Examination

TRAINING - ANSWERS

Everything must be fully justified on the basis of the definitions or the appendix.

Exercise *Rotating Disk*

1) Give the Minkowskian metric in cylindrical coordinates.

(R') inertial frame of the laboratory: $ds'^2=c^2dt'^2-dl'^2=c^2dt'^2-d\rho'^2-\rho'^2d\theta'^2-dz'^2$

2) From the metric of Minkowski and the appropriate change of coordinates, find the following metric on the rotating disk:

$$ds^2=\left(1-\frac{\rho^2\omega^2}{c^2}\right)c^2dt^2-2\,\rho^2\omega\,dt\,d\theta-d\rho^2-\rho^2d\theta^2 \qquad \text{with} \qquad z=0.$$

Change of coordinates: $\begin{cases}\rho=\rho' \\ \theta=\theta'-\omega t' \\ t=t'\end{cases}$, *(R)* reference frame of the disk.

$ds'^2=ds^2=c^2dt^2-d\rho^2-\rho^2(d\theta+\omega\,dt)^2$ from which we have the required formula.

3) Use the Lagrangian formalism to establish the equations of motion for $\rho(t)$ and $\theta(t)$.

$ds^2=c^2d\tau^2=c^2dt^2(1-\rho^2\omega^2/c^2-2\rho^2\omega\dot{\theta}/c^2-\dot{\rho}^2/c^2-\rho^2\dot{\theta}^2/c^2)=L\,c^2dt^2$, $d\tau=\int\mathscr{L}\,dt$, $\mathscr{L}=\sqrt{L}$

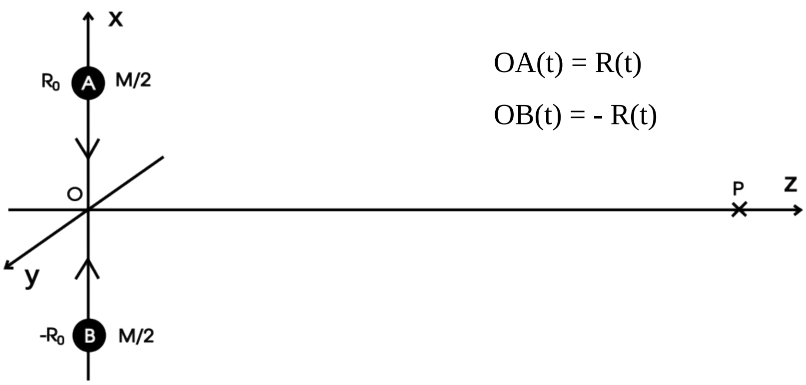
$$\begin{cases}\dfrac{\partial L}{\partial\rho}=\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\rho}} \\[2mm] \dfrac{\partial L}{\partial\theta}=\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{\theta}}\end{cases} \Rightarrow \begin{cases}\ddot{\rho}-\rho\omega^2-2\rho\omega\dot{\theta}-\rho\dot{\theta}^2=0 & (1) \\ \rho\ddot{\theta}+2\dot{\rho}\dot{\theta}+2\omega\dot{\rho}=0 & (2)\end{cases}$$

4) Show that there is no radial trajectory for a free particle in the disk frame.
Similarly, show that there are no circular trajectories in the disk reference frame, apart from the trivial case where the particle is at rest in the laboratory.

Radial: $\theta=cst \Rightarrow \dot{\theta}=0 \Rightarrow \ddot{\theta}=0$ then (2) $\Rightarrow \dot{\rho}=0 \Rightarrow \ddot{\rho}=0$, $(1) \Rightarrow \dot{\rho}=0 \Rightarrow \rho=0$, no radial trajectories.

Circular: $\rho=cst=R \Rightarrow \dot{\rho}=0 \Rightarrow \ddot{\rho}=0$, (2) $\Rightarrow \ddot{\theta}=0 \Rightarrow \dot{\theta}=cst=\Omega$. $(1) \Rightarrow R(\Omega+\omega)^2=0 \Rightarrow \Omega=-\omega$, particle at rest in the laboratory, else no circular trajectories.

