\mu\nu}$ in vacuum.
$T_{\mu\nu}=0$ (for example for dust $T_{\mu\nu}=\rho_0 u_\mu u_\nu=0$ because $\rho_0=0$ in vacuum), then $T=0$ and $R_{\mu\nu}=\Lambda g_{\mu\nu}$

If the cosmological constant exists, it is very small. This is linked to dark energy.

Part 2: 3D Spacetime Toy model

There is no gravity in 2D or 3D spacetimes, because then, all the components of the Riemann curvature tensor are null in vacuum. So, since matter cannot curve spacetime in these cases, let's add a cosmological constant Λ. The following metric is obtained in this theoretical 3D spacetime (with one time coordinate and only two dimensions of space):

$$ds^2=(1+\Lambda r^2)c^2dt^2-\frac{dr^2}{(1+\Lambda r^2)}-r^2d\phi^2, \qquad x^\mu(x^0=ct, x^1=r, x^2=\phi)$$

1) Calculate all the components of the Christoffel symbols.
$g_{00}=1+\Lambda r^2$, $g_{11}=-1/(1+\Lambda r^2)$ and $g_{22}=-r^2$. $g^{\mu\nu}(r)$ and metric diagonal, then only the $\partial_1 g_{\mu\mu}$

are different from zero. Also $g^{\mu\mu}=1/g_{\mu\mu}$ and all the connections with 3 different indices are nulls. $\partial_1 g_{00}=2\Lambda r$, $\partial_1 g_{00}=2\Lambda r/(1+\Lambda r^2)$ and $\partial_1 g_{22}=-2r$. Connections symmetric on the covariant indices.

So: $\Gamma^1{}_{00}=\frac{1}{2}\frac{1}{g_{11}}(\partial_0 g_{10}+\partial_0 g_{01}-\partial_1 g_{00})=-\frac{1}{2}\frac{1}{g_{11}}\partial_1 g_{00}=-\frac{1}{2}(-1)(1+\Lambda r^2)2\Lambda r=\Lambda r(1+\Lambda r^2)$

$$\Gamma^0{}_{10}=\Gamma^0{}_{01}=\frac{1}{2}\frac{1}{g_{00}}(-1)\partial_1 g_{00}=\Lambda r/(1+\Lambda r^2) \qquad \Gamma^1{}_{11}=\frac{1}{2}\frac{1}{g_{11}}\partial_1 g_{11}=-\Lambda r/(1+\Lambda r^2)$$

$$\Gamma^1{}_{22}=\frac{1}{2}\frac{1}{g_{11}}(-1)\partial_1 g_{22}=-r(1+\Lambda r^2) \qquad \Gamma^2{}_{12}=\Gamma^2{}_{21}=\frac{1}{2}\frac{1}{g_{22}}\partial_1 g_{22}=1/r$$

$$\Gamma^0{}_{00}=0 \quad \Gamma^0{}_{11}=0 \quad \Gamma^1{}_{21}=0 \quad \Gamma^1{}_{01}=0 \quad \Gamma^0{}_{22}=0 \quad \Gamma^2{}_{22}=0 \quad \Gamma^0{}_{02}=0$$

2) Express all the components of the Riemann curvature tensor for this metric.

6 independent components in 3D spacetime: $R^0{}_{101}$, $R^0{}_{202}$, $R^1{}_{212}$, $R^0{}_{102}$, $R^1{}_{012}$ and $R^0{}_{212}$.

Most of the terms are nulls (non nulls in purple):

