Relativity **Examination** semester 2

TRAINING

Everything must be fully justified on the basis of the definitions or the appendix.

Problem *Cosmology*

The Friedmann–Lemaître–Robertson–Walker metric describes a homogeneous, isotropic, expanding or contracting universe:

$$ds^2 = c^2 dt^2 - a(t)^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2 \right) \quad \text{with} \quad d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \quad \text{and} \quad (x^0 = ct, x^1 = r, x^2 = \theta, x^3 = \phi).$$

k is a constant representing the curvature of the space. k belong to the set {−1, 0, +1}. For $k=-1$, space has an open hyperbolic curvature, and after the Big Bang the universe will expand indefinitely. For $k=1$, space has a closed elliptic curvature, and after expansion, contraction will follow until a Big Crunch. Let's consider the critical case $k=0$ and the current scale factor $a(t)=e^{H_0 t}$ with H_0 the Hubble constant.

1) Calculate the components $\Gamma^0{}_{11}$, $\Gamma^2{}_{21}$ and $\Gamma^1{}_{23}$ of the Christoffel symbols.

2) Express the component $R^1{}_{212}$ of the Riemann curvature tensor for this metric. Is spacetime flat?

3) Then, find the Ricci tensor R_{00} component, and the Ricci scalar R.

4) Using Einstein's equation find the expression for the critical matter density ρ_c of the universe according to FLRW's model. Isolated particles are considered: $T^{\mu\nu} = \rho_0 u^\mu u^\nu$.

5) Let's consider the spatial part dl^2 of the metric. Find the space Riemann curvature tensor components. Is space flat?

6) Give an estimation of the numerical value for the critical density ρ_c with $H_0 \simeq 2\times10^{-18}\ \text{s}^{-1}$ and $G \simeq 7\times10^{-11}\ \text{m}^3\ \text{s}^{-2}\ \text{kg}^{-1}$. What future for the universe with the experimental value $\rho \simeq 7\times10^{-30}\ \text{kg/m}^3$?

APPENDIX:

Connections (Christoffel symbols): $\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (\partial_\mu g_{\beta\nu} + \partial_\nu g_{\beta\mu} - \partial_\beta g_{\mu\nu})$

Results given (non null connections): $\Gamma^1{}_{10} = \Gamma^2{}_{20} = \Gamma^3{}_{30} = H_0/c$, $\Gamma^3{}_{31} = \Gamma^2{}_{21}$, $\Gamma^0{}_{22} = \frac{H_0}{c} r^2 e^{2H_0 t}$,

$\Gamma^1{}_{22} = -r$, $\Gamma^0{}_{33} = \frac{H_0}{c} r^2 \sin^2\theta \, e^{2H_0 t}$, $\Gamma^1{}_{33} = -r\sin^2\theta$, $\Gamma^2{}_{33} = -\sin\theta\cos\theta$ *and* $\Gamma^3{}_{32} = \cos\theta/\sin\theta$.

Riemann tensor: $R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta} - \Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}$

Results given (non null components): $R^0{}_{202} = r^2 R^0{}_{202} = R^1{}_{212}$ *and* $R^0{}_{303} = R^1{}_{313} = R^2{}_{323} = \sin^2\theta \, R^1{}_{212}$.

Ricci tensor: $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$ *Ricci scalar:* $R = g^{\mu\nu} R_{\mu\nu}$

Results given: $R_{11} = 3\frac{H_0^2}{c^2} e^{2H_0 t}$, $R_{22} = r^2 R_{11}$ *and* $R_{33} = \sin^2\theta \, R_{22}$.

Einstein's Equation: $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$