Textbook for Undergraduate and Graduate Students in Physics

SPECIAL RELATIVITY

A Geometric Approach

Course with Exercises and Answers

followed by the conference

Interstellar travel and antimatter

Mathieu ROUAUD

SCIENCE AND DISCOVERIES

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Foreword

Special relativity, presented in the article published by Albert Einstein in June 1905, has deeply changed our physical concepts. The well-established theories of the time, Newton's old mechanics and Maxwell's brand new theory of electromagnetism, were fundamentally incompatible. In the first, there is an addition law for velocities, while in the second, an invariant speed is required: the speed of light in vacuum. In Newton's theory, in line with the relativity of motion introduced by Galileo, the speed of an object depends on the observational reference frame, so how could the speed of light in vacuum be a fixed fundamental constant? For inertial frames, special relativity reconciles mechanics and electromagnetism, at the cost of calling into question the absolute nature of space and time. Space and time are now relative and form a new absolute: the space-time. The theories of matter and light are thus unified in their natural spatiotemporal framework. Albert Einstein's historical approach is based on the constancy of the speed of light in a vacuum. The modern approach, which made it possible to build the Standard Model, is based on another logic: symmetries. This new approach is deeper and breaks free from the historical bias of the early 20th century. The structure of space-time imposes a speed limit. This maximum speed is specific to space-time and is not linked to a material object. This new constant is specific to the container, the space-time, and not to the content, for example, light rays. This new vision is conceptually very different and sheds light on the true nature of physical laws. In this book, we focus on visual and graphical methods that help develop understanding without the systematic use of equations. This geometrical approach will be highlighted and will allow the reader to make sense of the equations that will follow. The path followed is not academic, but pragmatic and utilitarian. From the first pages you will master the tools that will allow you to apply special relativity independently. We are not studying general relativity here. We specify this because confusion is frequent between the two theories. That said, for those who want to understand general relativity, you must first have understood the special. General relativity deals with gravitation and is based on its own principles. Small notable exception, we will sometimes make analogies with the black hole to help delimit the two theories.

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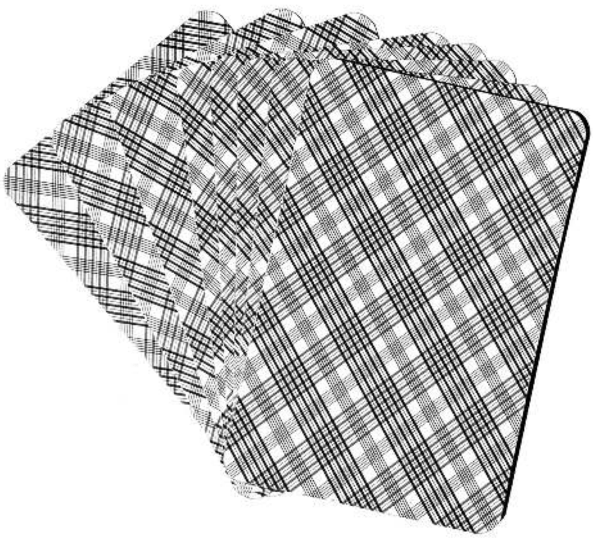

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Time dilation and length contraction

In this chapter, we introduce special relativity and we present the first geometrical tool: the triangle of times

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Units of time and distance

These two physical quantities, time and distance, are of different natures. Impossible, for example, to say if a second is greater or less than a meter, that makes no sense.

We can use a speed to link a distance to a time, but the speed depends on the observer; this link would therefore be perfectly arbitrary. It is always true in classical mechanics, but in special relativity we have a novelty, we have an invariant speed: the maximum speed. This fundamental constant makes it possible to unambiguously associate a distance with a time. This distance is called *light-time*.

For example, the light-year corresponds to the distance traveled in vacuum by light during a year.

The speed of light in vacuum is about a billion km/h, it is named *c* and is precisely fixed at:

$$c = 299\,792\,458 \text{ m/s}$$

It is the speed of any electromagnetic wave in vacuum, whether it be radio, infrared, visible, ultraviolet, X-rays or gamma rays.
We specify well, in vacuum, because in a transparent material, such as air, water or glass, the speed is lower and depends on the wavelength.
A light-year, denoted l.y., is worth about 10,000 billion km. The star closest to our Sun, Proxima Centauri, is located about 4 ly. Our Sun is 8 light-minutes from Earth, the Moon is one light-second, and an adult human measures between 5 and 6 light-nanoseconds:

$$1 \text{ l.ns.} \simeq 33 \text{ cm}$$

We can now freely compare distances and times, expressing the distances in units of light-time.

∞ FRAMES

Any measurement of a physical quantity is carried out in a given frame of reference.
The quantity can be a time, a distance, a velocity, an acceleration, a force, etc.
The reference frame, as in Newtonian mechanics, is defined by a reference solid considered fixed.
For example, a train can be taken as a reference. More precisely, a wagon of this train makes it possible to locate any object. We consider, arbitrarily, a point of the wagon as the origin. Then, from this point, we count how many times we have to move, end to end, a rigid ruler of one meter in the direction front-back, right-left and up-down to reach this object. We get a set of three numbers that uniquely defines the position of the object. If the object is fixed this will be sufficient, but if it moves, it will also be necessary to define a date. We then have a set of four numbers called event:

$$E\ (x, y, z, t).$$

For the date, we must proceed more precisely than in classical mechanics. Time is no longer absolute, and instead of a single clock we must have a set of synchronized clocks over the whole space.

Depending on the case, we can use the terrestrial reference frame, the heliocentric reference frame, the galactic reference frame, etc.

These frames of reference are in motion with respect to each other and for the same event we will have different sets of coordinates.

ↀ Einstein's postulates

Albert Einstein postulates in his article of June 1905[1] that the laws of physics are the same in all inertial frames of reference (1st postulate), and that in these same frames the speed of light in vacuum is invariant (2nd postulate) .

In Newtonian mechanics, for the statement of Newton's three laws, we did not speak of inertial frames but of Galilean frames, which amounts to the same thing. For example, in classical mechanics, in a frame rotating with respect to a Galilean frame, Newton's second law is no longer verified and new forces, called inertial, appear. A rotating frame with respect to an inertial frame is therefore not inertial.

How to define an inertial frame? A frame is inertial if the postulates are verified. The simplest is to have a inertial frame of reference, then all the frames in uniform rectilinear translation with respect to this first frame of reference are also of inertia.

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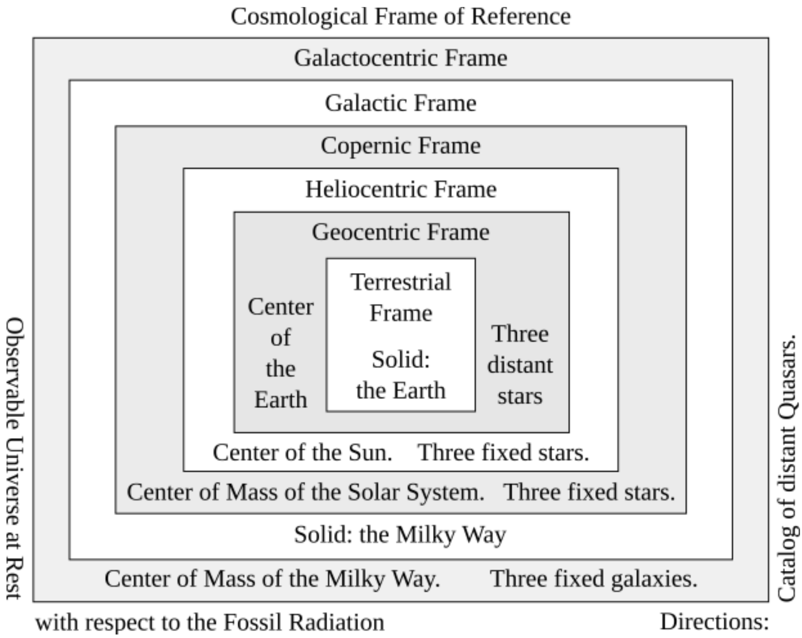
1 "*On the Electrodynamics of Moving Bodies*", June 30 1905, English Translation.

The farther away we aim at an object, such as a distant star, the more its motion can be neglected. For example, extremely massive and very distant quasars, several billion light-years away, are taken as fixed points and make it possible to define the cosmological reference frame. Fossil radiation, emitted 380,000 years after the Big Bang, 13.8 billion years ago, is homogeneous and isotropic in this frame of reference.

To come back to our train, if it runs in a straight line and at constant velocity in the terrestrial frame of reference, the reference frame of the train can be considered as inertial for an experiment of a few minutes. This duration is small compared to that of the rotation of the Earth on itself. This is a good approximation, and the terrestrial frame can be considered here as inertial. The more precise the measurements, the shorter the duration of the experiment for the approximation to remain valid.

For a satellite, the terrestrial frame of reference is no longer inertial. A low-Earth-orbiting satellite goes around in 1 hour 30 minutes, a not insignificant time compared to the Earth's rotation which lasts about 24 hours. We then consider the geocentric frame of reference, with the origin at the center of the Earth, and in which the Earth is in rotation around its own axis relative to distant stars assumed to be fixed.

In an inertial frame of reference, an object keep moving in a straight line at a constant speed when no forces act upon it.



Unable to position and date an event without landmarks. If you hide a treasure you will indicate its position relative to a point of origin: for example, "from the hundred-year-old oak tree, 22 steps west, 47 steps south and dig at three feet." If I say that I was born in 1992, it is in reference to an origin date, placed arbitrarily as a common reference point.

A reference frame is associated with a solid to which a chronology is added. A minimum of four fixed objects relative to each other is required. For chronology, in special relativity a single clock is no longer sufficient: one can imagine a solid made up of rigid bars of unit length, all placed perpendicular to each other in order to form a three-dimensional network, and at each node of this network we place a clock; all the clocks are synchronized, and the whole forms what is called a crystal of clocks.

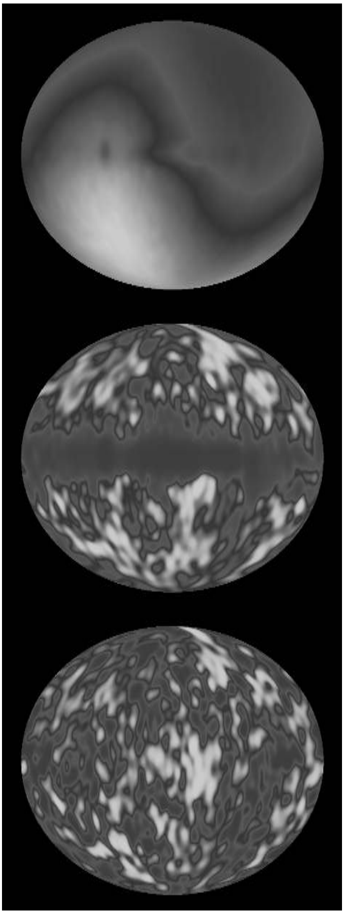
The largest object in the Universe is the Universe itself. Let's use it as a reference solid. In cosmology, the Universe can be seen as a fluid of galaxies which extends everywhere: any point of the Universe can be considered as the center. But, two remarks: first of all the Universe cannot be observed as a whole, because the further one looks far, the more one goes back in time. The oldest visible object is fossil radiation emitted 13.4 billion years ago when the Universe became transparent. Secondly, if we take a point where this fossil radiation is uniform, everything leads us to think that this point is motionless in the Universe.

Image opposite, the data collected by the COBE satellite on the cosmic diffuse background.

On the first image we visualize the anisotropy due to the displacement of the Earth in relation to the cosmological frame of reference, this is due to the Doppler effect and we thus evaluate a speed of 350 km/s.

In the second image, we have stray light from our own galaxy.

Finally, at the very bottom, we get an image of the Universe at its beginnings: it is homogeneous in the cosmological frame of reference and we can use quasars for the directions.



Thus the frames of reference nest one in the other: for the *Voyager* probe we consider the Copernic frame of reference, which has for origin the center of mass of the solar system and the directions of distant stars. For an interstellar journey to Proxima Centauri we will consider the galactic frame of reference. Indeed, over a journey of a few years or decades, the Milky Way and its stars can be assumed to be fixed; for example, our galaxy turns on itself in some 250 million years, much longer than our journey to the stars[2].

ꝏ THE TRIANGLE OF TIMES

There is not an absolute, unique and universal time. Times are multiple and relative. Each observer, or object, lives his own time. Times are plural, each time follows its course, and, when we compare them, we see that they evolve at different rates. These rhythms will be all the more different the greater the relative velocity between two inertial frames of reference. For each inertial frame we can define a unique time for a set of objects which are motionless with respect to each other.

Let us name *R* such an inertial frame of reference. Consider a fixed point $M_1(x_1, y_1, z_1)$ in *R*. At this point,

2 Continuation of the reflection on inertial frames of reference in the conclusion of the course on four-vectors.

two events occur at the date t_1 and t_2:

$$E_1(x_1, y_1, z_1, t_1) \quad \text{and} \quad E_2(x_2{=}x_1, y_2{=}y_1, z_2{=}z_1, t_2).$$

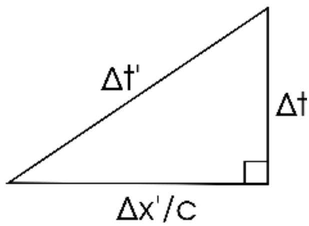
For example, a lamp that turns on and off. Second example, in the case of an interstellar journey, let us take for *R* the reference frame of a rocket, t_1 corresponds to the date of departure from the solar system, and t_2 indicates the date of arrival near Proxima Centauri. Dates measured on a clock fixed relative to the rocket.

The duration between the two events is $\Delta t = t_2 - t_1$.

If we now measure the four coordinates of these two events from a second inertial frame *R'*, in uniform rectilinear translational motion at the velocity $\vec{v}$ with respect to *R*, we measure a second duration $\Delta t' = t'_2 - t'_1$.

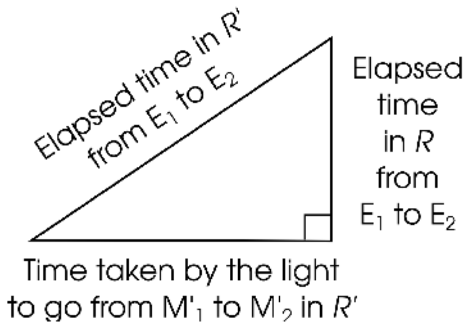
From the point of view of *R'*, the events E_1 and E_2 have space-time coordinates (x'_1, y'_1, z'_1, t'_1) and (x'_2, y'_2, z'_2, t'_2), and now occur at two distinct points $M'_1(x'_1, y'_1, z'_1)$ and $M'_2(x'_2, y'_2, z'_2)$. The first duration Δt is called *proper time*, because the events are at rest in *R*; the second duration $\Delta t'$ is called *relative time*, because the events are moving with respect to *R'*. The reference frame *R'* will have traveled, with respect to R, the distance $\Delta x' = x'_2 - x'_1$ during $\Delta t'$ (case where the x-axes are oriented along $\vec{v}$).

We then have *the triangle of times* which allows us to answer many of our questions:



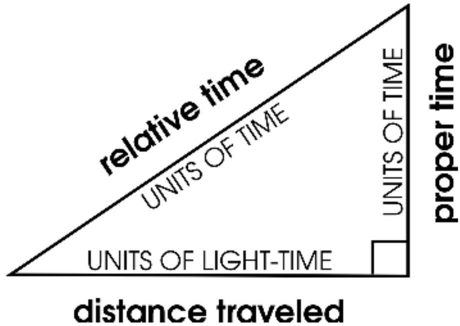
We use this triangle as a starting point to build special relativity. Later we can demonstrate its validity using Einstein's postulates or symmetries.

Each side of the triangle corresponds to a distinct time:



We can memorize it in the following form:

The Triangle of Times

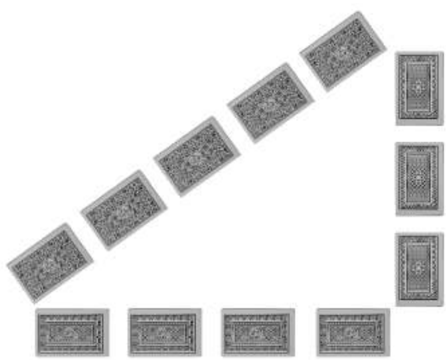


The *triangle of times* is easy to remember and apply. Take the case of an interstellar journey Sun - Proxima Centauri and use a card game to solve the problem.

The base of the right triangle is the distance in light-years. We place one card per light-year, so, here horizontally, four cards. Then we vertically place the number of cards that correspond to the travel time for the astronauts, one card per year.

We decide to complete the trip in three years, measured with a clock at rest in the frame of reference of the vessel.

How long will the journey measured from the galactic frame of reference last? It's simple we count the number of cards needed for the hypotenuse:



Relative time is 5 years and proper time 3 years. The *triangle of times* allows you to directly visualize the *time dilation*: $\gamma = \Delta t' / \Delta t$.

Here, the *gamma factor* is 5/3. The speed of the vessel is in *R'*: $v = \Delta x' / \Delta t'$. Here, the speed is 4/5 of the limit speed so 80% of *c*. As the hypotenuse is the longest side, time can only expand, and the speed of light in vacuum cannot be exceeded.

The first two exercises on page 25 allow you to familiarize yourself with these concepts.

ꝏ Length contraction

We previously envisioned a trip from the Sun to Proxima that lasts 3 years for astronauts. We could ask ourselves: «The ship takes three years while light takes four years, so we go faster than light!?» Question that comes up regularly among students at the time of the introduction to special relativity.

This is of course not the case. Rather, it should be reformulated as follows: if a terrestrial observer sends a light pulse with a laser, he will have to wait for his clock to indicate four years elapsed before the ray reaches Proxima Centauri; while an observer traveling at 80% *c* will have to wait for his clock to indicate three years elapsed before joining Proxima. And the terrestrial observer will observe well the vessel arriving after the ray, just as the astronaut leaving at the same time as the ray will never exceed it. To be logical, all reasoning must be carried out in a fixed frame of reference. If we change the frame of reference, we change our point of view, and we have to rethink the situation.

First of all, to measure a velocity in a given frame of reference, it is necessary to divide a distance by a time, taking care that the two quantities are measured in this same frame of reference.

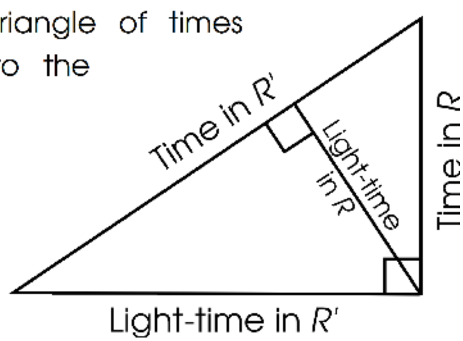
In the question asked by the student, he divides a distance measured in the galactic frame of reference by a time measured in the frame of reference of the vessel. It does not make sense[3], the quantity obtained does not correspond to the speed of an object.

The answer, to this apparent paradox, is that the Sun -Proxima distance measured from the frame of the vessel is not 4 ly, the length is contracted and is less than 3 ly.

The length contraction factor is equal to the time dilation factor.

Do we have the equivalent of the triangle of times?

Not really, because, if we are trying to construct a triangle of lengths, one of the sides does not correspond to a physical quantity, directly measurable. On the other hand, we can add a fourth time in the triangle of times which corresponds to the time taken by the light measured in the reference frame *R*:



3 The discussion will be prolonged and deepened when studying four-vectors and four-velocity.

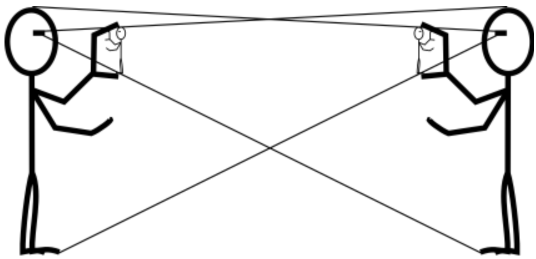
All the triangles are in the same proportions, and the light-time measured in *R* is the shortest.
The Sun-Proxima distance measured from the vessel is 2.4 ly.