$$R^0{}_{101}=\Gamma^0{}_{11,0}-\Gamma^0{}_{10,1}+\Gamma^0{}_{00}\Gamma^0{}_{11}-\Gamma^0{}_{01}\Gamma^0{}_{10}+\Gamma^0{}_{10}\Gamma^1{}_{11}-\Gamma^0{}_{11}\Gamma^1{}_{10}+\Gamma^0{}_{20}\Gamma^2{}_{11}-\Gamma^0{}_{21}\Gamma^2{}_{10}=-\Lambda/(1+\Lambda r^2)$$
$$R^0{}_{202}=\Gamma^0{}_{22,0}-\Gamma^0{}_{20,2}+\Gamma^0{}_{00}\Gamma^0{}_{22}-\Gamma^0{}_{02}\Gamma^0{}_{20}+\Gamma^0{}_{10}\Gamma^1{}_{22}-\Gamma^0{}_{12}\Gamma^1{}_{20}+\Gamma^0{}_{20}\Gamma^2{}_{22}-\Gamma^0{}_{22}\Gamma^2{}_{20}=-\Lambda r^2$$
$$R^1{}_{212}=\Gamma^1{}_{22,1}-\Gamma^1{}_{21,2}+\Gamma^1{}_{01}\Gamma^0{}_{22}-\Gamma^1{}_{02}\Gamma^0{}_{21}+\Gamma^1{}_{11}\Gamma^1{}_{22}-\Gamma^1{}_{12}\Gamma^1{}_{21}+\Gamma^1{}_{21}\Gamma^2{}_{22}-\Gamma^1{}_{22}\Gamma^2{}_{21}=-\Lambda r^2$$
$$R^0{}_{102}=\Gamma^0{}_{12,0}-\Gamma^0{}_{10,2}+\Gamma^0{}_{00}\Gamma^0{}_{12}-\Gamma^0{}_{02}\Gamma^0{}_{10}+\Gamma^0{}_{10}\Gamma^1{}_{12}-\Gamma^0{}_{12}\Gamma^1{}_{10}+\Gamma^0{}_{20}\Gamma^2{}_{12}-\Gamma^0{}_{22}\Gamma^2{}_{10}=0$$
$$R^1{}_{012}=\Gamma^1{}_{02,1}-\Gamma^1{}_{01,2}+\Gamma^1{}_{01}\Gamma^0{}_{02}-\Gamma^1{}_{02}\Gamma^0{}_{01}+\Gamma^1{}_{11}\Gamma^1{}_{02}-\Gamma^1{}_{12}\Gamma^1{}_{01}+\Gamma^1{}_{21}\Gamma^2{}_{02}-\Gamma^1{}_{22}\Gamma^2{}_{01}=0$$
$$R^0{}_{212}=\Gamma^0{}_{22,1}-\Gamma^0{}_{21,2}+\Gamma^0{}_{01}\Gamma^0{}_{22}-\Gamma^0{}_{02}\Gamma^0{}_{21}+\Gamma^0{}_{11}\Gamma^1{}_{22}-\Gamma^0{}_{12}\Gamma^1{}_{21}+\Gamma^0{}_{21}\Gamma^2{}_{22}-\Gamma^0{}_{22}\Gamma^2{}_{21}=0$$

3) Then, find the Ricci tensor $R_{\mu\nu}$ components, and the Ricci scalar R.

Riemann tensor antisymmetric on the two last indices. Covariant Riemann tensor, antisymmetric on the two first indices, and symmetry on the two pairs of indices: $R^\alpha{}_{\beta\mu\mu}=0$, $R_{\mu\mu\alpha\beta}=0$ and $R_{\alpha\beta\mu\nu}=R_{\mu\nu\alpha\beta}$.

$R_{01}=R^0{}_{001}+R^1{}_{011}+R^2{}_{021}=0+0+0=0$ because $R^0{}_{001}=R_{0001}/g_{00}=0$ and $R^2{}_{021}=R_{2021}/g_{22}=R_{0212}/g_{22}=g_{00}R^0{}_{212}/g_{22}=0$

$R_{00}=R^0{}_{000}+R^1{}_{010}+R^2{}_{020}=0+R^1{}_{010}+R^2{}_{020}=R_{1010}/g_{11}+R_{2020}/g_{22}=R_{0101}/g_{11}+R_{0202}/g_{22}$

$R_{00}=g_{00}/g_{11}R^0{}_{101}+g_{00}/g_{22}R^0{}_{202}=2\Lambda(1+\Lambda r^2)$ $R_{02}=R^0{}_{020}+R^1{}_{021}+R^2{}_{022}=0+0+0=0$

$R_{11}=R^0{}_{101}+g_{11}/g_{22}R^1{}_{212}=-2\Lambda/(1+\Lambda r^2)$ $R_{12}=R^0{}_{120}+R^1{}_{121}+R^2{}_{122}=R_{21}=0$ Ricci tensor symmetric

$R_{22}=R^0{}_{202}+R^1{}_{212}=-2\Lambda r^2$

g diagonal: $R=g^{\mu\mu}R_{\mu\mu}=R_{00}/g_{00}+R_{11}/g_{11}+R_{22}/g_{22}=6\Lambda$

4) In this spacetime, the Einstein's equation, in vacuum and with the cosmological constant, give a value for R. Is this the same value as that obtained from the metric in question 3)?

3D: $g^{\mu\nu}g_{\mu\nu}=\eta^{\mu\nu}\eta_{\mu\nu}=\eta^{00}\eta_{00}+\eta^{ii}\eta_{ii}=1^2+2\times(-1)^2=3$ then $R-\frac{1}{2}\times 3R+3\Lambda=\kappa T=0$ and $R=6\Lambda$, idem.

APPENDIX:

Connections (Christoffel symbols): $\Gamma^\alpha{}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu}+\partial_\nu g_{\beta\mu}-\partial_\beta g_{\mu\nu})$

Riemann tensor: $R^\alpha{}_{\beta\gamma\delta}=\Gamma^\alpha{}_{\beta\delta,\gamma}-\Gamma^\alpha{}_{\beta\gamma,\delta}+\Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}-\Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}$

Ricci tensor: $R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}$

Covariant derivative: $D_\mu A_\nu=\partial_\mu A_\nu-\Gamma^\lambda{}_{\mu\nu}A_\lambda$

Mass-energy scalar: $T=g^{\mu\nu}T_{\mu\nu}$