TRAINING - ANSWERS

Everything must be fully justified on the basis of the definitions or the appendix.

Problem *Cosmology*

The Friedmann–Lemaître–Robertson–Walker metric describes a homogeneous, isotropic, expanding or contracting universe:

$$ds^2=c^2dt^2-a(t)^2\left(\frac{dr^2}{1-kr^2}+r^2d\Omega^2\right) \quad \text{with} \quad d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2 \quad \text{and} \quad (x^0=ct\,,x^1=r\,,x^2=\theta\,,x^3=\phi).$$

k is a constant representing the curvature of the space. k belong to the set {−1, 0, +1}. For $k=-1$, space has an opened hyperbolic curvature, and after the Bing Bang the universe will expand indefinitely. For $k=1$, space has an closed elliptic curvature, and after expansion a contraction will follow to a Bing Crunch. Let's consider the critical case $k=0$ and the present-day scale factor $a(t)=e^{H_0T}$ with H_0 the Hubble constant.

1) Calculate the components $\Gamma^0{}_{11}$, $\Gamma^2{}_{21}$ and $\Gamma^1{}_{23}$ of the Christoffel symbols.

$g_{00}=1$, $g_{11}=-e^{2H_0t}$, $g_{22}=-r^2e^{2H_0t}$ and $g_{33}=-r^2\sin^2\theta\, e^{2H_0t}$. The metric is diagonal, only the $\partial_0 g_{ii}$, $\partial_1 g_{22}$, $\partial_1 g_{33}$ and $\partial_2 g_{33}$ are different from zero. Also $g^{\mu\nu}g_{\nu\sigma}=\delta^\mu_\sigma \Rightarrow g^{\mu\mu}=1/g_{\mu\mu}$ and all the connections with 3 different indices are null. $\partial_0 g_{11}=-2\frac{H_0}{c}e^{2H_0t}$ and $\partial_1 g_{22}=-2r\,e^{2H_0t}$. Connections symmetric on the covariant indices. So: $\Gamma^0{}_{11}=\frac{1}{2}\frac{1}{g_{00}}(\partial_1 g_{10}+\partial_1 g_{01}-\partial_0 g_{11})=-\frac{1}{2}\partial_0 g_{11}=-\frac{1}{2}(-1)2\frac{H_0}{c}e^{2H_0t}=\frac{H_0}{c}e^{2H_0t}$

$$\Gamma^2{}_{21}=\frac{1}{2}\frac{1}{g_{22}}(-1)\partial_1 g_{22}=1/r \qquad \Gamma^1{}_{23}=\frac{1}{2}\frac{1}{g_{11}}(\partial_2 g_{13}+\partial_3 g_{12}-\partial_1 g_{23})=0$$

2) Express the component $R^1{}_{212}$ of the Riemann curvature tensor for this metric. Is spacetime flat?

Most of the terms are null (non nulls in purple):

$$R^1{}_{212}=\Gamma^1{}_{22,1}-\Gamma^1{}_{21,2}+\Gamma^1{}_{01}\Gamma^0{}_{22}-\Gamma^1{}_{02}\Gamma^0{}_{21}+\Gamma^1{}_{11}\Gamma^1{}_{22}-\Gamma^1{}_{12}\Gamma^1{}_{21}+\Gamma^1{}_{21}\Gamma^2{}_{22}-\Gamma^1{}_{22}\Gamma^2{}_{21}+\Gamma^1{}_{31}\Gamma^3{}_{22}-\Gamma^1{}_{32}\Gamma^3{}_{21}$$

$R^1{}_{212}=-1+\frac{H_0^2}{c^2}r^2e^{2H_0t}+1=\frac{H_0^2}{c^2}r^2e^{2H_0t}$ Spacetime is flat if all the components of the Riemann curvature tensor are null, $R^1{}_{212}\neq 0$ then the spacetime for this metric is not flat, but curve.

3) Then, find the Ricci tensor R_{00} component, and the Ricci scalar R.

Riemann tensor antisymmetric on the two last indices. Covariant Riemann tensor, antisymmetric on the two first indices, and symmetry on the two pairs of indices: $R^\alpha{}_{\beta\mu\mu}=0$, $R_{\mu\mu\alpha\beta}=0$ and $R_{\alpha\beta\mu\nu}=R_{\mu\nu\alpha\beta}$.

$$R_{00}=R^0{}_{000}+R^1{}_{010}+R^2{}_{020}+R^3{}_{030}=0+R_{1010}/g_{11}+R_{2020}/g_{22}+R_{3030}/g_{33}=R_{0101}/g_{11}+R_{0202}/g_{22}+R_{0303}/g_{33}$$

$$R_{00}=g_{00}/g_{11}R^0{}_{101}+g_{00}/g_{22}R^0{}_{202}+g_{00}/g_{33}R^0{}_{303}=-e^{-2H_0t}\frac{H_0^2}{c^2}e^{2H_0t}-e^{-2H_0t}/r^2\times\frac{H_0^2}{c^2}r^2e^{2H_0t}-\frac{H_0^2}{c^2}=-3\frac{H_0{}^2}{c^2}$$

g diagonal: $R=g^{\mu\mu}R_{\mu\mu}=R_{00}/g_{00}+R_{11}/g_{11}+R_{22}/g_{22}+R_{33}/g_{33}=-12H_0^2/c^2$

4) Using Einstein's equation and the information given in the introduction, detail the demonstration to find the expression for the critical matter density ρ_c of the universe according to FLRW's model. Isolated particles are considered: $T^{\mu\nu}=\rho_0 u^\mu u^\nu$.