ꝏ Spatiotemporal perspective effect

Suppose the astronaut's heart beats at 60 beats per minute. If the time dilation is two, from the point of view of observers on Earth, his heart beats more slowly, once every two seconds. And if for another observer the gamma is equal to three, there will be a beat every three seconds according to the latter. But it goes without saying that for the astronaut, from his point of view, his heart beats quite normally, once a second. Its frame of reference is inertial as for the other two observers.
Also, by the relativity of the motion, the astronaut who observes the inhabitants of the Earth will have the impression of a symmetrical slowing down.
It should be noted that this slowing down of the clocks is the same whether one moves away or that one approaches. This phenomenon is different in nature from the Doppler effect, where, when a source approaches, the received signal is of higher frequency, and when the source moves away, it is lower.
A classic confusion consists in confusing what we see with what is. When you look at a star, you see

the light that it emitted many years ago, possibly not there anymore, or even not existing. Yet spontaneously when we look at the starry sky we feel united to the cosmos, here and now. This illusion stems from our daily habits in a world where maximum speed is very high compared to our routine motions. We can assume the instantaneous propagation of light, we see what is. If the speed limit were 10 km/h, we would be used to these differences. Often we imagine ourselves watching, with the naked eye or with a terrestrial telescope, the astronaut in his vessel moving away and performing his motions in slow motion, but this thought experiment is false, it is not about that.
We do not «see», we measure with the crystal of clocks. The first time you apprehend it, the approach may seem somewhat conceptual, but with practice it becomes natural, and you stop saying that you *see* the clocks slowing down. It is necessary to have in mind the two reference frames of inertia such as meshes, one immobile, and the other in motion, and imagine the two successive events and the dates recorded locally by each of the synchronized clock crystals.

However, we can make analogies with spatial perspective effects. When you look at someone in the distance, he is very small, you can look at him from head to toe between two fingers. He can do the same, it's symmetrical. There is a contraction of the lengths, and nobody imagines the phenomenon as real, the other is not small like a smurf.



The contraction takes place in all directions.
Another perspective effect that produces a contraction of the lengths: the rotation. When I show you a book from the front, then I turn it 90° on a vertical axis, you only see its edge, and the cover has reduced in size to zero during the rotation. The apparent contraction occurred horizontally only.

In special relativity, the two observers are in motion with respect to each other, and it is this motion that simultaneously creates the contraction of lengths and the dilation of time. The lengths are only contracted in the direction of the relative velocity. We recall that, unlike previous analogies, it is not what we see but what we measure.

Contrary to what we sometimes hear, it is not a spatiotemporal rotation. We will see the transformation to be performed between the coordinates (x, y, z, t) of *R* and (x', y', z', t') of *R'* in the chapter *Changing reference frame*, this is not a rotation.

ꝏ Twin experiment

This is a thought experiment proposed by Paul Langevin in 1911. We hope that one day we will have a space-time ship to make it happen! Although not performed with real twins, it has, for the moment, been performed with atomic clocks. We sometimes talk about the twin *paradox*, but it is a reality, not a paradox; this misleading name comes from misunderstandings. Langevin, the main defender of relativity in France at the beginning of the 20th century, did not speak, at the Bologna Philosophy Congress in 1911, of paradox, or of twins... but of a Jules Verne-style voyage by cannonball! It is the mathematician and physicist Hermann Weyl who speaks of twins in 1918. It is the philosopher Henri Bergson who devotes an entire book, published in 1922, on special relativity, which speaks of paradox and gives an erroneous interpretation of the experience.

Now let's explain this experiment. We take two twins as they celebrate their 20th birthday on Earth. Right after the birthday, they leave each other, one stays on Earth and the other leaves for Proxima at 80% *c*. According to the *triangle of times*, we have 5 years elapsed for the twin who remained on Earth and 3 years for the one who travels to Proxima. Then the traveling twin returns to Earth, which doubles the times. The twin on Earth is now 30 years old and the one who has made the round trip 26 years. Our twins are no longer the same age.

The image is striking because the two twins can directly compare their two clocks with a difference of four years. It is less abstract than a measurement via a crystal of clocks. The postulates of special relativity consider inertial frames of reference. We can at some point have the clocks of two different frames coinciding, but then they just move away from each other at constant speed. Thus, the twin experiment cannot be understood on the basis of Einstein's first two postulates alone.

We see a cumulative effect of time dilation on the round trip, why not also a cumulative effect of contractions: a younger and flattened astronaut...!? Time and space do not have equivalent natures: a left-right motion can be compensated by a right-left motion, for time it is impossible, there is the principle of causality and one can only go from the past to the future, one can only move forward in time and the proper times are added.

Before concluding on the twins' experiment one last point. Doesn't it seem absurd to you that the traveler leaves just like that at 800 million km/h, implied instantaneously? It is of course impossible, a physicist is only interested in physically acceptable situations, it would require infinite energy and the force due to the acceleration exerted would also be infinite. In short, even if the acceleration phase lasted a few seconds, it is not conceivable that such a powerful reactor could exist, and the occupants would simply be crushed... The spaceship actually sees its speed increase continuously, which can be modeled by a succession of inertial reference frames of increasing velocities.

A new postulate completes special relativity, it is the *clock hypothesis* which has been verified experimentally:
Two clocks of the same instantaneous speed *v*, one being accelerated and the other not, undergo the same time dilation factor γ.
The clock measures the *proper time* and we add the times of the traveler over the whole of his space-time round trip:

$$\tau=\int d\tau=\int \frac{dt'}{\gamma}$$

The *proper time* is the time measured by a clock at rest in relation to the phenomenon to be studied. We had called it Δt, but often to emphasize its peculiarity we use the Greek letter τ. On the other hand, measuring a relative time requires *two* different clocks previously synchronized.

It is thus possible, without ambiguity, to calculate the actual time taken by the traveler for the round trip. Calculation made from the galactic inertial frame *R'*. Note that if we do the calculation from another inertial frame of reference *R''* we would find the same proper duration τ.

On the other hand, a direct calculation is impossible from the reference frame of the vessel because this one is not of inertia[4].

Joseph Hafele and Richard Keating, in 1971, experimentally verify the «clock hypothesis», the third postulate of special relativity. With few resources and a lot of perseverance, they went around the world twice, one to the east and the other to the west. They were in commercial planes with atomic clocks and many passengers. On the way back, they compare with a clock that has remained on the ground[5].
Photo: Time Magazine, October 18, 1971.

Training: exercises 3, 4 & 5 on page 26.

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4 The calculation can be done from the point of view of the accelerated reference frame using the non-Minkonskian metric given on page 143.

5 *L'expérience cruciale de Hafele et Keating* by Pierre Spagnou, pdf, 27 pages, March 2018.

ꝏ USE OF EQUATIONS

The triangle of times, page 10, gives by application of the Pythagorean theorem :

$$(\Delta t')^2 = (\Delta t)^2 + \left(\frac{\Delta x'}{c}\right)^2 \quad \text{besides} \quad \gamma = \frac{\Delta t'}{\Delta t} \quad \text{and} \quad v = \frac{\Delta x'}{\Delta t'}$$

$$\text{then} \quad (\gamma \Delta t)^2 = (\Delta t)^2 + \left(\frac{v}{c}\gamma \Delta t\right)^2 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

we also note beta : $\beta = \frac{v}{c}$ which expresses the speed with respect to *c*,

So, we have the following relation for gamma:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

Knowing this expression of the gamma factor by heart makes it possible to do without the triangle of times.

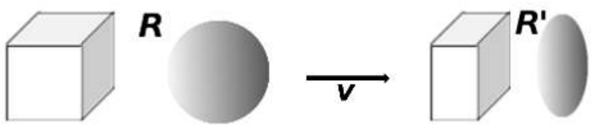
Training: exercises 6 to 9 on page 28.

○ Transformation of volumes and angles

• **Volumes :** Only the lengths along the direction of the relative velocity between the two frames of reference are contracted. Let us take the case of a rectangular parallelepiped along the axes *(Oxyz)* at rest in *R*, then if $\vec{v}=v\vec{i}$: $\Delta x'=\Delta x/\gamma$, $\Delta y'=\Delta y$ and $\Delta z'=\Delta z$,

from where : $\boxed{V'=\frac{V}{\gamma}}$.

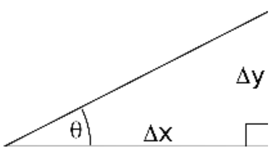
True relationship whatever the shape of the object. Indeed, any object can be decomposed into infinitesimal parallelepipedal volumes each contracted by the same factor γ, the integral, sum of infinitesimals, is therefore also.
A cube in *R* flattens in *R'* while keeping the same section perpendicular to $\vec{v}$. A sphere in *R* flattens in the direction of $\vec{v}$ in *R'*.



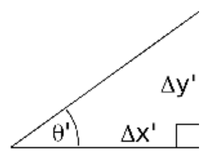
The distance measurement protocol ensures that each position of the object is measured at the same time *t'* in *R'*.
This is of course only a perspective effect, nothing physical here, if for example the cube is a box which contains a gas, this one is not compressed and no risk that this one liquefies!
Concerning what is seen by an observer, there is a new deformation due to the propagation of light

rays to the point of observation. The distance from a point of the object to the observation point varies and the object photographed on a sensor consists of light points which correspond to different instants t' at the level of the object, the measurements are not then simultaneous in R'. This more subtle aspect is discussed in the chapter *The Appearance of Things*.

• **Angles :** Consider a right triangle. A side of length Δx along $\vec{v}$, and a second perpendicular along y and of length Δy. We measure the angle θ between the side of length Δx and the hypotenuse. The triangle is at rest in R and $\tan\theta=\Delta y/\Delta x$. In R': $\tan\theta'=\Delta y'/\Delta x'$. $\Delta x'=\Delta x/\gamma$ and $\Delta y'=\Delta y$ then:



$$\boxed{\tan\theta'=\gamma\tan\theta}$$



When you see a star in the sky, you measure its position using angles. These angles are modified by the motion of the Earth in its orbit in the galactic frame of reference. The apparent angle θ_a under which we see a star is not simply the angle θ' because we must also take into account the propagation of light rays to our telescope. The color of a star is also modified, see the chapter *The Appearance of Things* for more details.

Exercises

Methods of resolution:

- ♥ (card game)
- ✎ (ruler, triangle, protractor and compasses)
- √ (equations)

Difficulty : ▲△△ (simple) / ▲▲△ / ▲▲▲ (complex)

Data :

Speed of light (vacuum) $\simeq$ 300 000 km/s
Distance Sun-Proxima $\simeq$ 4 light-years
Distance Sun-Barnard $\simeq$ 6 light-years
Distance Sun-Sirius $\simeq$ 9 light-years
Radius of the Earth $\simeq$ 6 400 km

1. ♥ ▲△△ The Crystals of the Pop exomoon

In the galactic year 2110, you undertake the Sun-Barnard voyage to study the crystals of the Pop exomoon. After eight years in your rocket, you land on Pop. In what galactic year are we then, and what was the speed of your rocket?

Answers page 337.

2. ♥ ▲△△ One-way ticket for Sirius

It is decided, in 2154, for your 30 years, you will leave for Sirius with the antique ship β6 of your friend Zu. Too eager to change air and make a new start. The ship is not very fast, but spacious and comfortable. At what age will you arrive, and will you be able to attend the festival of the two suns of 2168, or will you have to wait for the one of 2178?

Dream Series β6: model 2110-2124 / Speed 60% of c.

Answers page 337.

3. ♥ ▲△△ Parcel delivery

Your job? The delivery of parcels throughout the galaxy. And you are the first on the market because you have the fastest SpaceTruck!
"... to trade between the Sun and Proxima, I only need 4 years of travel time for the round trip. And a profit of 5 million Blings, imagine how much money I make !!"
How long does it take to deliver, what is the speed of the ship and the time dilation?

Answers p338.

4. ♥ ▲▲△ Twin on his way to Sirius

Twins are 20 years old in 2132, the most intrepid leaves for Sirius and returns in 2156.
How old are the twins then?
What was the speed of the rocket?

Answers page 338.

5. ✎ ▲▲△ Cruel dilemma?

We are in 3021. Denys lives in the galactic center. He has just received terrible news: during his stay in the spiral arm of Perseus, he caught a virus, he will die in exactly 32 years, and there is no cure ...
In addition to that, he has just received a very precise mission order: to defuse a gamma ray bomb located at 26 ly before it destroys the whole galaxy, explosion planned in 3052.
And most important of all, to be there, at the center of the galaxy, for the great secular galactic celebration of 3082!
Denys has a ship with a gamma equal to two.
What can you offer Denys?

Answers p339.

√ The use of equations is the most complete and general method. Nevertheless, we believe that its systematic use, from the very beginning of learning, makes it difficult to understand phenomena intuitively. Moreover, the mathematical language to be mastered unnecessarily blocks many people who are passionate about physics.
The equations are very practical in the two cases where the triangle of times is very stretched: for slow motions where the speed is very low in front of that of light, or, on the contrary, for fast motions where the speed is very close to the maximum speed (ultra-relativistic cases).

6. √ ▲▲△ Muons

Cosmic rays are made up of high-energy particles. Many of those that hit the Earth's atmosphere are protons. They come from the Sun, our galaxy and beyond. Fortunately for life on Earth, many of these particles are destroyed in the upper atmosphere and create showers of other, less energetic particles. We are interested here in the case of muons created in this way. When you are by the sea, an average of 170 muons reach the ground per square meter per second. Every second that you take to read the statement of this exercise dozens of muons pass through you.

Muons have a half-life $t_{1/2}$ of 1.5 microseconds. This means that if you take a large number of muons at rest, only half of them will remain after 1.5 µs, and since they do not age, only a quarter will remain after 3 µs, and so on.
Let's take the example of a muon created at an altitude of 10 km and which moves vertically towards the ground with a speed of 99.9% c.

What do you think about the probability of this muon reaching the ground (sea level)?

Answers page 340.

7. √ ▲▲▲ High-speed train journey

In 2012, the longest high-speed train line is in China and connects Beijing to Guangzhou. Its length is 2300 km and the journey time is eight hours.
You have two atomic clocks. You synchronize them, then, you leave one of them at the station in Beijing, and, the other one accompanies you for your round trip Beijing-Guangzhou.
On the return trip, what will be the time difference between the two clocks?

∘ *Accuracy of on-board atomic clocks: 10^{-14}s/s.*

∘ *The trip is considered at constant speed, which will give a good approximation.*

∘ *A necessary mathematical tool here, a series expansions: if epsilon is very small compared to one, $\epsilon \ll 1$, then $(1+\epsilon)^{\alpha} \simeq 1+\alpha\epsilon$. Here $1/\sqrt{1-\beta^2}=(1-\beta^2)^{-1/2} \simeq 1+\frac{1}{2}\beta^2$.*

Answers page 341.

8. √ ▲▲▲ Satellite

Let's consider a low altitude satellite, such as, the International Space Station. The satellite is placed at an altitude of 500 km and travels at 27,000 km/h in the geocentric reference frame. This frame of reference is considered to be inertial in this exercise.
One clock is placed in the International Space Station and a second is kept motionless in the

geocentric frame of reference. Synchronization and time comparison protocols are perfectly respected.
What is the time difference after a revolution?

∘ *The satellite's frame of reference is not inertial and we apply the clock hypothesis.*

∘ *Unlike Hafele and Keating's experience, the clocks remain at a constant altitude, so we don't have to take into account the effects of gravity.*

Answers page 342.

9. √ ▲▲▲▲ Hafele-Keating experiment

Here we will try to find the results of Hafele and Keating established in 1971.
For a round-the-world trip to the east, they found that the onboard clock aged less than about 60 ns compared to the clock on the ground, on the other hand, for a round-the-world trip to the west, the onboard clock aged more by about 300 ns.
We simplify the problem, only one plane is enough to go around the world. The flight is equatorial at an altitude of 10 km. The plane has a speed of 1000 km/h from the ground. At the equator, the ground is moving at 1674 km/h relative to the geocentric reference frame, here considered Galilean. The takeoff and landing phases are considered fast enough to be neglected.

Concerning gravitation, time slows down when gravitation increases:

$$\Delta t' = \left(1 + \frac{g\,h}{c^2}\right)\Delta t,\ h \text{ : altitude},\ g = 9.81\ \text{m/s}^2$$

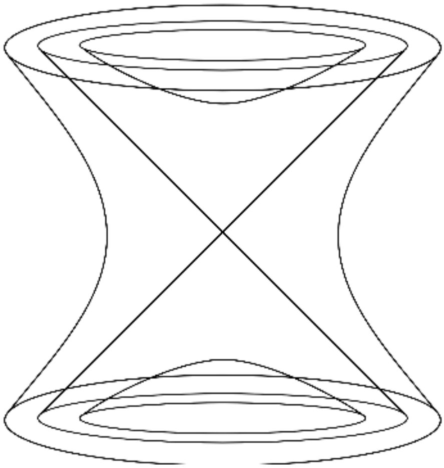
$\Delta t'$ is the time spent in altitude, Δt on the ground.

(general relativity in the weak-field limit)

You can imagine three clocks, the first stationary in the geocentric reference frame, the second at rest in the plane and the third on the ground.

Are your results in agreement with those of the experiment carried out in 1971?

Answers page 342.



2

SPACETIME DIAGRAM

After the *triangle of times*, we present here a second geometrical tool, a diagram, which broadens our vision of space-time, gives a synthetic representation of situations and makes it possible to answer a very large set of questions.

∞ WORLDLINES

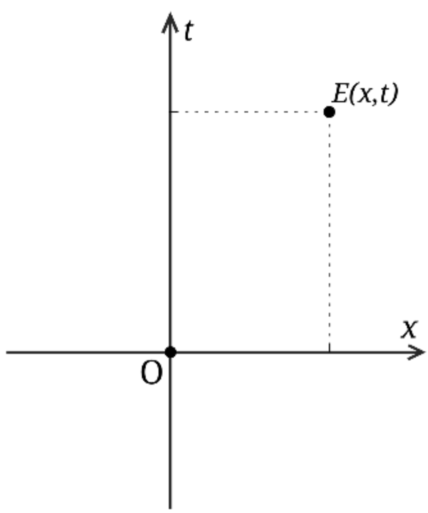
The triangle of times is enough to study the motion of a single moving object with constant velocity. When the velocity of the object varies, or we have several moving objects, we prefer space-time diagrams. For example, for the twin experiment, the traveling twin's direction of velocity changes between the outward and return journey.

The world-line of an object contains all of its physical information: all of its positions through time, and therefore the evolution of its velocity, acceleration and force exerted on the particle.

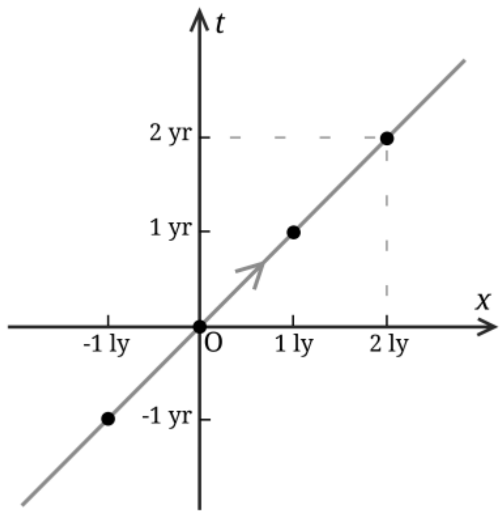
A worldline represents the set of events experienced by an object.

ꝏ MINKOWSKI DIAGRAM

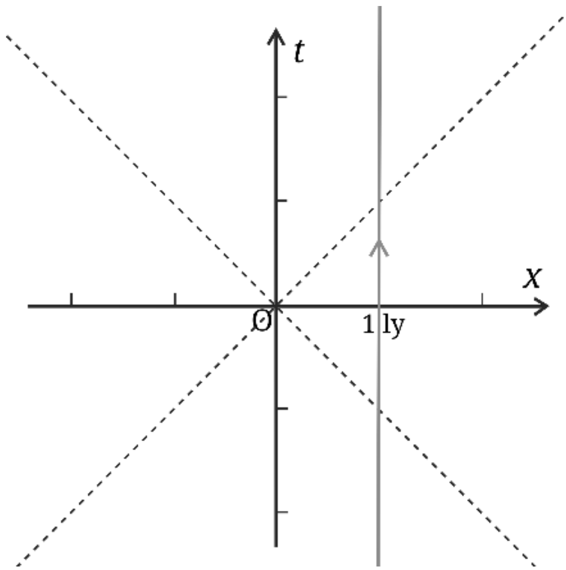
The spacetime diagram is often called a Minkowski diagram in the context of special relativity. In the case of a rectilinear motion, a spatial axis is sufficient and the diagram will be represented in a plane. The horizontal axis represents the *x*-coordinate of the object and the vertical axis the time *t*. Each point in the diagram corresponds to an event. Point O corresponds to the origin event — both temporal and spatial.