Problem *Collapsing Star*



Let's consider a simple model of gravitational wave (GW) emission during the collapse of a star into a black hole. The mass of the star is denoted *M*. The star is assumed to be split into two masses *M/2*. At *t=0*, the masses are at rest and at a distance $2R_0$. Then, the two parts of the star fall radially with respect to each other until they reach the Schwarzschild distance.

1) Calculate the quadrupole moment I_{ij}.

Two punctual sources: $I_{ij}=\sum m x_i x_j = m_A (x_i)_A (x_j)_A + m_B (x_i)_B (x_j)_B$

$A(x_1=x=R(t), x_2=y=0, x_3=z=0)$ and $B(-R(t),0,0)$

$I_{11}=M/2\,R^2+M/2(-R)^2=M\,R(t)^2$

$I_{22}=M/2\,y_A^2+M/2\,y_B^2=0$ and all the other terms are null:

$$I_{ij}=M\,R(t)^2\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

2) Express the GW amplitude h_{ij} observed at P ($r=OP >> R_0$).

$$h_{ij}=\frac{2GM}{rc^4}\ddot{R^2}\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

3) Is h_{ij} Transverse and Traceless? If not, make the necessary changes.

h_{ij} is transverse because $h_{i3}=0$ and $h_{3j}=0$. Not traceless: tr(A)=1+0+0=1≠0 with $A=\begin{pmatrix}1&0\\0&0\end{pmatrix}$.

$A_{ij}=A_{ij}-1/2\,\delta_{ij}\,tr(A)+1/2\,\delta_{ij}\,tr(A)$ so $A^T=A-\frac{1}{2}I\,tr(A)=\begin{pmatrix}1/2&0\\0&-1/2\end{pmatrix}$

and $h_{ij}^{TT}=\frac{GM}{rc^4}\ddot{R^2}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$.

4) What's the polarization of the GW? Polarization rectilinear "plus" +.

Data:
$M=30M_S$
$M_S=2\times10^{30}kg$
$G\sim7\times10^{-11}IS$
$1ly\sim10^{16}m$
Milky Way size: ~100 000 ly
Andromeda Galaxy: distance ~3Mly

5) Use Newton's mechanics to prove that $\ddot{R^2}=GM\left(\frac{3}{4R}-\frac{1}{R_0}\right)$.

$\ddot{R^2}=d/dt(2R\dot{R})=2(\dot{R}^2+R\ddot{R})$

$m\vec{a}=\vec{F}$ ⇒ applied to the system A: $M/2\,\ddot{R}\,\vec{i}=-G(M/2)^2/(2R)^2\,\vec{i}$ then $\ddot{R}=-GM/8R^2$.

Conservation of energy: $1/2(M/2)\dot{R}^2-G(M/2)^2/2R=cst$ then $\dot{R}^2=GM/2(R-R_0)$

$\ddot{R^2}=2(GM/2R-GM/8R-GM/2R_0)=3GM/4R-GM/R_0$

6) Find the maximal amplitude h of the wave emitted at the end of the collapse ($R_0 >> r_S$).
The minimal amplitude detected on Earth is $h_{min}\simeq10^{-23}$. Could this collapse be detected in our galaxy or in others?

$(\ddot{R^2})_{max}\simeq 3GM/4r_S=3c^2/8$ then $(h_{ij})_{max}\sim\frac{GM}{rc^2}$ and $r_{max}\sim\frac{GM}{h_{min}c^2}\sim\frac{7\times10^{-11}\times30\times2\times10^{30}}{10^{-23}\times9\times10^{16}}$

eventually $r_{max}\sim10^{27}$m~100Gly, this kind of events could be detected in our galaxy and beyond.

APPENDIX:

GW, weak field: $h_{ij}^{TT}(t)=\frac{2G}{rc^4}\frac{d^2 I_{ij}(t-r/c)}{dt^2}$ *with* $I_{ij}=\int_{source} x_i x_j \rho\, dV$

Schwarzschild Radius: $r_S=\frac{2GM}{c^2}$