$R_{00}-\frac{1}{2}g_{00}R=\frac{8\pi G}{c^4}T_{00}$, $u^\mu=(\gamma c,\gamma\vec{v})$, $u^0=\gamma c$, $u_0=g_{00}u^0=\gamma c$ and $\rho_0=\rho_c$,

$\Rightarrow\ -3\frac{H_0^2}{c^2}-\frac{1}{2}\times 1\times(-1)12\frac{H_0^2}{c^2}=\frac{8\pi G}{c^4}\rho_c\gamma^2c^2$, $v<<c$ and $\gamma\simeq 1$, then $\rho_c=\frac{3H_0^2}{8\pi G}$.

5) Let's consider the spatial part dl^2 of the metric. Find the space Riemann curvature tensor components. Is space flat?

Into 3D, 6 independent components to check:

$(R^1{}_{212})_{space}=R^1{}_{212}-\Gamma^1{}_{01}\Gamma^0{}_{22}+\Gamma^1{}_{02}\Gamma^0{}_{21}=(H_0^2r^2e^{2H_0t}-H_0\times H_0r^2e_0^{2H_0t}+0)/c^2=0$

$(R^1{}_{313})_{space}=R^1{}_{313}-\Gamma^1{}_{01}\Gamma^0{}_{33}+\Gamma^1{}_{03}\Gamma^0{}_{31}=(H_0^2r^2\sin^2\theta\, e^{2H_0t}-H_0\times H_0r^2\sin^2\theta\, e_0^{2H_0t}+0)/c^2=0$

$(R^2{}_{323})_{space}=R^2{}_{323}-\Gamma^2{}_{02}\Gamma^0{}_{33}+\Gamma^2{}_{03}\Gamma^0{}_{32}=(H_0^2r^2\sin^2\theta\, e^{2H_0t}-H_0\times H_0r^2\sin^2\theta\, e_0^{2H_0t}+0)/c^2=0$

$(R^3{}_{212})_{space}=R^3{}_{212}-\Gamma^3{}_{01}\Gamma^0{}_{22}+\Gamma^3{}_{02}\Gamma^0{}_{21}=0-0+0=0$

$(R^1{}_{323})_{space}=R^1{}_{323}-\Gamma^1{}_{02}\Gamma^0{}_{33}+\Gamma^1{}_{03}\Gamma^0{}_{32}=0-0+0=0$

$(R^1{}_{312})_{space}=R^1{}_{312}-\Gamma^1{}_{01}\Gamma^0{}_{32}+\Gamma^1{}_{02}\Gamma^0{}_{31}=0-0+0=0$

All the components of the Riemann curvature space tensor are null, space is flat, as expected for *k=0.*

6) Give an estimation of the numerical value for the critical density ρ_c with $H_0\simeq 2\times 10^{-18}$ s^{-1} and $G\simeq 7\times 10^{-11}$ m^3 s^{-2} kg^{-1}. Which future for the universe with the experimental value $\rho\simeq 7\times 10^{-30}$ kg/m³ ?

$\rho_c\simeq\frac{3\times 4\times 10^{-36}}{8\times 3\times 7\times 10^{-11}}\simeq 10^{-26}$ then $\rho\ll\rho_c$, the gravity is not enough to counter the expansion, the universe will expand indefinitely (*k=-1*, space has an opened hyperbolic curvature).

APPENDIX:

Connections (Christoffel symbols): $\Gamma^\alpha{}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}(\partial_\mu g_{\beta\nu}+\partial_\nu g_{\beta\mu}-\partial_\beta g_{\mu\nu})$

Results given (non null connections): $\Gamma^1{}_{10}=\Gamma^2{}_{20}=\Gamma^3{}_{30}=\frac{H_0}{c}$, $\Gamma^3{}_{31}=\Gamma^2{}_{21}$, $\Gamma^0{}_{22}=\frac{H_0}{c}r^2e^{2H_0t}$,

$\Gamma^1{}_{22}=-r$, $\Gamma^0{}_{33}=\frac{H_0}{c}r^2\sin^2\theta\, e^{2H_0t}$, $\Gamma^1{}_{33}=-r\sin^2\theta$, $\Gamma^2{}_{33}=-\sin\theta\cos\theta$ *and* $\Gamma^3{}_{32}=\cos\theta/\sin\theta$.

Riemann tensor: $R^\alpha{}_{\beta\gamma\delta}=\Gamma^\alpha{}_{\beta\delta,\gamma}-\Gamma^\alpha{}_{\beta\gamma,\delta}+\Gamma^\alpha{}_{\sigma\gamma}\Gamma^\sigma{}_{\beta\delta}-\Gamma^\alpha{}_{\sigma\delta}\Gamma^\sigma{}_{\beta\gamma}$

Results given (non null components): $R^0{}_{202}=r^2R^0{}_{101}=R^1{}_{212}$ *and* $R^0{}_{303}=R^1{}_{313}=R^2{}_{323}=\sin^2\theta\, R^1{}_{212}$.

Ricci tensor: $R_{\mu\nu}=R^\alpha{}_{\mu\alpha\nu}$ *Ricci scalar:* $R=g^{\mu\nu}R_{\mu\nu}$

Results given: $R_{11}=3\frac{H_0^2}{c^2}e^{2H_0t}$, $R_{22}=r^2R_{11}$ *and* $R_{33}=\sin^2\theta\, R_{22}$.

Einstein's Equation: $R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=\frac{8\pi G}{c^4}T_{\mu\nu}$