Let's start by considering the motion of a photon which "passes" by O and which goes to the right. The successive events "experienced" by the photon create its worldline. We graduate the axes in natural space-time units and we choose the year as the unit of time.

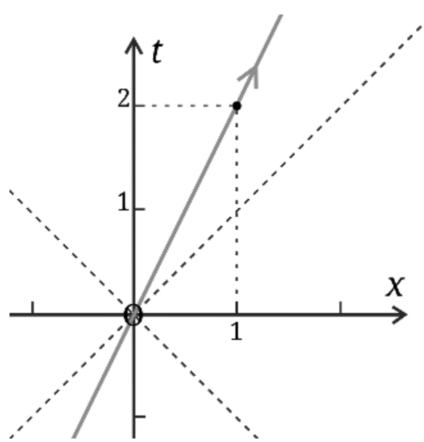


A year ago the photon was located one light-year to the left, it is now here, it will be one light-year to the right in a year, etc.
In addition, we consider a second photon, which also passes through the origin, but which moves in the other direction, from right to left. The two photon worldlines are shown in dotted lines and are often present to aid the reading of Minkowski diagrams. In the case of an immobile particle in the observational frame of reference, the worldline is a vertical line oriented upwards.
On the following diagram we have the world line of an object at rest in the observational frame of reference and located one light year to the right of the origin of the frame.

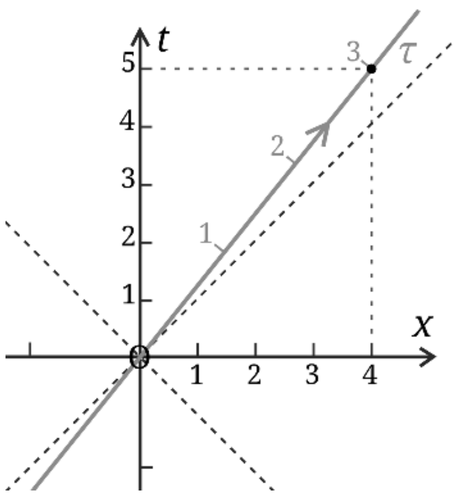


We now consider the general case of a particle which passes through *O* and moves to the right with a constant velocity *v*. As a particle cannot go faster than light, the worldline is represented by a straight line of inclination intermediate between the vertical line (time axis) and the dotted line of the corresponding light ray.

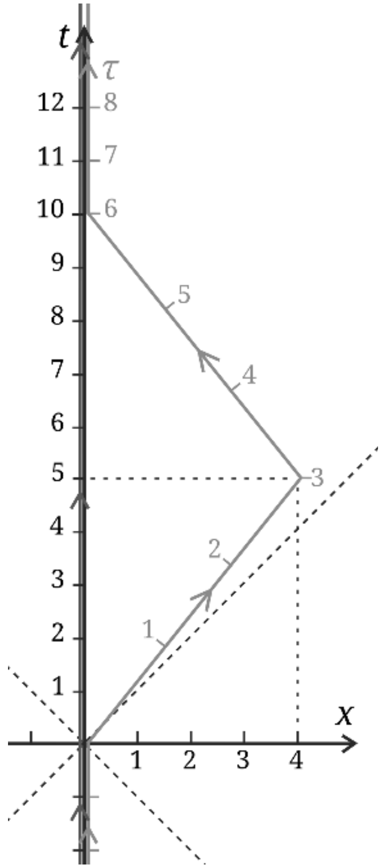
On this example, the object moves at 50% of *c*, it travels one light-year in two years.



We now know that there is dilation, the time for a moving object is not the same as for an object at rest. We take the example of a trip at 80% *c*. With the triangle of times we obtain the proper time τ which we add on the worldline of the moving object. The dilation of time appears clearly.



For the twin experiment we visualize the two worldlines of each on the same diagram:



The worldlines are represented in the frame of reference of the twin who remained on Earth, more precisely the galactic frame of reference which is an excellent inertial frame of reference. We cannot directly reason from the reference frame of the traveler, the latter is not inertial because his velocity varies.

ꝏ Use of equations

Equation of worldlines

These straight line equations are used to determine dates and positions, appointments and reception of spatial messages.

Ship passing through *O* and heading to the right at speed *v*:

$$v=\frac{x}{t} \quad \text{then} \quad t=\frac{1}{\beta}\frac{x}{c} \quad \text{with} \quad \beta=\frac{v}{c}$$

Ship passing through *A* and heading left at speed *v'*:

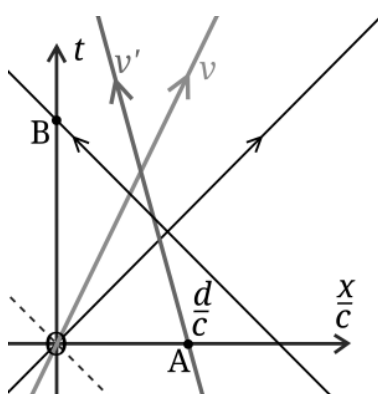
$$t=-\frac{1}{\beta'}\frac{(x-d)}{c} \quad \text{with} \quad \beta'=\frac{v'}{c}$$

Photon passing through *O* and heading to the right:

$$t=\frac{x}{c}$$

Photon which passes by *B* and goes to the left:

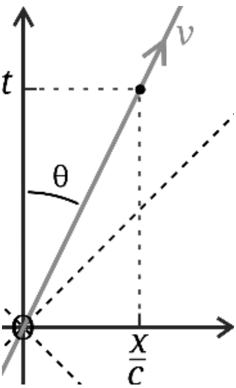
$$t=-\frac{x}{c}+t_B$$



Angles

The more the speed increases, the more the world-line of the spaceship, initially vertical, inclines at an angle θ which tends towards 45° when the speed approaches the maximum speed *c*.

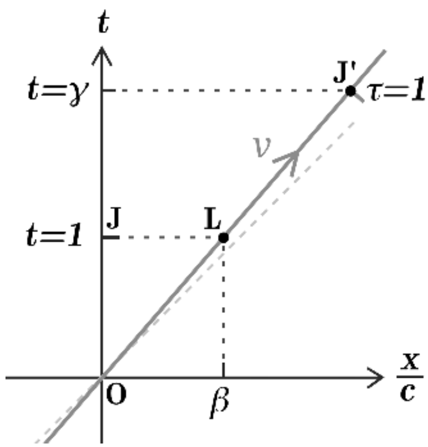
$$\tan\theta = \frac{x/c}{t} = \beta$$



β	0	0.1	0.25	0.5	0.6	0.8	0.9	0.94	1
θ	0°	6°	14°	27°	31°	39°	42°	43.3°	45°
γ	1	1.005	1.03	1.15	1.25	1.67	2.3	3	+∞

Scale factor

On the worldline of the ship, the proper time axis, time passes more slowly and the graduations are more spaced.



$$OJ=1$$

$$OL=\sqrt{1+\beta^2}\,OJ$$

$$OJ'=\gamma\,OL$$

$$OJ'=\gamma\sqrt{1+\beta^2}\,OJ$$

$$\frac{OJ'}{OJ}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}=\sqrt{2\gamma^2-1}$$

v % of *c*	50	60	75	80	87	95	99	99.5
γ	1.15	1.25	1.51	5/3	2	3.2	7	10
OJ (t=1)	1	1	1	1	1	1	1	1
OJ' (τ=1)	1.29	1.46	1.89	2.13	2.6	4.4	10	14

Exercises

┼ : resolution by Minkowski diagrams.

1.┼ ▲△△ Draw the Minkowski diagrams of chapter 1 exercises 1 to 5.

Answers p344.

2.┼√ ▲△△ **Interstellar communications**

In the Twins Experiment page 17, when the traveling twin lands on the planet Proxima b, it takes a photo and sends it to Earth as an electromagnetic wave.
When will the twin on Earth receive the photo?
Throughout the journey, the twin on Earth follows his brother's journey using a very powerful telescope. When will he see his brother land on the planet in his telescope?
If the twin on Earth looks through his telescope the moment his brother lands on Proxima b, 5 years after his departure, what does he see?
To send a birthday message to his brother when he lands on the exoplanet, when should he send it?
Make a Minkowski diagram that represents the worldlines of the twins and those of the photons that transmit the photo, the telescope images and the message.

Answers p347.

3.√ ▲▲△ Call for help

A cruise ship with more than 10,000 people on board undertakes the Proxima - Earth crossing at the speed of 50% of light.
Halfway through the journey, the ship calls for help. An emergency rescue shuttle leaves Earth at 90% *c* as soon as the electromagnetic distress message is received.
How long will the passengers have to wait before the arrival of the help ?

Answers p349.

4. ▲△△ Tim, Tam, Tom

We are in a slow universe where the maximum speed inherent to space-time is 20 km/h.
Tom, Tim and Tam are in the living room, the clock indicates 10 o'clock. They decide to meet there at 11 o'clock. Tom stays there. Tim leaves to run at 10 km/h. Tam goes to work at his office 10 light-minutes away with a bicycle that travels at 15 km/h
Tim has to be back by what time indicated on his watch?
How much work time will Tam have at his office? What time will his watch show when he returns?

Answers p350.

CHANGING REFERENCE FRAME

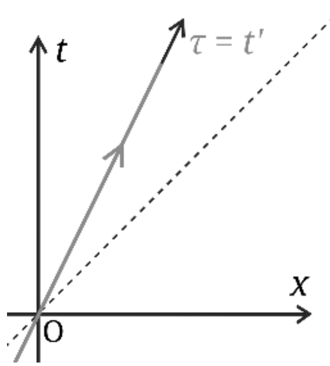
We will consider a second inertial reference frame. The first observational reference frame was the reference frame *R* of axes *(x, y, z, t)*, a frame often associated with the galactic frame of reference in the context of interstellar journeys.

The second frame of reference *R'* is in motion with respect to *R*, moved at a constant velocity. We say that *R'* is in uniform rectilinear translational motion with respect to *R*. For *R'* the origin is denoted *O'* and the axes *(x', y, 'z', t ')*.

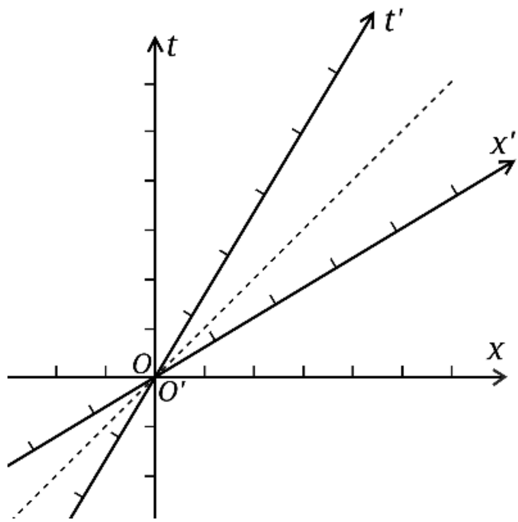
R' is then also a reference frame of inertia, where the principles of special relativity apply. This frame of reference *R'* will often be associated with the spaceship.

SPACETIME DIAGRAM

We will build step by step the axes of *R'* in the Minkowski diagram of *R*. The proper time τ on board the space-time vessel corresponds to the time *t'*. The axis *O't'* is thus identified with the ship's worldline.

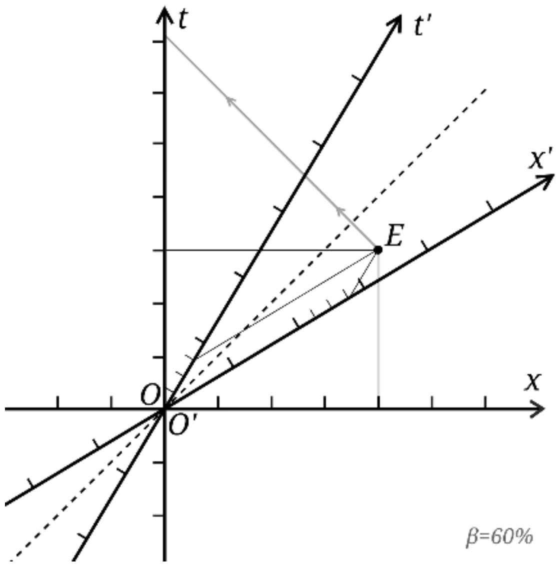


The speed limit is the same in *R* and *R'*. This invariant shows that the axis *O'x'* is necessarily symmetrical with respect to the worldline of a light flash that moves to the right and passes through O. We thus have the reference frame *R'* seen from the reference frame *R* :



Let's show on an example how the coordinates are read. From Earth, we record, 3 years after the departure of the spacecraft, a huge stellar eruption produced by the star Proxima Centauri located 4 light-years away. The spacecraft is moving at 60% of *c*. In the reference frame of the spaceship where and when does the eruption occur?
In the galactic frame of reference *R* the event *E* has coordinates *(x=4, t=3)*.
In the vessel frame of reference *R'* we read on the

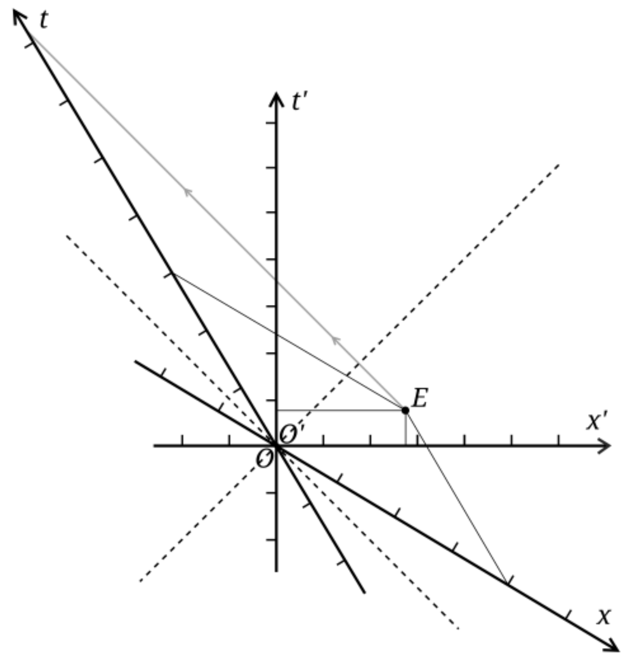
Minkowski diagram that the event *E* has coordinates *(x'=2.75, t'=0.75)*. The occupants of the ship will determine that the eruption occurred 9 months after their departure at a distance of 2.75 light-years.



Nevertheless, the astronauts will see the flare in their telescope well after 9 months. Indeed, following the eruption, it is also necessary to allow time for the light to propagate to the telescope and to the eye of the observers. To complete this we have drawn in gray the worldline of a light beam emitted by the flare. It will first be observed in the spaceship after about 3 and a half years of travel, and it will then be observed on Earth 7 years after departure.

In Minkowski's diagrams, the coordinates indicated for an event are taken from local recordings made using the reference solid and the associated clock crystal. Propagation times are not included.

All inertial frames of reference are equivalent in special relativity and we can also represent R from R':

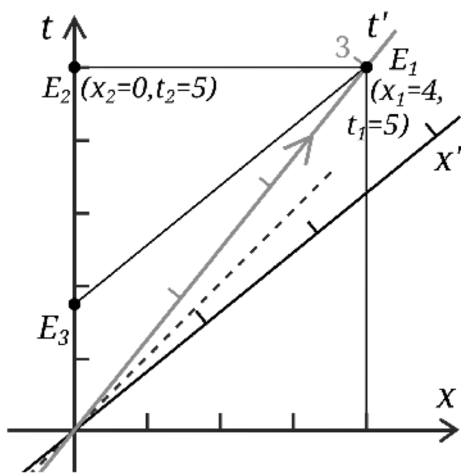


⚭ Relativity of simultaneity

In the case of the ship heading towards Proxima at 80% c we had a 3-4-5 triangle of times. When the ship is at the level of Proxima 4 light-years away, before reducing its speed, 3 years have elapsed in the starship and 5 years on Earth.

Consider these two events, E_1, the ship is at the level of Proxima at 80% *c*, and, E_2, the clock on Earth indicates 5 years since departure. An earthling can say to himself: "there, now that 5 years have elapsed for me, the ship is at this very moment at the level of Proxima. If I cannot see it directly with a telescope it is for technique reasons of finite speed information propagation time", he can then have the emotion of the moment shared with the astronauts. If we look at these two events in a space-time diagram they are effectively at the same time *t*, they are simultaneous events in *R*.

For this to be true, this emotion of a common moment must be shared, but for the observers on board the ship it is not. For them in *R'* the simultaneous event on Earth is E_3, a much earlier event, less than two years after departure.



Simultaneity is relative to the frame of reference considered. In Newtonian mechanics, simultaneous events remained so in all frames of reference, in special relativity simultaneity is not an absolute notion. In R, E_1 and E_2 are simultaneous, in R', E_1 is earlier than E_2.

ꝏ CAUSALITY

We can only go from the past to the future. It is pure logic, the cause produces an effect and not the opposite! The world is *One*, and this is only an obvious principle of consistency. If you could go back in time and travel in the past, you would destroy the present...

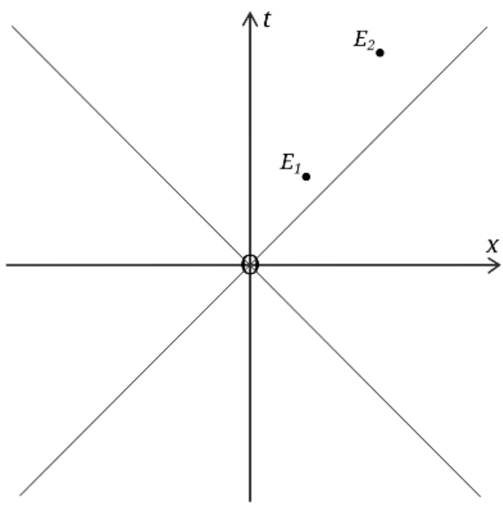
For example, you go 50 years in the past and during this time travel you die in a car accident, or just your actions do that your parents don't actually meet, etc. If you want to travel to the past at all costs, then you would need several presents and suppose parallel worlds which realize all possibilities.

In physics, we choose the simplest theory to explain the facts, there is only one world, *One* reality, the past cannot be changed, the future does not pre-exist, we cannot go back and the *arrow of time* is constantly advancing from the present to the future.

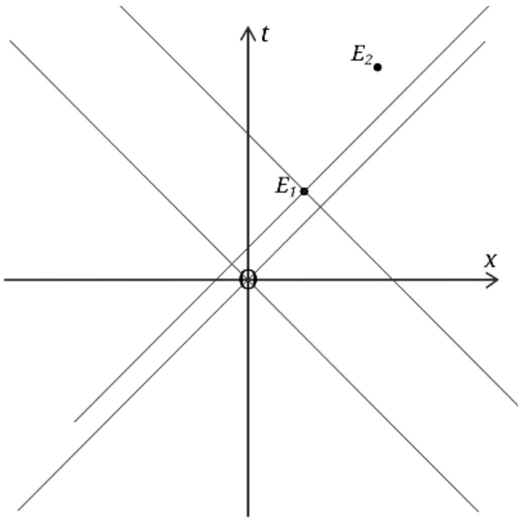
Special relativity of course respects the principle of causality. Not as simply as in the old mechanics, but just as rigorously. The fact that there are several times, the possibility of traveling in the future, a relative simultaneity, can create a confusion that we will clarify immediately.

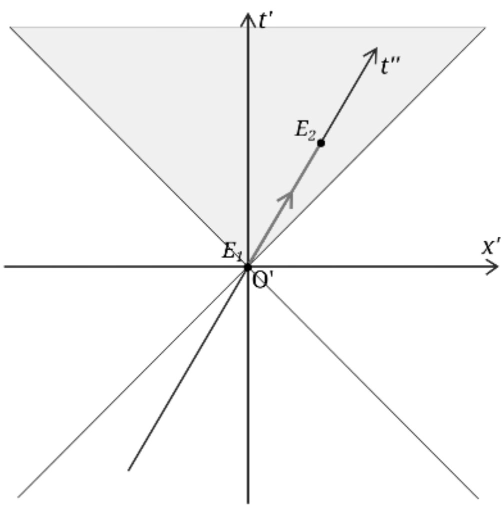
Let us take any two events E_1 and E_2. If there is a causal link between them we can determine which event is prior, and this temporal ordering must be independent of the observation frame of reference. Two different cases can occur, let us represent the events on a diagram in an arbitrary observation frame *R*.

First case: there is a possible causal link between E_1 and E_2. The two events have a constant temporal order whatever the observational reference frame.



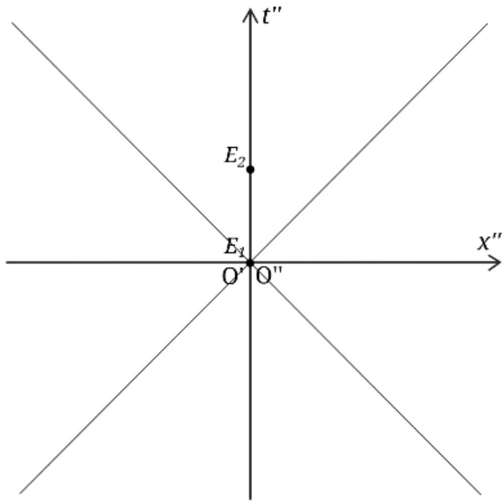
In *R*, E_2 is subsequent to E_1 because $t_2>t_1$.
We then consider *R'*, a frame that is immobile with respect to *R* but with a new origin $O' = E_1$.





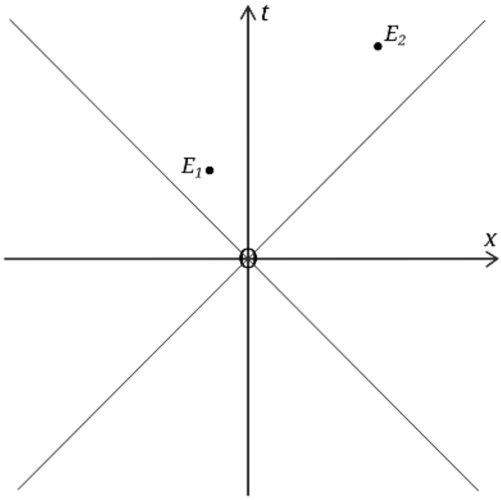
We note a possible causal link between the two events, for example a ship can connect the two points (its speed would not have to exceed the maximum speed), or a succession of events which propagate step by step like in a line of dominoes that fall and establish a causal chain.

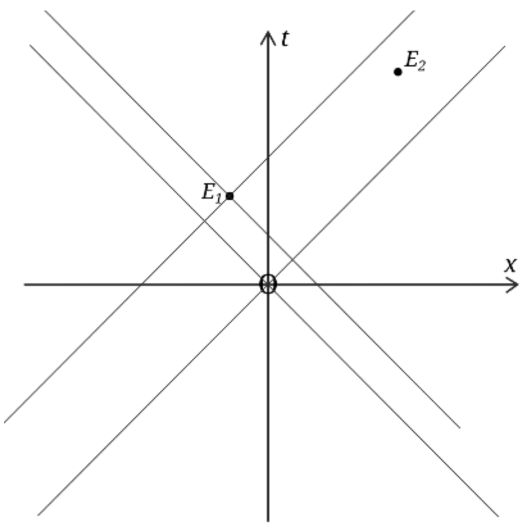
We can then place ourselves in the ship's proper frame R'', the chronological order is not changed and we always have E_2 later than E_1 and $t_2''>t_1''$.



Events E_1 and E_2 occur at the same place in R''. It is in this proper frame of reference that the time interval between events is minimal: $t_2'' - t_1'' = \Delta t'' = \tau < \Delta t' = \Delta t$.

Second case: there is no possible causal link between E_1 and E_2. The temporal order is not defined, E_1 is prior to E_2 in one frame of reference, the reverse in another, and the events are simultaneous in a third. This does not call into question the principle of causality, because there is no possible cause and effect link between these two events.


t
E_2
E_1
x
O


t
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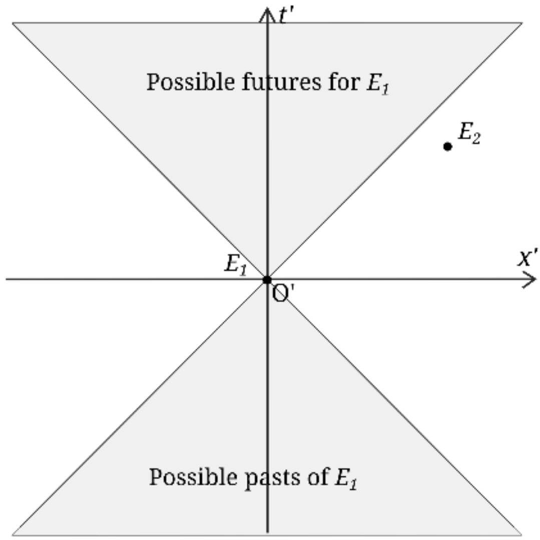


No material object or luminous object passing through E_1 can join E_2, and vice versa. No object can go faster than light. These two events are independent and cannot interact. Looking for a timeline between them does not make sense. There is no proper frame where these two events are at rest.

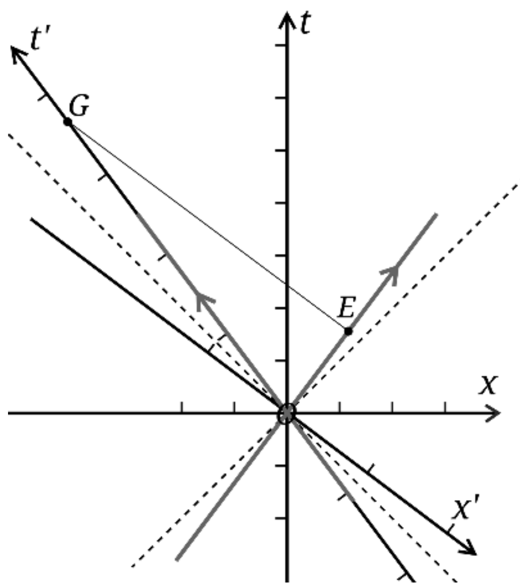
ꝏ Composition of velocities

Two ships hurtle towards each other at 75% of maximum speed. If you get into one of the ships, how fast will you see the other ship coming towards you?

If we had the additivity of the speeds as in classical mechanics we would find 150% of c, speed above the limit, which is, in fact, impossible.

We are going to represent on a diagram the worldlines of the two vessels in the galactic reference frame *R*. The two vessels approach, cross in *O*, then move away.

From the frame of reference *R'* of one of the two vessels, we measure the coordinates of the second and we will simply have its speed in *R'*.

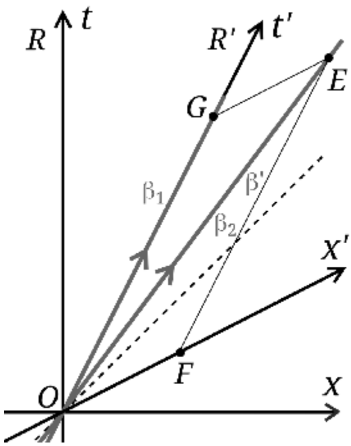


The distance *OG* corresponds to *t'* and measures 4.8 cm on the drawing. The distance *EG* corresponds to *x'* and measures 4.6 cm on the drawing. By dividing *EG* by *OG* we get the relative speed of the vessels:

$$v' = 96\% \text{ of } c$$

Let us now take a second case where the vessels move in the same direction, one at 50% and the other at 75% of *c*:

We divide *EG* by *EF* and we find a relative speed between the two vessels of 40% *c*.



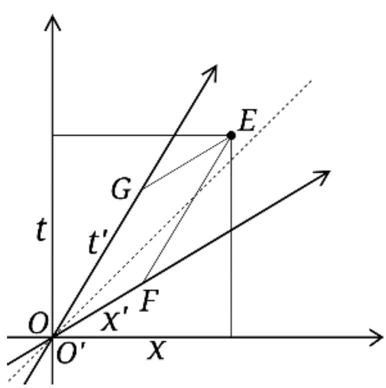
Clearly, for relativistic speeds, the speeds do not add or subtract.

The law of composition of the speeds is different in special relativity and ensures that the speeds of the objects are indeed subluminic.

ↀ Use of equations

∘ Lorentz transformation

For an event *E*, we want to express its coordinates *(x',t')* in *R'* in relation to that *(x,t)* in *R*.



For t' : $\quad t'=\frac{OG}{\gamma\sqrt{1+\beta^2}}$

We have applied the scaling factor to go from *R* to *R'*, a factor established in the previous chapter.

The coordinates of point *G* are given by the intersection of the two following lines:

$$t=\frac{1}{\beta}\frac{x}{c} \quad \text{and} \quad t-t_E=\beta\left(\frac{x}{c}-\frac{x_E}{c}\right)$$

(t' axis and straight line parallel to the x' axis which passes through *E* with a slope inverse to that of the t' axis)

After solving this system of equations :

$$t_G=\gamma^2\left(t_E-\beta\frac{x_E}{c}\right) \quad \text{and} \quad \frac{x_G}{c}=\beta t_G$$

So : $$OG=\sqrt{t_G^{\ 2}+\left(\frac{x_G}{c}\right)^2}=\gamma^2\sqrt{1+\beta^2}\left(t_E-\beta\frac{x_E}{c}\right)$$

And finally : $$t'=\gamma\left(t_E-\beta\frac{x_E}{c}\right)$$

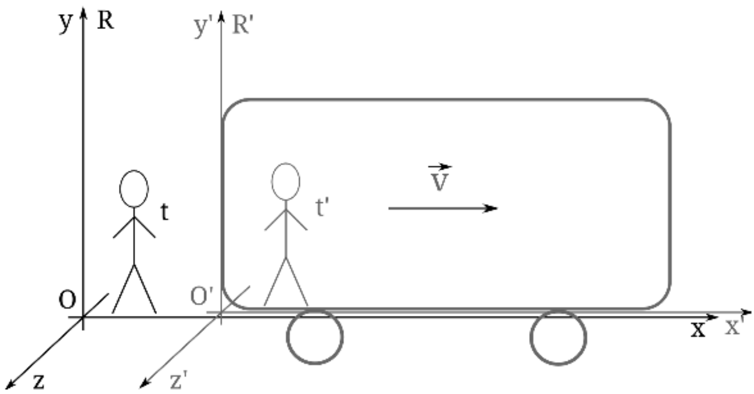
Proceeding in a similar way for x', we find :

$$\frac{x'}{c}=\gamma\left(\frac{x_E}{c}-\beta t_E\right)$$

We obtain what is called the Lorentz transformation of the coordinates of an event. For a motion of *R'* with respect to *Ox* and the setting to zero of clocks and spatial coordinates when they coincide in *O=O'*, we can, without losing generality, write :

Lorentz transformation

$$\begin{cases} \dfrac{x'}{c} = \gamma\left(\dfrac{x}{c} - \beta t\right) \\ y' = y \\ z' = z \\ t' = \gamma\left(t - \beta \dfrac{x}{c}\right) \end{cases}$$



At *t=0* and *t'=0*, *O* and *O'* coincide, then *O'* moves away to the right as time passes. On a Minkowski diagram, in full agreement with the one above, *O* and *O'* are no longer points but worldlines, the axis of *t* and the axis of *t'*. The origins *O* and *O'* indicated are the spatio-temporal positions at *t=t'=0*.

To get the coordinates in *R* from those in *R'*, simply change the sign of the speed and thus β :

$$\begin{cases} \frac{x}{c}=\gamma\left(\frac{x'}{c}+\beta t'\right) \\ y=y' \\ z=z' \\ t=\gamma\left(t'+\beta\frac{x'}{c}\right) \end{cases}$$

Within the limits of low speeds we find the classical transformation of coordinates. Spatial and time coordinates are then disconnected to let space and time both absolute:

Galilean transformation

$$\begin{cases} x'=x-vt \\ y'=y \\ z'=z \\ t'=t \end{cases}$$

In this book we made the pedagogical choice to start from the triangle of times to construct the special relativity. We could also start from the Lorentz transformation. In what follows we find the time dilation, the length contraction and the existence of a relativistic invariant using this transformation.

• **Time dilation:** we have events that occur at the same location in *R*, so $x_2=x_1$ and $\Delta x=x_2-x_1=0$, separated by a time interval $\Delta t=t_2-t_1$. What happens to this time interval in *R'* ? $\Delta t'=(\gamma\Delta t-\beta\Delta x/c)$ then $\Delta t'=\gamma\Delta t$. QED

• **Length contraction:** we can imagine a ruler at rest in *R* placed on the *x*-axis, $L=\Delta x=x_2-x_1$. The protocol for measuring a length in a given frame of reference requires to determine the positions of the ends at the same time in this frame. Measurement of the relative length *L'* in *R'* : $\Delta x=\gamma(\Delta x'+\beta c\Delta t')$ and $t'_2=t'_1$ then $L=\gamma L'$, and $L'=L/\gamma$. QED

• **Lorentz invariant:** In classical mechanics we had two invariant quantities: length $L=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ and duration Δt. Whatever the observational frame of reference, we had the Euclidean distance and the duration unchanged. This is no longer the case in special relativity. But we have another quantity that verifies this property: $\Delta s^2=c^2\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2$. Δs is the *spacetime interval* between any two events, it corresponds to a sort of Minkowskian distance.
Its property of invariance is verified by carrying out the calculation of Δs in a second inertial reference frame *R'* :

$$\Delta s'^2=c^2\Delta t'^2-\Delta x'^2-\Delta y'^2-\Delta z'^2$$
$$\Delta s'^2=\gamma^2(c\Delta t-\beta\Delta x)^2-\gamma^2(\Delta x-\beta c\Delta t)^2-\Delta y^2-\Delta z^2=\Delta s^2$$

We can write Δs^2 as a function of the speed *v* of an object which joins the two events along a rectilinear and uniform trajectory:

$$\Delta s^2 = c^2 \Delta t^2 \left(1 - \frac{v^2}{c^2}\right)$$

Δs^2 can be of different signs, if there is a possible causal link between the events, $v \leqslant c$, Δs^2 is positive or null and the interval is said to be timelike or lightlike (null vector), if it is negative, $v > c$, Δs^2 is spacelike. When Δs^2 is not spacelike, we can link the interval Δs to the proper time τ :

$$\tau = \frac{\Delta s}{c} = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

Proper time is the fundamental notion on which special and general relativity is built. This measure of the aging of a particle is invariant and absolute, unlike the space-time coordinates *(ct, x, y, z)* which are relative and have no physical meaning in themselves.

○ **Composition of velocities**

We use the notations in the figure on page 61. β_1 and β_2 are the speeds in *R* of starships 1 and 2 expressed as a percentage of *c*. β' is the speed of vessel 2 in *R'*.

$$\begin{cases} t = \dfrac{1}{\beta_1} \dfrac{x}{c} \\[2ex] t - t_E = \beta_1 \left(\dfrac{x}{c} - \dfrac{x_E}{c}\right) \\[2ex] t_E = \dfrac{1}{\beta_2} \dfrac{x_E}{c} \end{cases}$$

The first equation is for the world line of ship 1 in *R*, the second for the line *(EG)* and the third the relationship between the coordinates of a point *E* on the worldline of ship 2 .

The first equation applied to point G gives:

$$OG=\sqrt{t_G^{\ 2}+\left(\frac{x_G}{c}\right)^2}=\sqrt{1+\beta_1^2}\ t_G$$

Besides :

$t_G-t_E=\beta_1(\beta_1 t_G-\beta_2 t_E)$ and $t_G(1-\beta_1^2)=t_E(1-\beta_1\beta_2)$

After some calculus, we have *EG* as a function of β_1, β_2 and t_G. We calculate the relative speed:

$$\beta'=EG/OG.$$

Then :

$$\beta'=\frac{\beta_2-\beta_1}{1-\beta_1\beta_2}$$ (vessels in the same direction),

$$\beta'=\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$ (vessels in opposite directions).

We find the good results for the two examples of the course:

$\beta'=\frac{0.75+0.75}{1+0.75\times 0.75}=0.96$ and $\beta'=\frac{0.75-0.5}{1-0.5\times 0.75}=0.4$

In terms of speeds we have: $\boxed{v'=\frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}}$

If the speeds are small compared to *c*, the denominator tends towards 1 and $v'=v_1+v_2$, we find the classical additivity of the velocities again.

<u>Second method</u> : We reasoned with objects which move at constant velocities. We can do a more general calculation using the Lorentz transformation. Definition of the instantaneous velocities with respect to *(x,t)* and *(x',t')* in *R* and *R'* :

$$v=\lim_{\Delta t\to 0}\frac{\Delta x}{\Delta t}=\frac{dx}{dt} \quad \text{and} \quad v'=\frac{dx'}{dt'}$$

these quantities should be noted v_x and $v_{x'}$, we will write *v* and *v'* for ease of reading.

From Lorentz's transformation:

$$x'=\gamma(x-\beta c t) \quad \text{and} \quad ct'=\gamma(ct-\beta x) \quad \text{with} \quad \beta=u/c$$

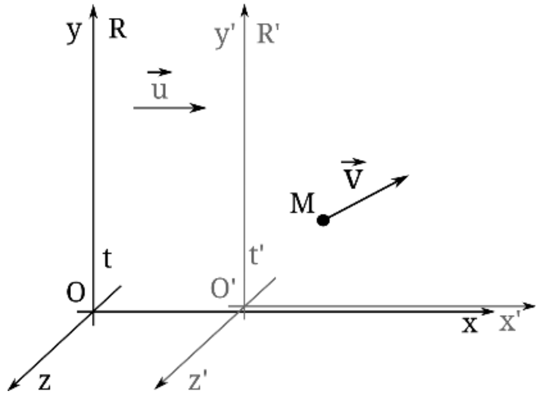
hence for infinitesimal variations:

$$dx'=\gamma(dx-\beta c\,dt) \quad \text{and} \quad c\,dt'=\gamma(c\,dt-\beta\,dx)$$

And by dividing the two equations:

$$\frac{dx'}{c\,dt'}=\frac{dx-\beta c\,dt}{c\,dt-\beta\,dx} \quad , \quad \frac{v'}{c}=\frac{\frac{v}{c}-\beta}{1-\beta\frac{v}{c}} \quad \text{and} \quad v'=\frac{v-u}{1-\frac{uv}{c^2}}$$

u is the speed of *R'* compared to *R*.



We can easily obtain the two other components of the velocity for *y* and *z* [6], but we limit ourselves here to the rectilinear motion.

- **Transformation of accelerations**

With respect to *x* and *x'* : $a_x = \frac{d v_x}{d t}$ and $a_{x'}' = \frac{d v_{x'}'}{d t'}$

Simply noted *a* and *a'* thereafter.

$$a' = \frac{d v'}{d t}\frac{d t}{d t'} = \frac{a\left(1-\frac{u v}{c^2}\right)+(v-u)\frac{u}{c^2}a}{\left(1-\frac{u v}{c^2}\right)^2}\frac{1}{\gamma\left(1-\frac{u v}{c^2}\right)}$$

(quotient rule)

Then : $a' = \frac{1}{\left(1-\frac{u v}{c^2}\right)^3\gamma^3}a$

In the case where *M* is initially at rest in *R'* the initial velocity *v* is zero and $\boxed{a' = \frac{a}{\gamma^3}}$.

6 Done in exercise on page 101 (composition of velocities and accelerations in 3D).

Exercises

1. ▲△△ Composition of velocities

a - Two vessels are heading towards each other at 50% of *c*. What is their relative speed?

b - Two vessels are moving in the same direction, one at 80% of *c* and the other at 50% of *c*. What is their relative speed?

Answers p351.

2. ▲▲△ Two vessels

Two spaceships *A* and *B* produce the following events in the galactic frame *R* :

$$E_{A,1}(x_A=0, y_A=0, z_A=0, t_1=0)$$

$$E_{B,1}(x_B=2, y_B=2, z_B=2, t_1=0)$$

$$E_{A,2}(2,0,0,t_2=4) \qquad E_{B,2}(4,4,4,t_2=4)$$

$$E_{A,3}(4,0,0,8) \qquad E_{B,3}(5,5,5,8)$$

Distances and times in light-years and years.

R considered of inertia.

a - What are the average velocities of the ships between t=0 and t=4 ?
Same question between t=4 and t=8.

b - What are the average accelerations of the vessels between t=0 and t=8 ?

c - Vessel *A* has a translational, rectilinear and uniform motion. We call *R'* the reference frame of vessel *A*. Is the frame of reference *R'* inertial?
Determine the coordinates of the events of vessel *B* as seen from vessel *A*.
Can the trajectory of vessel *B* in *R* be rectilinear? Is it the same for the trajectory of *B* seen from *R'* ?

d - In *R'*, determine the average velocity of vessel *B* between t'_1 and t'_2, then between t'_2 and t'_3.

e - In *R'*, determine the average acceleration of vessel *B*.

f - Could you determine the average acceleration felt by the passengers of vessel *B*?

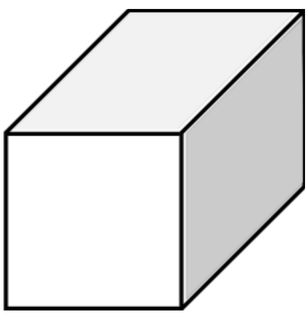
g - Accelerations are calculated here in ly/yr^2, how to convert them into m/s^2?
Deduce the acceleration to which the astronauts are subjected as a percentage of the Earth's gravity field at sea level: $g=9.81\ m/s^2$.

Answers p351.

3.√ ▲▲△ Low speeds limit

Two cars drive face to face at 90 km/h. What is their relative speed? Determine the difference with the classical limit.

Answers p356.



4

The appearance of things

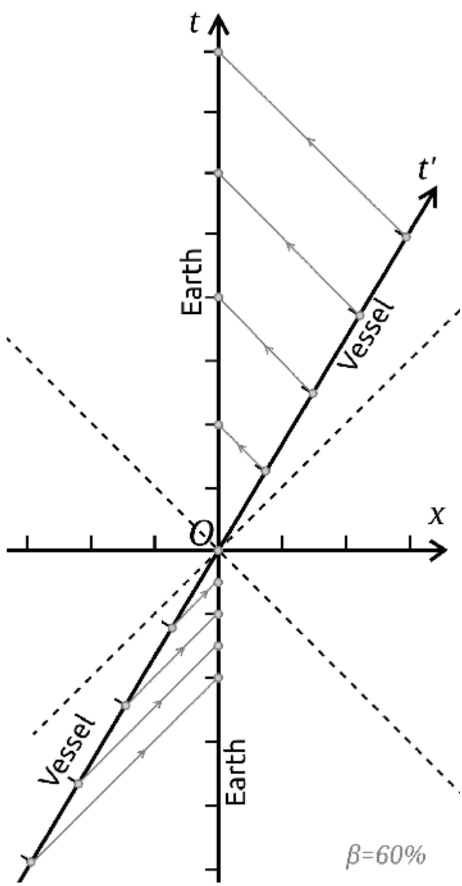
Sometimes we naively forget to take into account the duration of the propagation of the signals to our eye, as if we were spontaneously seeing spacetime as a whole.

We will begin by studying the Doppler effect where, due to relative motion, the color of objects is modified. The color of light depends on the period of the light wave. This quantity is a time, and we could think that it is sufficient to take into account the time dilation. The perceived period would simply be multiplied by γ as the travel time in the twin experiment. Except that the twins end up in the same place and there is no delay due to the propagation of a signal at finite speed. For the Doppler effect the frequency will not simply be divided by γ, and moreover, it will differ depending on whether the vessel is moving closer or further away.

After studying the Doppler effect, we will take pictures of a relativistic ruler, followed by a contemplation of the starry sky in a starship each time faster.

ꝏ Doppler effect

The Doppler effect can be experimented with all kinds of waves: sound waves, electromagnetic waves, waves on water, etc... In all cases, we have a wave propagating at a finite speed, and a source and a receiver in relative motion. For example, for a sound wave that propagates in the air, if you get closer to the source the frequency is heard higher, and if the source moves away the sound is, on the contrary, perceived deeper.



Here we will focus on an electromagnetic wave that propagates in vacuum, or more precisely in spacetime.

In this case, in addition to the delays or advances due to the propagation of the wave towards a moving object, the effect of spacetime perspective is added.

Let's take the example of a yellow

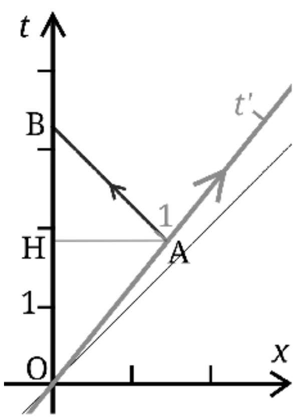
light, with a wavelength of 600 nm, emitted by a vessel moving at 60% c. To simplify, let's imagine that the vessel emits regular flashes of light at the frequency of the wave. We have drawn the world-lines of these flashes on a Minkowski diagram. We see on Earth the flashes closer when the ship is approaching and further apart when the ship is moving away. The time between the reception of two flashes corresponds to the period of the signal, we measure on the diagram, when the ship is approaching the Earth:

T=T'/2 so f=2f' and λ=λ'/2 then λ=300 nm,
the light received is in the ultra-violet.

When the ship moves away:

T=2T' so f=f'/2 and λ=2λ' then λ=1200 nm,
the light received is in the infrared.

- **Use of equations**



A periodic signal is emitted in *R'* with a period *T'*, and, is received in *R* with a period *T*. We call *r* the ratio between these two periods : $r=T/T'$.

In the event that the source and receiver move away:

$r_+=OB_R/OA_{R'}$

For $OA=1$ on the axis of *t'* we have $OH=\gamma$ on the axis of *t*.

In *(x, t)* : $OA=\gamma\sqrt{1+\beta^2}$ (scale factor).

The triangle *BHA* is right isosceles in *H* : $AH=BH$.

Pythagorean theorem in *OHA* :

$$r_+ = OB = OH + HB = \gamma + \sqrt{\gamma^2(1+\beta^2) - \gamma^2}$$

When moving away : $r_+ = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}$

When getting closer : $r_- = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}$

In terms of frequencies, *f=1/T* :

$$f' = \sqrt{\frac{1\pm\beta}{1\mp\beta}}\, f \quad \text{and} \quad \boxed{T = \sqrt{\frac{1\pm\beta}{1\mp\beta}}\, T'}$$

Interval : $0 < r < 2\gamma$

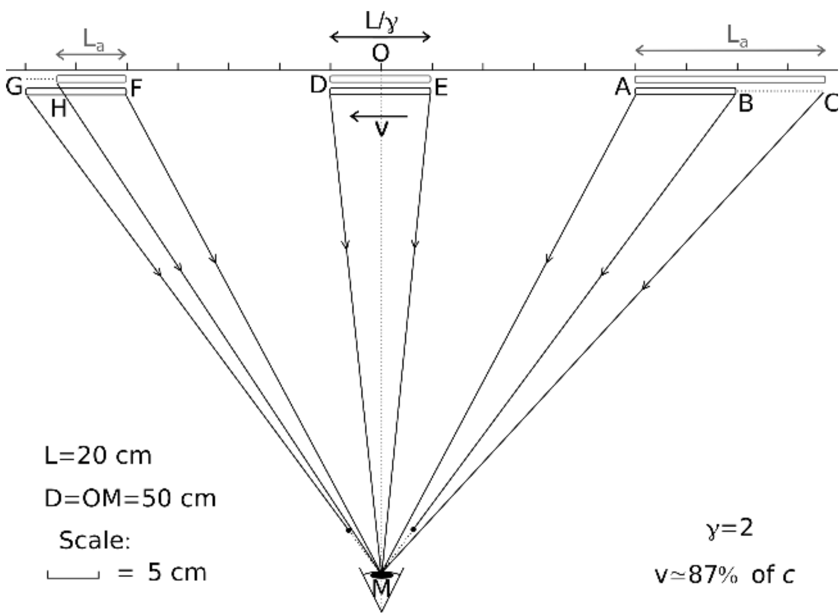
In the example of the course, $\beta=0.6$, $\gamma=1.25$ and the numerical application gives correctly *f=2f'* when transmitter and receiver are approaching, and *f=f'/2* when they are moving away.

The Doppler effect shows that the color of a photon is not an absolute quantity. A photon is neither red, blue, nor yellow, it all depends from where you look at it. Its wavelength depends on the observational inertial frame of reference and there is no privileged observer.

A photon has other characteristics, such as chirality, which is intrinsic. A photon is either left or right and, unlike its color, it does not depend on the point of view.

ꝏ Photograph of a moving ruler

A ruler moves at the velocity $\vec{v}$ in the observational reference frame *R*. A graduated optical bench, fixed in *R*, makes it possible to locate the position of the two ends of the ruler. The proper length of the ruler, in the frame of reference *R'* where it is at rest, is denoted L. The length in the laboratory is the contracted length L/γ. We take different pictures of the ruler as it passes. On each photograph, we note the apparent length L_a, the difference between the abscissas of the two ends marked on the bench.



The contracted length corresponds to measurements at the same instant *t* of the position of the ends, while the image of the ruler which appears on the photographic plate is formed by photons which reach the sensor at the same instant and which, due to time of different routes, were not emitted at the same time at the object level.

We do not yet know how to make a camera with such sensitivity and such a short shutter time, but it is not out of reach given current advances in opto-electronics. Second challenge, to animate a macroscopic object at a relativistic speed. The thinking exercise is excellent anyway, as it allows us to deepen our understanding of the theory.

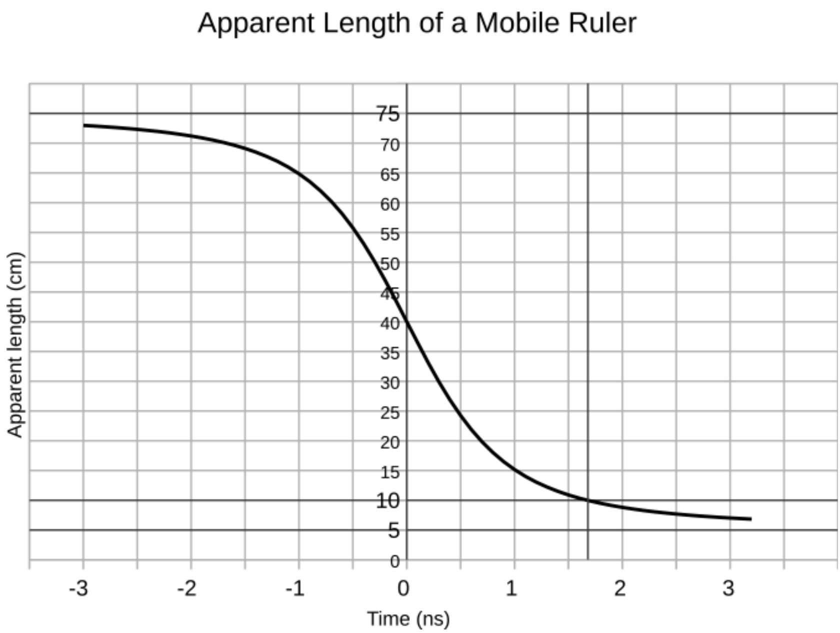
Let us think in the laboratory reference frame *R*. The ruler of length L/γ comes from the right. The light rays emitted at the same time from the *A* and *B* ends will not reach the eye at the same time and will therefore not be in the same image. The ray emitted by *B* will arrive late.

There is an earlier moment when the ray emitted by this end compensates this delay, it is the case of point *C* on the diagram. The apparent length is then greater than the contracted length.

When at $t=0$, the ruler is centered on *O*, the rays are emitted symmetrically and the apparent length is equal to the contracted length. This occurs for a photo taken at $t \simeq 1.7$ ns (light travel time from *D*, or *E*, to *M*).

For *t* positive, when the ruler moves away, the apparent length is instead smaller than the contracted length.

Below we have the curve of the apparent length versus time *t* :



We can easily find the extreme values. When the ruler is still far away, the delay of the light beam from C is about AC/*c*. Moreover the ruler moves at the speed *v*, so, to catch up, BC is worth *v* times the delay.

Then : $L_a = AB + BC = \frac{L}{\gamma} + v\frac{L_a}{c}$

So : $L_a = \frac{L}{\gamma(1-\beta)} = L\sqrt{\frac{1+\beta}{1-\beta}} \simeq 75\,cm$

On the contrary, when the ruler moves away, to the limit of t→+∞,

$$\frac{HF}{c}=\frac{L_a}{c}=\frac{L/\gamma-L_a}{\beta c}, \quad \text{and} \quad L_a=L\sqrt{\frac{1-\beta}{1+\beta}}\simeq 5\,cm.$$

We finally find the same kind of formula as for the Doppler effect with inverted effects :

$$L_{a,\,t\rightarrow\pm\infty}=L\sqrt{\frac{1\mp\beta}{1\pm\beta}}$$

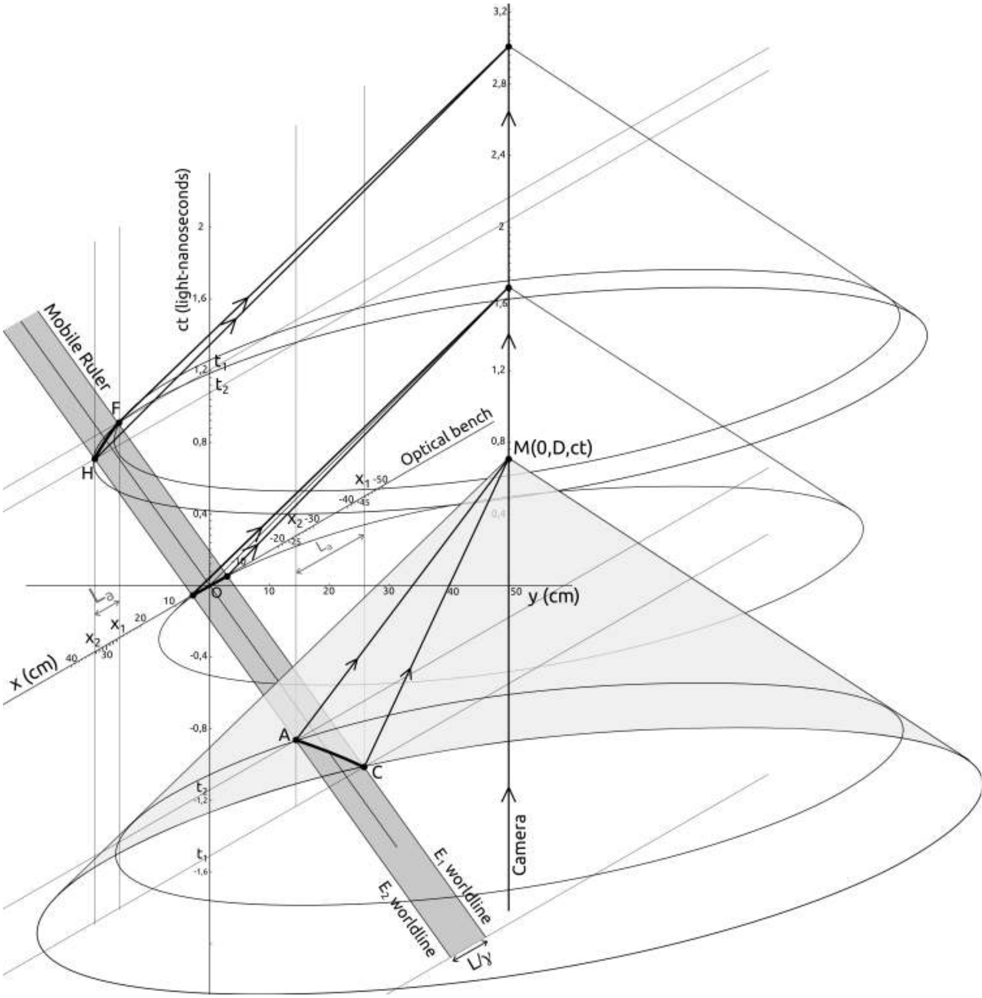
When an object gets closer, the perceived period is shorter and its length seen, in the direction of motion, is greater, on the contrary, when it moves away the perceived period is greater and its length seen smaller.

We also had an inversion of behavior between time and space with time dilation and length contraction.

We just did the long-distance calculations. To find the complete curve of the length on the photograph as a function of time, we consider a three-dimensional Minkowski diagram (x, y, ct).

The camera is represented by a vertical world line $(0, D, ct)$. The optical bench by the world plane $y=0$. The ruler by an inclined world strip. The resolution of the problem is in principle simple: find the intersection between the past light cone from the eye at time t with the world strip of the mobile ruler.

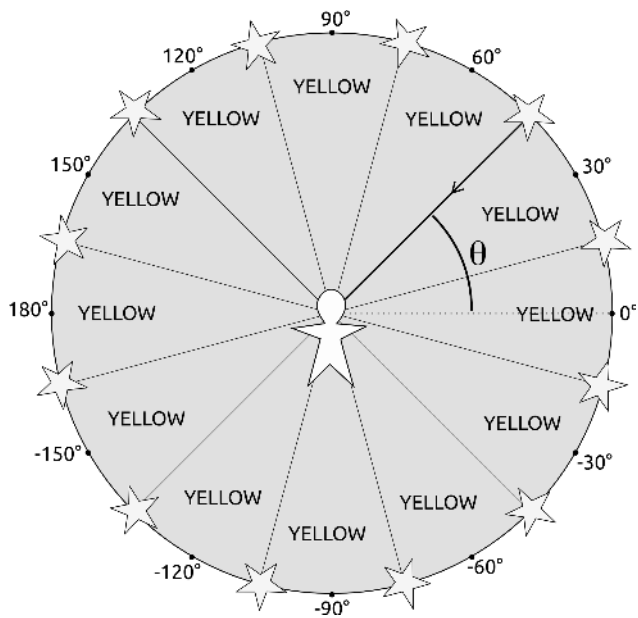
The intersection gives the position of the two ends in R : $E_1(x_1, 0, ct_1)$ and $E_2(x_2, 0, ct_2)$. We then have the apparent length $L_a=x_2 - x_1$. Except at O, we have well $t_1\neq t_2$.



The detailed calculation is left in exercise. The explicit expression $L_a(t)$ is then given. The computation, although it only uses notions of space geometry as seen in high school, is a bit long.

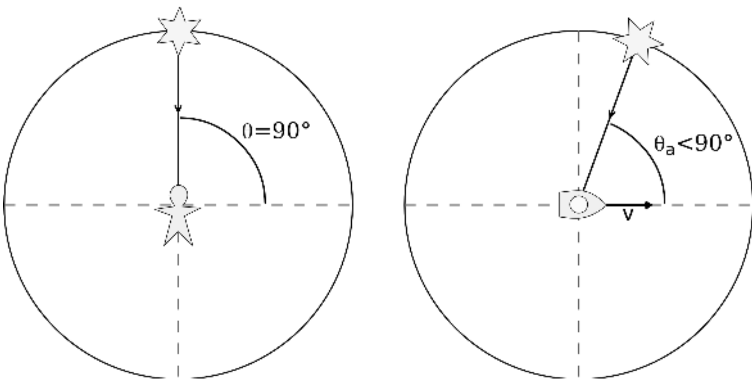
ꝏ THE STARRY SKY SEEN FROM THE SHIP

Let's determine the change in the perception of the starry sky as a function of the speed of the ship. In addition to the change in the perceived color of the stars by Doppler effect, their position in the sky is modified, this is called the aberration of light. When we are at rest in the galactic frame of reference, the stars are, as a whole, motionless. To simplify, we will consider yellow and homogeneously distributed stars.



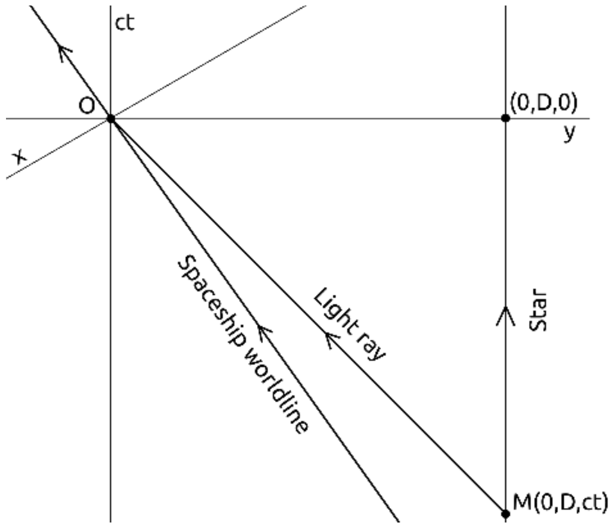
Let's take the case of a star seen at rest in the galactic frame of reference perpendicular to the direction of motion of the spacecraft. Under which angle θ_a is this same star seen from the ship's frame of reference?

We can make an analogy with the rain that falls, seen through the windshield of a car it looks like the rain comes from the front, even if from the road it falls vertically. The demonstration in classic mechanics is quite simple, just apply the addition of the velocities. You can imagine that here the result will be, at least quantitatively, different.



We have to think again in a three-dimensional Minkowski diagram *(x, y, ct)*. As soon as we measure an angle, there are at least two dimensions of space. However, there is no need to use the third dimension of space, because there is invariance by rotation according to the direction of the vessel, otherwise, in addition to colatitude θ, we would have had to use longitude φ and we would have had to work in a four-dimensional Minkowski diagram *(x, y, z, ct)*.

We consider the galactic frame of reference and we start by studying the case θ=90°. The ship is moving in the direction of increasing *x* and the star is located on the *y* axis at a distance D. We have three world lines, one for the spacecraft in the *(x, ct)* plane, one for the star, vertical, and one for the light ray in the *(y, ct)* plane.



We define a straight line by the intersection of two planes defined in Cartesian coordinates.

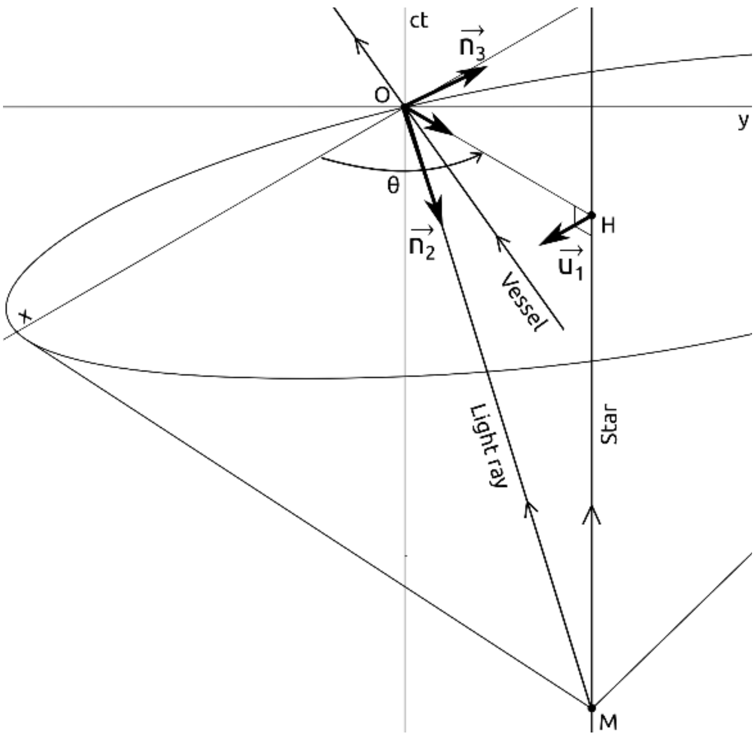
Light ray worldline : $\begin{cases} x=0 \\ y+ct=0 \end{cases}$

We then use the Lorentz transformation to obtain this equation in the ship's reference frame *R'* :

$$\begin{cases} x'=\gamma(x-\beta ct) \\ y'=y \\ ct'=\gamma(ct-\beta x) \end{cases} \quad \text{and} \quad \begin{cases} x'=\gamma\beta y' \\ \gamma y'+ct'=0 \end{cases}$$

also $\tan(\theta_a)=\dfrac{y'}{x'}$ then $\tan\theta_a=\dfrac{1}{\gamma\beta}$

In the case of a starship moving at 87% *c*, we find for $\theta=90°$, $\theta_a=30°$. We notice that the result does not depend on the distance D. The effect is accentuated with respect to the Newtonian formula where $\tan(\theta_a)=1/\beta$ and $\theta_a \simeq 49°$.
Now let's look for any angle θ between 0° and 180°.



A unit vector parallel to $\overrightarrow{OH}$ has for coordinates $(\cos\theta, \sin\theta, 0)$. The vector $\vec{u}_1$ orthogonal to the OHM plane has the coordinates $\vec{u}_1(\sin\theta, -\cos\theta, 0)$.
As collinear vector to the light beam we can choose $\vec{n}_2(\cos\theta, \sin\theta, -1)$. We verify that $\vec{n}_3(\cos\theta, \sin\theta, 1)$ is orthogonal to $\vec{u}_1$ and $\vec{n}_2$.

Hence the world line of the light ray:

$$\begin{cases} \sin\theta\, x - \cos\theta\, y = 0 \\ \cos\theta\, x + \sin\theta\, y + ct = 0 \end{cases}$$

Using the same Lorentz transformation as the one used in the previous case, we obtain, after a somewhat long but simple calculation:

$$y' = \frac{\sin\theta}{\gamma(\beta + \cos\theta)} x' .$$

Thus the expression for tan(θ_a), or, simpler to use, after some mathematical manipulations, detailed in exercise, the expression of tan($\theta_a/2$) :

$$\tan\left(\frac{\theta_a}{2}\right) = \sqrt{\frac{1-\beta}{1+\beta}} \tan\left(\frac{\theta}{2}\right)$$

For the color of the star, we give the expression of the wavelength perceived in the vessel which takes into account the transverse Doppler effect :

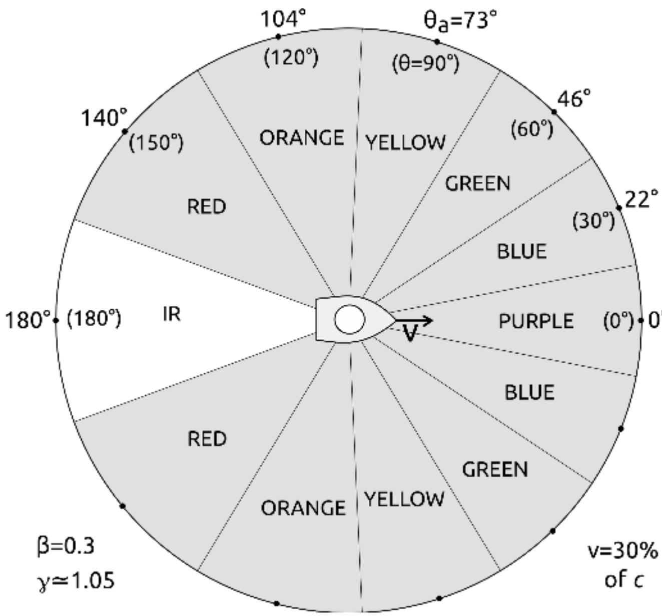
$$\lambda_a = \frac{1 - \beta\cos\theta_a}{\sqrt{1-\beta^2}} \lambda$$

For example, for β=0.3 and λ=600 nm, we have the results in the following table which we then reported in a circular diagram.

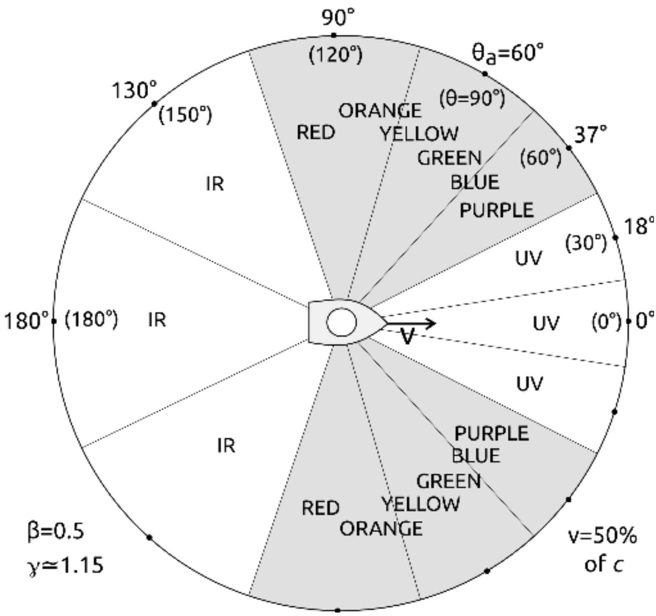
Angles in degrees and wavelengths in nm:

θ	180	165	150	135	120	105	90	75	60	45	30	15	0
θ_a	180	160	140	121	104	87	73	59	46	34	22	11	0
λ_a	818	806	773	726	673	621	572	531	498	472	454	444	440

As the ship gains speed, the stars in the front turn blue and those in the back red. Laterally we have all the spectral shades with an zone where the stars remain yellow. The forward hemisphere, under which we saw the stars at rest, is narrowing. Some stars present in the rear hemisphere appear in the front of the vessel, for example for θ=105°>90°, we have θ_a=87°<90°.



For even higher speeds, the stars fade in the front by passing in the UV, and in the back as they pass in the infrared. At 87% of *c*, only a visible ring is left in the front around 50°. However, new objects will appear, celestial objects in the infrared in the galactic frame of reference will be visible at the bow of the ship and objects in the UV will become visible at the stern.



From the galactic frame of reference, the light intensity received from the different parts of the sky is homogeneous. On the other hand, in the frame of the vessel, the overall energy received is greater and the luminosity dominates forward.

The energy received from the starry sky depends on the speed of the vessel according to two factors, light aberration and the Doppler effect. A star sees its position and its intensity change. The intensity of a star varies according to the following formula:

$$I_a = \frac{1-\beta^2}{(1-\beta\cos\theta_a)^2} I$$

The intensity corresponds to the energy received per m^2 and per second.

The energy comes from photons, of individual energy $e=h\nu_a=hc/\lambda_a$. Due to the Doppler effect, the photons see, on the one hand, individually, their frequency, and thus their energy modified, and on the other hand, taken as a whole, they arrive with a different rhythm. The two effects have the same Doppler factor $\sqrt{1-\beta^2}/(1-\beta\cos\theta_a)$, hence the square in the expression of I_a. The photons shoot more frequently and violently at the front, and more slowly and softly at the back.

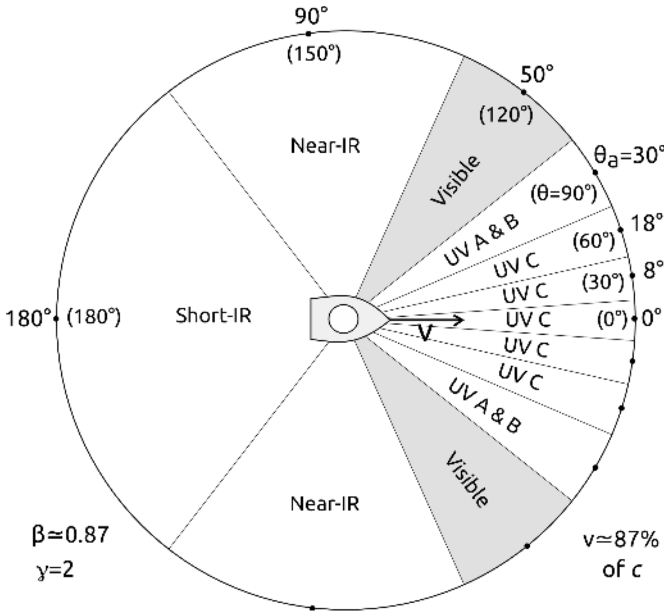
Now let's look at a group of stars, they occupy a certain area, also called a solid angle, on the celestial vault. As the ship speeds up one group of stars in the front tightens and another, in the rear, stretches. To calculate the total energy received, we must also take into account this density of stars which varies.

To find the total energy received, we integrate the light intensity on a spherical surface S of radius R, centered on the vessel. We have the following results, established in exercise:

$$E_a=\int_{\theta_a=0}^{\pi} I_a\, d\Omega_a=\gamma^2(1+\beta^2/3)E$$

with $$E=\int_{\theta=0}^{\pi} I\, d\Omega=4\pi I=E(\beta=0)$$

Ω is the solid angle, it corresponds by definition to the cut surface on a unit sphere, $\Omega=S_{\mathbb{1}}$, R=1.



To illustrate, at 30% of *c*, the frontal solid angle, of vertex angle 30° in *R*, reduces to 22° in *R'*, thus the apparent density of stars in this frontal part of the sky becomes 80% greater[7]. In addition, the photons received have a higher energy, from yellow they become blue, and moreover they are received in greater number.

At 50% of *c*, the stars become even more rare at the back, and 91% of the light energy comes from the front hemisphere.

At 95% of *c*, the sky is 13 times brighter.

Now what about the number of photons arriving on the ship? We have N photons which arrive on the ship during a proper time interval. From the galactic

7 ratio of the surfaces seen under the solid angles $\Omega=2\pi(1-\cos\theta)$.

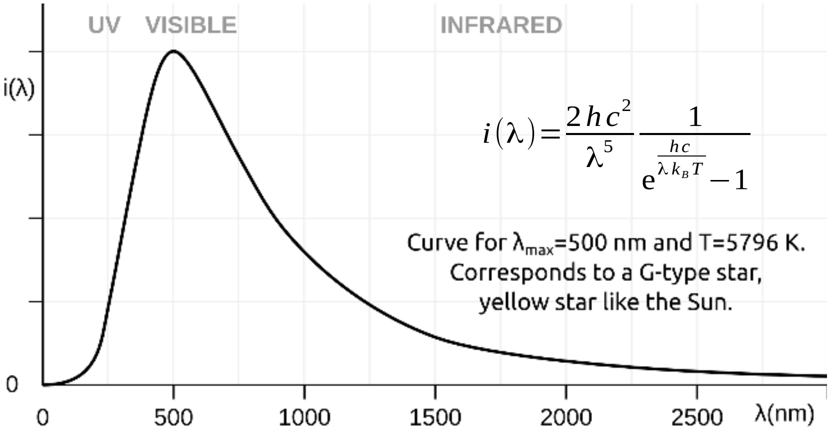
frame of reference, we observe these same photons arriving on the vessel during a relative dilated interval. Thus, the more the vessel gains speed, the more the number of photons received per second by the astronauts increases with the factor γ.

At 50% *c*, the vessel receives 15% more photons, and 84% of the photons come from the front hemisphere.

At 95% of *c*, the vessel receives 3 times more photons, the front celestial hemisphere is 26 times brighter, and the back hemisphere 350 times less. Now let's focus on a half-degree disk, which is the apparent size of the Moon or Sun as seen from Earth. This disc located at the zenith of the ship will have a luminosity 1500 times greater than that of the sky at rest. For comparison with what is observed from the Earth's ground, this luminosity is 40,000 times less than that of the Sun, and 12 times greater than that of the full Moon[8]. But beware, this central disk emits in the ultraviolet, the visible corona is located between 34 and 52°.

Of course the stars are not all the same color, the Sun is yellow, but Rigel is blue and Betelgeuse red. In addition, a star does not emit only one wavelength but a continuous spectrum given by what is called the spectrum of the black body:

8 Data: Starry sky 0.002 lux / Moon 0.25 / Sun 120,000 lux.



Thus stars of the Sun type, such as Alpha Centauri A or B, can be seen with the naked eye at the front of the ship even at 50% of c, because they also emit in the IR which shifts in the visible by Doppler effect, and, although the emitted intensity is lower in the IR, this is compensated by an increase in the perceived intensity towards the front. So, no navigation problem by finding your way in the starry sky to reach Proxima Centauri. On the other hand, towards the rear of the ship, the stars will fade much faster.

Regarding the energy and the number of total photons received the results do not change because they do not depend on the wavelength. The Doppler factor does not depend on λ and the aberration displaces all the chromatic components of a star's spectrum by the same angle. There is no dispersion, as in the phenomenon of refraction of light rays (through a prism the frequency components are deflected differently and create an iridescence in the form of a rainbow).

Exercises

01 10 : resolution by numerical simulation.

1. ▲△△ The suicidal physicist

A driver arrives at a crossroads and the traffic light is red. The driver, who is going crazy after reading a physics book, decides, instead of stopping, to increase his speed so that by Doppler effect, the light of the traffic light appears green to him.
What speed should his vehicle reach?
λ_{red}=700 nm, λ_{green}=500 nm

Answers p357.

2. ▲▲△ Laser sail

A Terajoule laser battery based on the ground bombs photons for 10 minutes on a sail placed in orbit. The sail reaches a speed of 20% of *c*.

a - What is the radiation pressure exerted on the sail depending on the light power Φ received?

b - For a constant luminous power incident on the sail in the terrestrial reference frame, will the force felt on the sail remain constant?
By what factor is it modified?

c - By what factor is the radiation pressure modified at the end of the acceleration phase?

Answers p357.

3. ▲▲△ Optical molasses

To slow down atoms and thus cool them we place two identical lasers face to face. If an atom placed between these two beams is stationary, it remains so, because the radiation pressures are in equilibrium.

a - Show that, for an atom moving along the axis of the lasers, a force appears that causes it to come to a standstill.

This force is similar to viscous friction, hence the name optical molasses for this phenomenon. Atomic clocks use optical molasses to cool atoms.

b - Show that, for low speeds compared to *c*, this force is analogous to the friction force of viscous fluids in laminar regime: $\vec{f}=-\alpha\vec{v}$.

The radiation pressure can be explained at the microscopic scale by the absorption then emission of a photon by the atom. The momentum of the atom is modified, in the direction of the laser during absorption and in a statistically isotropic manner during spontaneous emission. The atom is thus slowed down and confined. The resonance frequency of the atom is slightly higher than that of lasers.

c - As with viscous friction, we have an energy dissipation phenomenon. Explain qualitatively how the process of absorption/emission of a photon by the atom allows it to lose kinetic energy and thus to cool down.

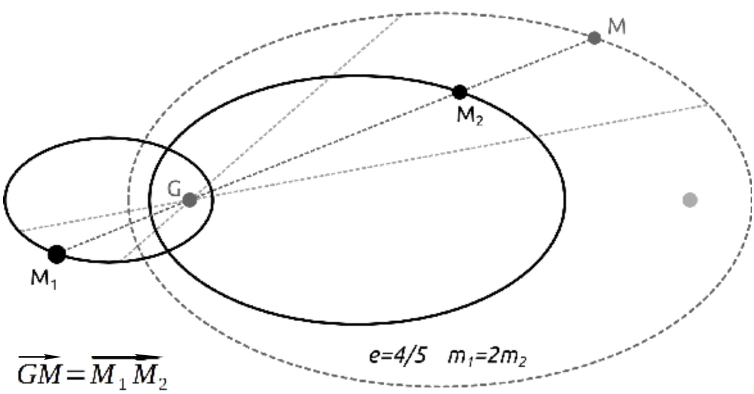
d - In the context of perfect gas, the mean kinetic energy of an atom is given by the relation $e=\frac{3}{2}k_B T$, where T is the temperature in Kelvin. Once slowed down, the atom will have a minimum kinetic energy of the order of the difference of energy between the absorbed photon and the photon emitted during de-excitation. The line width of the laser is very small compared to that of the atom, which predominates. In the extreme case, at rest, the line width of the atom is just below that of the laser. The distance between the two lines then corresponds to the width of the atomic line. The lifetime τ of the excited level of the atom is related to the energy difference by the Heisenberg uncertainty relation. From this an approximation of the temperature of the atom obtained by Doppler cooling can be deduced.
Numerical application: τ=27ns for a rubidium 87 atom.

e - Give the speed of an atom thus cooled.

Answers p358.

4. ▲△△ Detection of exoplanets by Doppler effect

A large number of exoplanets have been detected until now and their known number continues to increase. One method of detection, called Doppler method, or radial-velocity method, consists in observing the periodic variation of the wavelength of a star. The motion of the star is due to the presence of an exoplanet. When the star is moving towards us, and thus the planet backwards, the characteristic lines of its spectrum move towards blue, and when the star is moving away, towards red.



We consider a two-body system consisting of a star and a planet. The two masses are in a gravitational bound state. Let's conduct a Newtonian study. Each of the bodies revolves around the center of mass G of the system. We can fictitiously return to a problem with one body M of reduced mass μ which orbits around G, a fixed point of origin in the center-of-masse frame:

$\mu = \frac{m_1 m_2}{m_1 + m_2}$. Kepler's law for the fictive particle M:

$$\frac{a^3}{T^2} = \frac{\alpha}{4\pi^2 \mu} \quad \text{with} \quad \alpha = G m_1 m_2$$

a: semi-major axis of the ellipse traveled by M.
T: period of revolution around G.
We then find the trajectories of the two bodies M_1 and M_2 by applying the following homothetic factors:

$$\overrightarrow{GM}_1 = -\frac{m_2}{m_1 + m_2}\overrightarrow{GM} \quad \text{and} \quad \overrightarrow{GM}_2 = \frac{m_1}{m_1 + m_2}\overrightarrow{GM}$$

We will consider the cases of a two-body system with circular orbits and a plane of revolution that contains the long-distance observation site of the Doppler effect.
Let's take the example of a star slightly smaller than the Sun around which a giant Jupiter orbits. The Sun is a small star, a yellow dwarf, here we will take an orange dwarf of 0.8 solar mass. We will have a supermassive giant planet of 80 Jovian masses (this planet may be similar to a brown dwarf, not very luminous and not detectable by direct methods). The star in this case has a mass ten times greater than that of the planet. There are many stellar systems of this type: HD 87883, HD 4747, Epsilon Eridani, etc.

a - Determine the speed of the star on its orbit around the system's center of gravity. Show that this speed is indeed non-relativistic.

b - Give the classical limit of the Doppler effect formula.

c - What will be the relative wavelength variation $\Delta\lambda/\lambda$ of the light emitted by the orange dwarf observed from the Earth in its plane of revolution?

Data: $G=6.67\times10^{-11} N.m^2/kg^2$, $M_S=2\times10^{30}$ kg, $M_J=M_S/1000$, $d_{G\text{-}Planet}=540\times10^6$ km

Answers p359.

5.√√ ▲▲▲ Calculations for the moving ruler

We detail the calculations that allow us to find the exact expression of the apparent length of the moving ruler on the photographic plate as a function of time. We rely on the notations given in the course.

a - Determine the equations of the worldlines for the E_1 and E_2 ends of the ruler.

b - We seek to express the equation of the past cone of $M(0, D, ct_M)$. We consider the vector $\vec{u}=(a,b,1)$ with $\sqrt{a^2+b^2}=1$ and collinear with a generatrix line of the cone. Let be $C=(x,y,ct)$ a point of the cone.

We have two constraints, $\overrightarrow{MC}$ collinear to $\vec{u}$ and point C belongs to the ends of the worldsheet of the ruler. Deduce the apparent length L_a as a function of t.

Answers p360.

6.√ ▲▲△ Velocity transformation and aberration of the light

a - From the Lorentz transformation determine the three components of the velocity in *R'* as a function of those in *R*.

$\vec{v}=(v_x,v_y,v_z)$, $\vec{v}\,'=(v_x',v_y',v_z')$ and $\vec{\beta}=\frac{\vec{u}}{c}=\frac{u}{c}\vec{i}$

From the transformation of velocities we can quickly find the formula of the relativistic aberration of light which gives θ_a as a function of θ.

b - Give the components of the velocity of a photon that arrives in O at an angle θ with respect to Ox.

c - Give the expression of $\vec{v}\,'$ and check that we have $\vec{v}\,'\cdot\vec{v}\,'=c^2$.

d - Express $\tan\theta_a$ as a function of θ.

Answers p361.

7.√ ▲▲△ Composition of velocities and accelerations. 3D generalization

a - Two vessels move at 50% *c* and cross perpendicularly in O in R.
What is their relative speed?

b - In the general case of two vessels animated by velocities $\vec{v_1}$ and $\vec{v_2}$, one does not lose in generality by taking $\vec{i}$ co-directed with $\vec{v_1}$, $\vec{j}$ co-directed with $\vec{v_1}\wedge\vec{v_2}$ and $\vec{k}=\vec{i}\wedge\vec{j}$.

The angle between the velocities is $\theta=\widehat{(\vec{v}_1,\vec{v}_2)}$. Express the relative velocity v' as a function of v_1, v_2 and θ.
Numerical application for two vessels of $\gamma=2$ and trajectories that make an angle of 30°.

c - We continue the exercise *Two vessels* on page 71.
1 - Starting from the velocity $\vec{v}$ in *R*, find again, with the velocity transformation laws, the velocity v' of vessel *B*.
2 - Establish the law of transformation of accelerations in three dimensions. From the velocity $\vec{v}$ and acceleration $\vec{a}$ in *R*, find again the acceleration *a'* of the starship *B*.

Answers on page 362.

8.√ ▲▲△ Starry sky at the halfway point

We start our journey to Proxima Centauri with an acceleration of one g. As we will show in the next chapter the speed is then 95% of c at mid-course (after 2 ly traveled in the galactic frame of reference). We wonder if the Sun and Proxima Centauri are at that moment visible to the naked eye from the spacecraft. In astronomy we use the apparent magnitude to determine the brightness of a star. A star of magnitude greater than 6 is invisible to the naked eye. The star Vega is taken as a reference with a magnitude of zero. A star brighter than Vega has a negative magnitude.

Magnitude formula: $M=-2.5\log(L/L_0)$.

L and L_0 are the luminosities of the star and Vega perceived at the point of observation.
The luminosity L_V of Vega, which corresponds to the total power emitted, is expressed as a multiple of the luminosity L_S of the Sun: $L_V = 37L_S$.
Distance Vega-Sun: D_{VS}=25 ly.
For Proxima Centauri: $L_P = 5\times10^{-5}L_S$.
The perceived luminosity of a star decreases with distance, and is inversely proportional to the square of the distance.

a - Determine the apparent magnitude of the star Proxima Centauri from Earth. Is the star visible to the naked eye?

b - Determine the apparent magnitude of Proxima Centauri at midpoint if the spacecraft was motionless with respect to the stars. Would the star be visible to the naked eye?

c - Determine the apparent magnitude of Proxima Centauri at mid-course when the spacecraft is at 95% of *c*. Will the star be visible to the naked eye?

d - Determine the apparent magnitude of the Sun at mid-course if the spacecraft was stationary. Would the Sun then be visible to the naked eye?

e - Determine the apparent magnitude of the Sun at the halfway point when the spacecraft will be at 95% of c. Will the Sun then be visible to the naked eye?

f - Here you are on the exoplanet *Proxima b* orbiting the star Proxima Centauri. A well-deserved rest. Will you see the Sun in the night sky? *Answers p366.*

9. ▲▲△ **Numerical simulation of the sky**

In the analytical model of the course we have a continuous distribution of light energy to model the starry sky. Here we will have a discrete distribution of point stars. We will take N=10,000 stars, identical, monochromatic, and, randomly and uniformly distributed.
This numerical model allows us to better understand the perception of the sky from the moving vessel, to better understand the meaning of the integrals calculations and to verify them.

a - <u>Uniform spherical probability law</u>: We place stars on the celestial sphere using two angles θ, the colatitude, and φ, the longitude. These are the spherical coordinates. The positioning is analogous to the one used to find our bearings on the surface of the Earth. The colatitude is zero at the celestial North Pole, 90° at the celestial equator and 180° at the South Pole. The longitude is 0° at a meridian taken for origin and returns to it after a full 360° turn. Propose laws of probabilities Θ and Ψ which ensure a uniform distribution on the celestial sphere as a function of the continuous uniform law $\mathrm{U}(0,1)$[9].

b - We use a spreadsheet and the function that generates a random number between 0 and 1. On the first two columns we have N values for θ and for

9 For the laws of probability and their simulation, see, for example, the book *Probability, Statistics and Estimation*, by the same author, on pages 109 and 118.

φ. Then we calculate for the N stars θ_a and I_a with the formulas of the course. You can thus find the values, for a speed of 50% of *c*, of the energy and the total number of photons received with respect to rest.

Answers on p367.

10.√√ ▲▲▲ A bit of math...

To do physics in higher education you have to be comfortable with math and I prefer to put everything on the table in the same book to be clear and avoid multiple tergiversations. Nature is logical, logic is mathematical, so let's indulge in a little trigonometry.

According to the relation between *y'* and *x'* given page 88:

θ belong to $]0, \pi[$ and $\tan\theta_a = \dfrac{\sin\theta}{\gamma(\beta+\cos\theta)}$ if the denominator is positive. θ_a then belong to $]0, \pi/2[$ and in this case: $\beta+\cos\theta>0$ so $0\leq\theta<\theta_0$ with $\theta_0 = arcos(-\beta)$.

If the denominator is negative. θ_a belong to $]\pi/2, \pi[$ and in this case $\theta_0<\theta\leq\pi$ then:

$$\tan\left(\pi-\theta_a\right)=\frac{\sin\theta}{-\gamma(\beta+\cos\theta)}$$

This is very complicated. The tangent function is made up of an infinity of branches, and, therefore, for one value of the tangent there are an infinity of possible angles. A traditional calculator gives the value of the angle on the central branch on]-π/2, π/2[. Our star observation angle is between -π and π, and by symmetry we restrict to]0, π[. We are then on two branches of the tangent. To solve this thorny and exciting (!) problem we prefer to have tan(θ/2), because θ/2 belongs to]0, π/2[. We stay on the same central branch whose values are given by the calculators.

a - After recalling the expressions of cos(a+b) and sin(a+b) give the expression of tan(a+b) as a function of tan(a) and tan(b).

b - Deduce tan(θ) as a function of tan(θ/2).

c - Solve a quadratic equation to show that

$$\tan(\theta_a/2)=\sqrt{\frac{1-\beta}{1+\beta}}\tan(\theta/2).$$

Answers p368.

11.√√ ▲▲▲ Energy distribution

We establish here the formulas giving the energy received from the starry sky in the reference frame of the vessel as a function of β.

a - Use the relationship between θ_a and θ to express dθ as a function of $d\theta_a$ and θ_a. Deduce how the solid angle $d\Omega=2\pi\sin\theta\,d\theta$ transforms in the vessel's frame of reference. You will be able to express $d\Omega$ as a function of $d\Omega_a$ and θ_a. The factor gives us the star density *n* as a function of θ_a. Express this density at the stern and bow as a function of β, then make a numerical application for β=0.5.

b - Verify by integrating over the whole space that the number of stars remains well constant when the ship gains speed.

c - Find again the expression of E_a as a function of β of the course.

d - Determine how the energy is distributed between the front and back hemispheres of the vessel. Expression as a function of β, and numerical application for β=0.5.

Answers p369.

12.√√ ▲▲▲ Number of photons

The number of photons reaching the vessel every second is proportional to gamma. Within the framework of the model of yellow photons uniformly emitted in the galactic frame of reference, in the moving frame of reference, the photons are more numerous and of different frequencies. They are each time less numerous and of low energy towards the rear and each time more numerous and energetic towards the front.

a - By a complete integral calculation find the factor: $N_a/N=\gamma$. You can use symbolic computation software.

b - What proportion of photons is received from the front hemisphere? Calculation as a function of β, then numerical application for β=0.5.

Answers p371.

13.√ ▲▲▲ Power emitted by a star

To obtain the total power emitted, we integrate the luminance *i* on all wavelengths, solid angles and surfaces:

$$P=\int i(\lambda)\,d\lambda\,d\Omega\,dS$$

The expression of the luminance is given on page 93. For a black body, an infinitesimal area *dS* emits uniformly over a half-space, i.e. an integrated solid angle of 2π.

a - In the case of the Sun, do you find the known total emitted power of 4×10^{26} W? The surface temperature is taken equal to T_S=5000 K and the solar radius R_S=700 000 km. You can estimate the integral by a numerical integration.

b - How is the power emitted by the Sun divided between visible, infrared (>800 nm) and UV (<400 nm)?

c - For Proxima Centauri, we take T=3000K and R=0.14 R_S. We read on the Wikipedia page of Proxima Centauri that "*Its total luminosity over all wavelengths is 0.17% that of the Sun*". Does your calculation confirm this assertion?

Answers p372.

110

ACCELERATED MOTION

We have so far studied vessels in uniform rectilinear motion: an object animated at a constant speed and which moves along a straight line. For realistic interstellar travel the trajectory can remain rectilinear, but, on the other hand, the speed necessarily varies. We are going to be interested in uniformly accelerated rectilinear motion: the vessel has a constant acceleration, the speed constantly varies by the same amount. We can thus create an artificial gravity in the rocket: we will consider the case where the speed increases (or decreases) by 10 m/s every second.

STUDY OF AN ACCELERATED FRAME

The basic principles of special relativity are stated for inertial frames of reference. Once we have a starting inertial frame of reference, all frames of reference in uniform rectilinear translation with respect to it are also inertial frames of reference. A frame of reference accelerated with respect to a frame of inertia does not belong to this set, which does not prevent the application of special relativity indirectly if we know the motion of this reference frame with respect to an inertial reference frame, which we will name *R*. We proceed in the same way in Newtonian mechanics, the fundamental relationship of dynamics $\vec{F}=m\vec{a}$ is only valid in inertial frames of reference and therefore Newton's laws are used to study any type of motion in any type of frame of reference.

Classical mechanics is used to construct special relativity by using it as the limit of low speeds. In addition, the principle of additivity of the proper times on the particle worldline is added as a construction element. With this principle, we are not limited to inertial frames of reference: the particle proper frame of reference can have any motion (it is the clock hypothesis seen page 19).

Then $\tau=\int d\tau=\int\frac{dt}{\gamma}$ where τ is the proper time in the particle proper reference frame, t is the time in the inertial frame of reference and γ is expressed as a function of the instantaneous speed v of the particle in this same frame of reference.

At any time t there is always an inertial frame of reference named *R'* which coincides with the proper reference frame R_p. The frame *R'* has a constant velocity v with respect to *R* and, at the moment it coincides with the proper frame R_p, the particle has a zero velocity in *R'*. Its acceleration is *a'* and that in *R* is then $a=\frac{a'}{\gamma^3}$ (demonstrated page 69). This is where classical mechanics comes in, indeed, the particle then has a low speed in *R'* between t and

$t+dt$. It is like if an accelerated vessel passed a vessel moving at constant velocity. If at the moment they are at the same level their velocities are equal, their relative velocity is zero. The vessel accelerated by the thrust of its engines then moves away slowly with respect to the speed of light and we can use classical mechanics to study the motion of the accelerated vessel from the other vessel taken as a reference.

Let's take the example of a car that first stands still at a red traffic light and then accelerates to green. From the reference frame of the road, the acceleration of the mobile is $\vec{a}$, but what is the acceleration felt by the passenger in the proper reference frame of his car?

According to the classical acceleration transformation formula: $\vec{a}=\vec{a}_r+\vec{a}_e+\vec{a}_c$ where we have the absolute acceleration $\vec{a}$ in *R*, relative acceleration $\vec{a}_r$ in R_p, coincident acceleration $\vec{a}_e$ [10] and Coriolis acceleration $\vec{a}_c$.

Let's write Newton's second law in *R'* :

$\vec{F}=m(\vec{a}_r+\vec{a}_e+\vec{a}_c)$ and $m\vec{a}_r=\vec{F}+\vec{F}_{ie}+\vec{F}_{ic}$.

In an accelerated, non-Galilean frame of reference, we feel new forces, called inertial forces. Here the accelerations $\vec{a}_r$ and $\vec{a}_c$ are null because the passenger is motionless in his car. The driver feels a

10 Advanced remark: $\vec{a}_e=\vec{a}_R(C)$, $C(t=t_0)=M(t_0)$ and $\vec{v}_{R_p}(C)=\vec{0}$

$$\vec{a}_e=\vec{a}_R(O')+\frac{d\vec{\Omega}_{R_p/R}}{dt}\wedge\overrightarrow{O'M}+\vec{\Omega}_{R_p/R}\wedge(\vec{\Omega}_{R_p/R}\wedge\overrightarrow{O'M})$$

The coincident point C coincide instantaneously with M. For a non-rotating frame $\vec{a}_e=\vec{a}_R(O')$. For a uniformly rotating frame we obtain the centrifugal acceleration. *e* for *entraînement* in French.

inertial force $\vec{F}_{ie}=-m\vec{a}_e$ that pushes him to the bottom of his seat when starting. This is due to the inertial acceleration which equals that of the car: $\vec{a}=\vec{a}_e$. For the same reason, the acceleration felt by the particle in its proper frame also worth $\vec{a}\,'$, acceleration of the particle in *R'*.

ꝏ ARTIFICIAL GRAVITY

When the car accelerates at the green traffic light, it is as if a force exerted at a distance pulls the driver towards the rear of the car. Like a non-contact force, analogous in these effects to a gravitational force due to a mass placed at a distance at the back of the car. When a spaceship starts at the green traffic light at an interstellar crossroads, the passengers first in weightlessness are then pressed during the acceleration phase to the walls perpendicular to the displacement. In our case the acceleration is maintained and the vessel has a *uniformly accelerated rectilinear motion*.

The acceleration is equal to the Earth's surface gravity g, thus:

$$a=\frac{dv}{dt}=\frac{g}{\gamma^3} \quad \text{and} \quad \tau=\int\frac{\gamma^2}{g}dv=\frac{c}{g}\int\frac{d\beta}{1-\beta^2}$$

then $\tau=\frac{c}{g}\int_0^\beta\left(\frac{1/2}{1-\beta}+\frac{1/2}{1+\beta}\right)d\beta$ and $\boxed{\tau=\frac{c}{2g}\ln\left(\frac{1+\beta}{1-\beta}\right)}$

where $v=\beta c$ is the speed reached in *R* after a proper duration τ.

Let us express the distance x traveled in *R* as a function of $\mathbf{v}$:

$$v = \frac{dx}{dt} \quad \text{then} \quad x = \int dx = \int \frac{\gamma^3}{g} v\, dv = \frac{c^2}{g} \int \frac{\beta}{\left(1-\beta^2\right)^{3/2}} d\beta$$

and after integration: $\boxed{x = \frac{c^2}{g}\left(\frac{1}{\sqrt{1-\beta^2}} - 1\right)}$

Let's calculate the galactic time t :

$$t = \int dt = \int \frac{\gamma^3}{g} dv = \frac{c}{g} \int \frac{1}{\left(1-\beta^2\right)^{3/2}} d\beta$$

We perform the change of variable

$\beta = \sin\theta$ and we find:

$$\boxed{t = \frac{c}{g} \frac{\beta}{\sqrt{1-\beta^2}}}$$

We can now express the position, speed and acceleration as a function of time t :

$$\begin{cases} x=\dfrac{c^2}{g}\left[\sqrt{1+\dfrac{g^2t^2}{c^2}}-1\right] \\ v=\dfrac{c}{\sqrt{1+\dfrac{c^2}{g^2t^2}}} \quad \text{and} \quad \gamma=\sqrt{1+\dfrac{g^2t^2}{c^2}} \\ a=\dfrac{g}{\left(1+\dfrac{g^2t^2}{c^2}\right)^{3/2}}=\dfrac{g}{\gamma^3} \end{cases}$$

We can also express the proper time τ as a function of galactic time t :

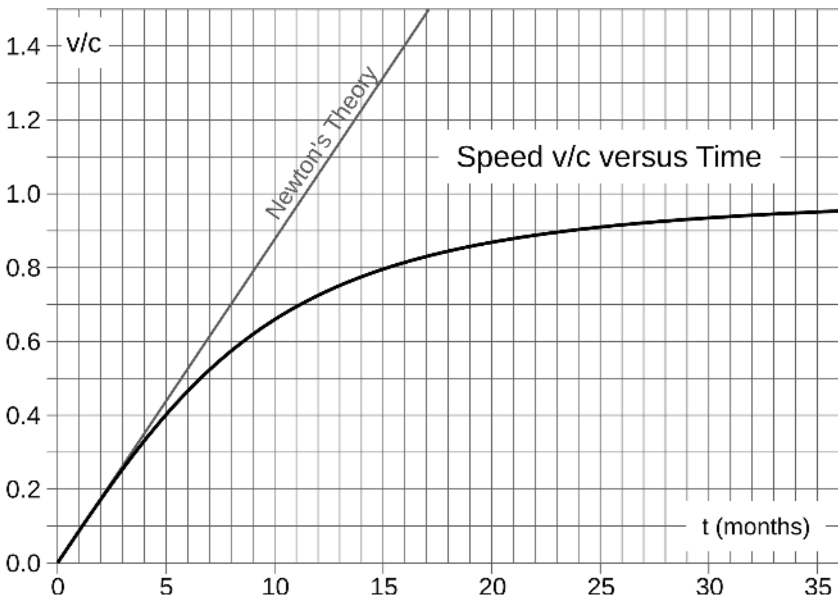
$$t=\frac{c}{g}\gamma\beta \quad \text{then} \quad \tau=\frac{c}{g}\ln\left(\sqrt{1+\frac{g^2t^2}{c^2}}+\frac{g\,t}{c}\right)$$

$$\text{and} \quad \tau=\frac{c}{g}\,argsh\left(\frac{g\,t}{c}\right)$$

$$t=\frac{c}{g}sh\left(\frac{g\,\tau}{c}\right) \qquad x=\frac{c^2}{g}\left[ch\left(\frac{g\,\tau}{c}\right)-1\right]$$

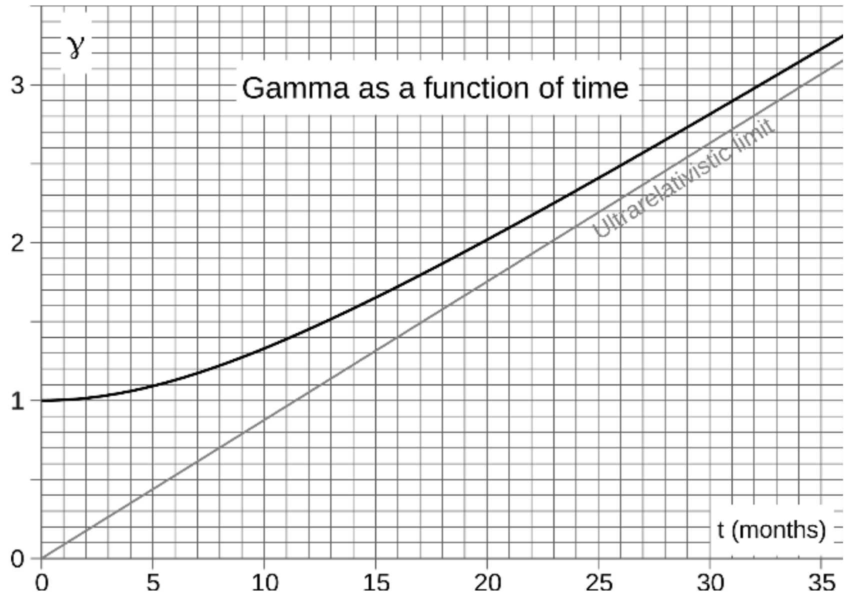
$$(c\,t)^2-\left(x+\frac{c^2}{g}\right)^2=\left(\frac{c^2}{g}\right)^2$$

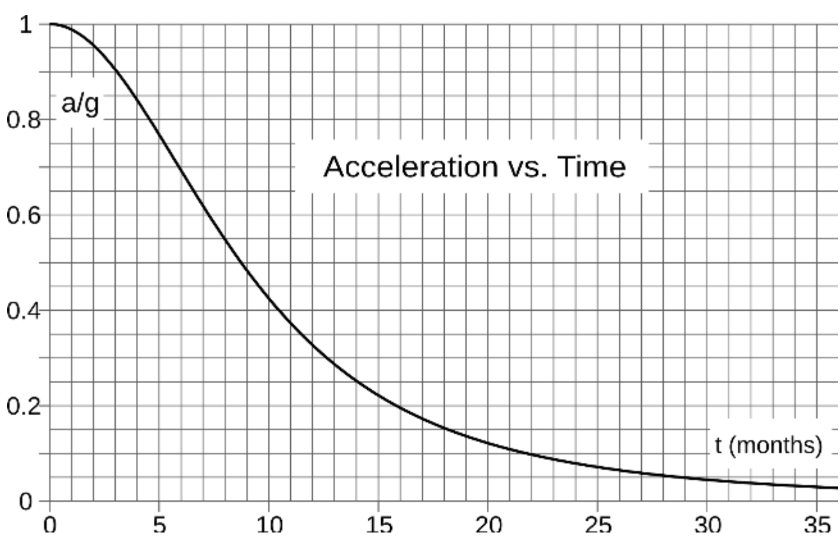
Curves :



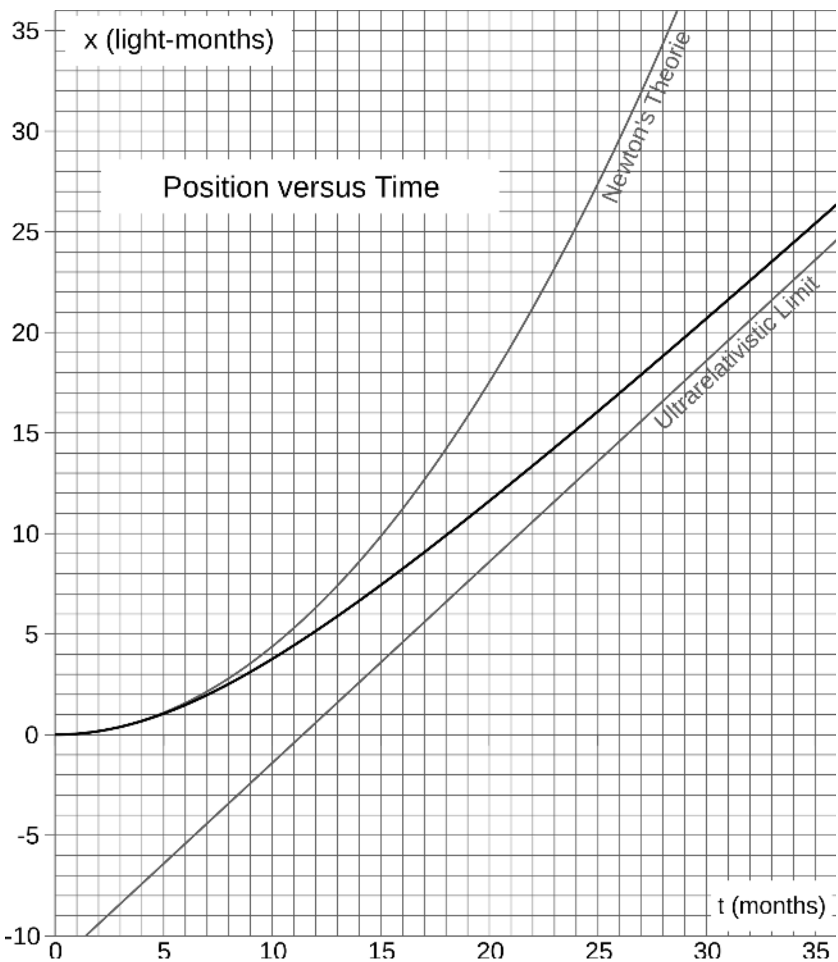
The speed tends towards the maximum speed *c*. For low speeds, the speed increases linearly with time, we find the classic limit $v = gt$.

Next page, the variation of the temporal dilation factor as a function of galactic time. We have a horizontal tangent at low speeds. When the speed increases, we tend towards the ultrarelativistic asymptote $\gamma \sim gt/c$, γ then varies linearly with galactic time.


γ
Gamma as a function of time
Ultrarelativistic limit
t (months)
0
1
2
3
0
5
10
15
20
25
30
35

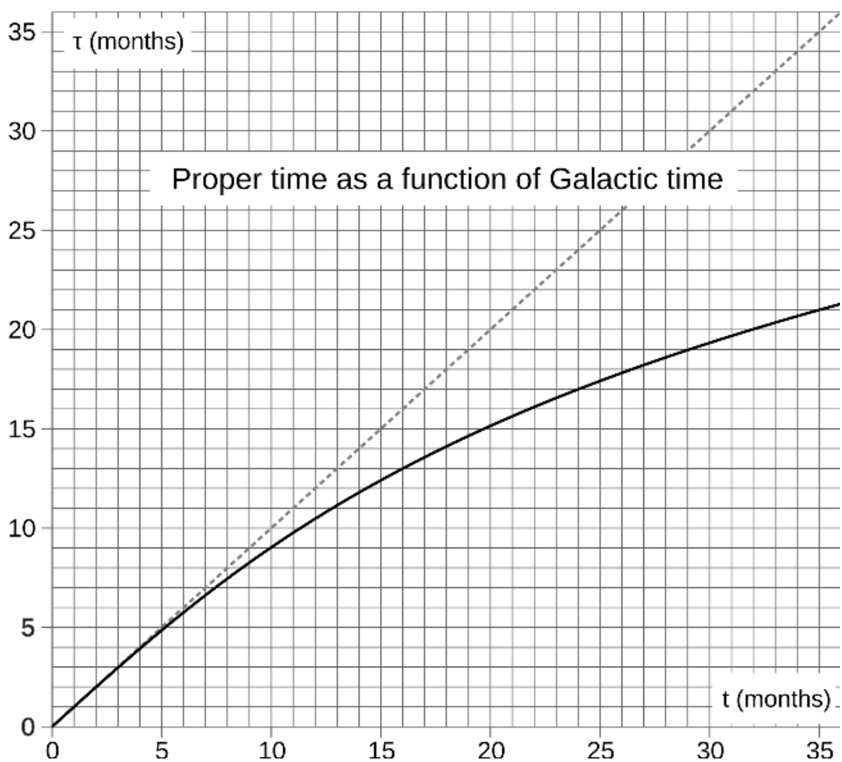

a/g
Acceleration vs. Time
t (months)
0
0.2
0.4
0.6
0.8
1
0
5
10
15
20
25
30
35

Previous page, the acceleration of the ship seen from the starting frame of reference. Although the acceleration remains constant in the proper frame, observed from the Earth, the speed reaches a ceiling and the acceleration decreases in gamma cubed. We have a horizontal tangent at low speeds, a zero infinity limit, and an inflection point at $t = c/2g$.



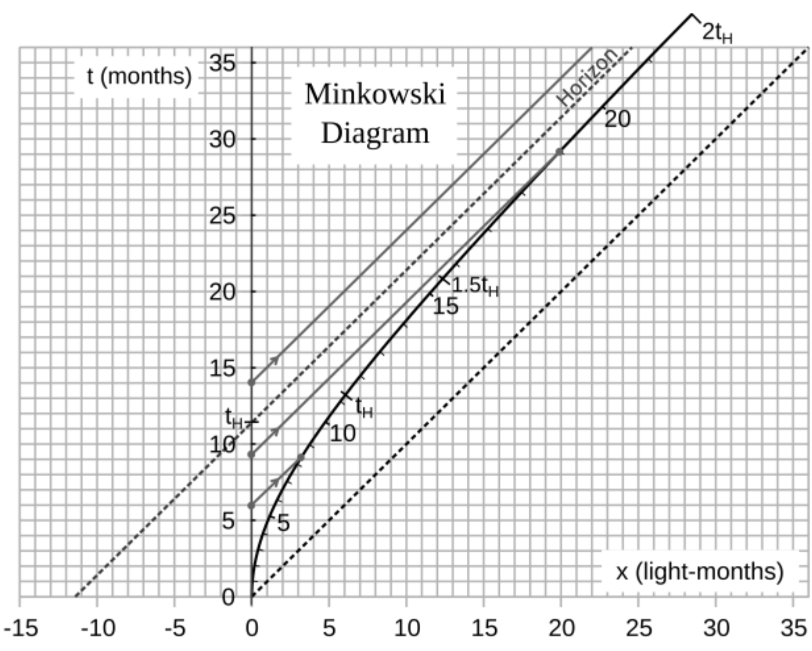
Previous page, we see, after 6 months, the position move away from the forecasts of classical mechanics. In Newton's theory we had a parabolic branch while in the context of special relativity we have a hyperbolic branch with an ultrarelativistic asymptote $x = ct - c^2/g$ where the galactic distance traveled increases linearly with time.

Below, the traveler's time accelerated according to that of those who remained on Earth:



Horizon concept:

We get the Minkowski diagram by simply reversing the x and t axes. We find that the asymptote $t = x/c + c/g$ represents a horizon. For terrestrial observers, it is impossible to communicate with the vessel after a period of time $t_{\text{lim}} = c/g$ (approximately 11.4 months). Indeed, after this period, a photon will no longer be able to reach the vessel. On the other hand, the occupants of the accelerated vessel will be able to continue to send us messages throughout their journey. They will also be able to permanently receive messages from Earth, but they will be earlier than t_{lim}.



As the proper time τ increases, the astronauts see the inhabitants of the Earth slow down their motions and freeze at the time limit $t_{\lim}$.

ꝏ Round trip

We want to join an exoplanet at a distance D from our planet Earth. We will be under artificial gravity for the entire round trip. We accelerate the first half of the trip and then, after turning the ship around, decelerate to the exoplanet. We repeat the reverse procedure for the return.

First phase: $\frac{D}{2}=\frac{c^2}{g}\left(\frac{1}{\sqrt{1-\beta_{max}^2}}-1\right)$

Maximum speed halfway:

$$\beta_{max}=\sqrt{1-\frac{1}{\left(1+\frac{g\,D}{2\,c^2}\right)^2}}$$

(for D=4 light-years, $\beta_{max}\simeq 95\%$ and $\gamma\simeq 3$)

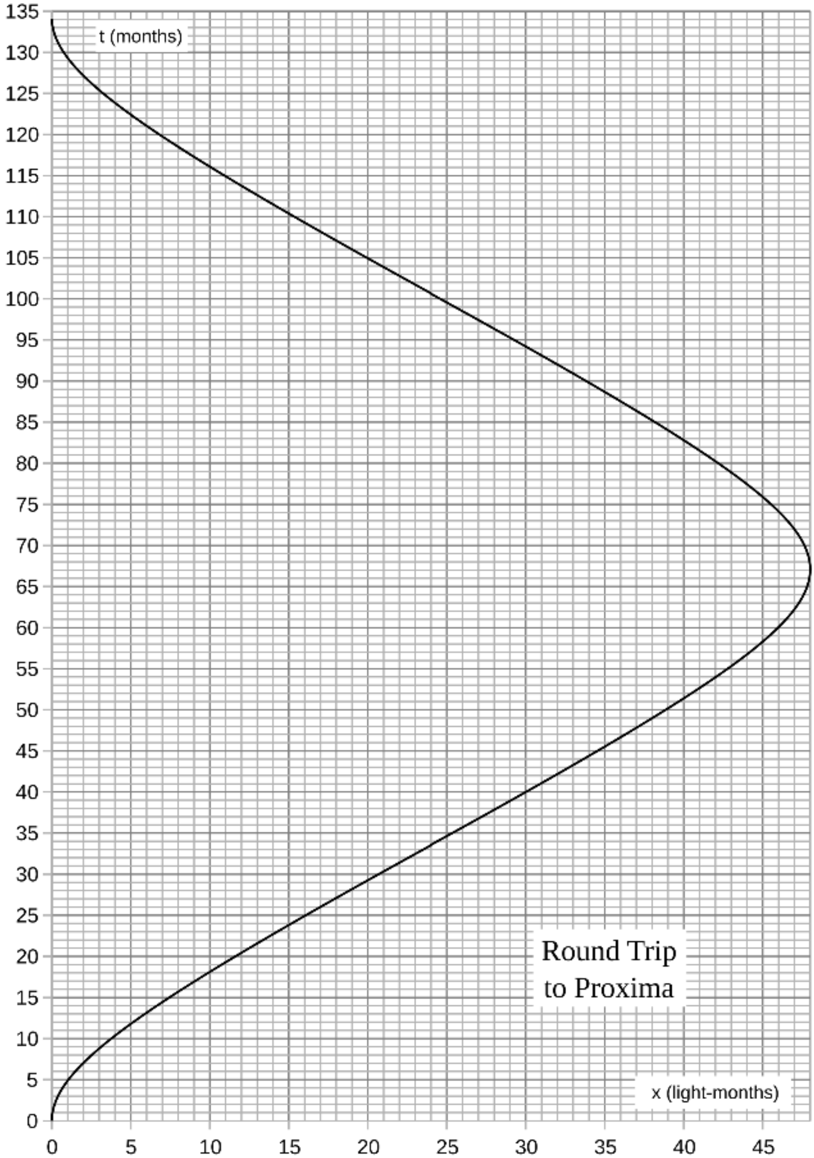
Duration T for the round trip:

$$\frac{T}{4}=\frac{c}{g}\frac{\beta_{max}}{\sqrt{1-\beta_{max}^2}} \quad \text{and} \quad T=\frac{4\,c}{g}\sqrt{\left(1+\frac{g\,D}{2\,c^2}\right)^2-1}$$

Proper time τ for the round trip:

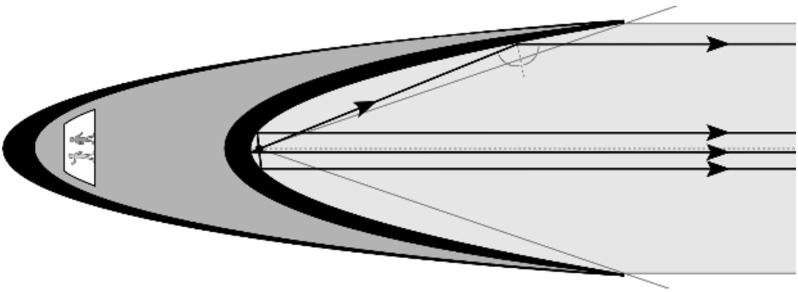
$$\tau=\frac{2c}{g}\ln\left(\frac{1+\beta_{max}}{1-\beta_{max}}\right)=\frac{4c}{g}\,argth\,\beta_{max}$$

(for D=4 l.y., T$\simeq$11.2 years and $\tau\simeq$6.84 years)



Photon rocket:

A light beam, created by the rocket, propels it by reaction. For example, matter and antimatter, in equal parts, are placed at the focus of a parabolic mirror, and, by annihilation, produce pure energy projected backwards in a parallel beam.



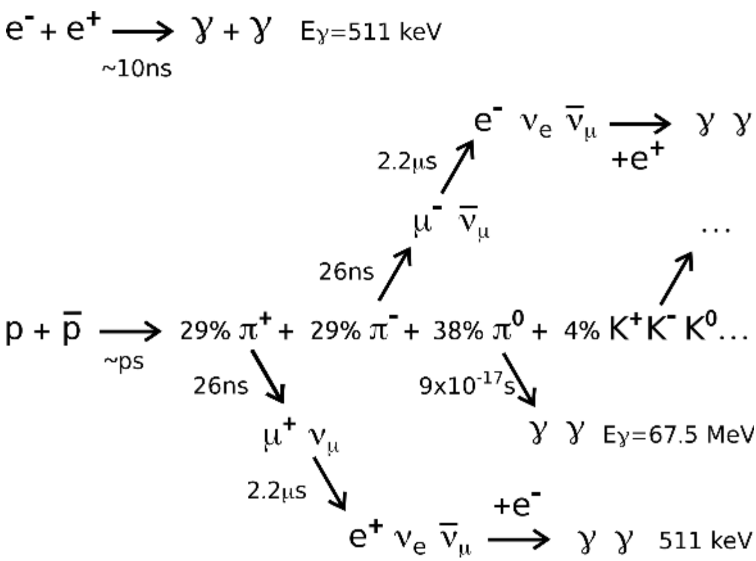
Consider the following case, a particle and its antiparticle meet and create two photons which go in opposite directions. One goes backwards and the other forwards. The backward one does not contribute to the propulsion, on the other hand, the second one contributes doubly, because the reflection on the mirror is supposed to be perfect. On average, each photon transfers its impulse to the rocket. Ultra-relativistic particles are just as efficient as their mass energy converted into light.

More realistically, a photon is sometimes absorbed by the gamma *shield*. The efficiency is then 50%. Also, part of the energy of the absorbed *gamma rays* can be reused to heat a gas to a very high temperature. The thermal agitation generates a very important ejection speed[11].

On the contrary, if a neutrino is created by the reaction, it carries away energy that is lost for propulsion.

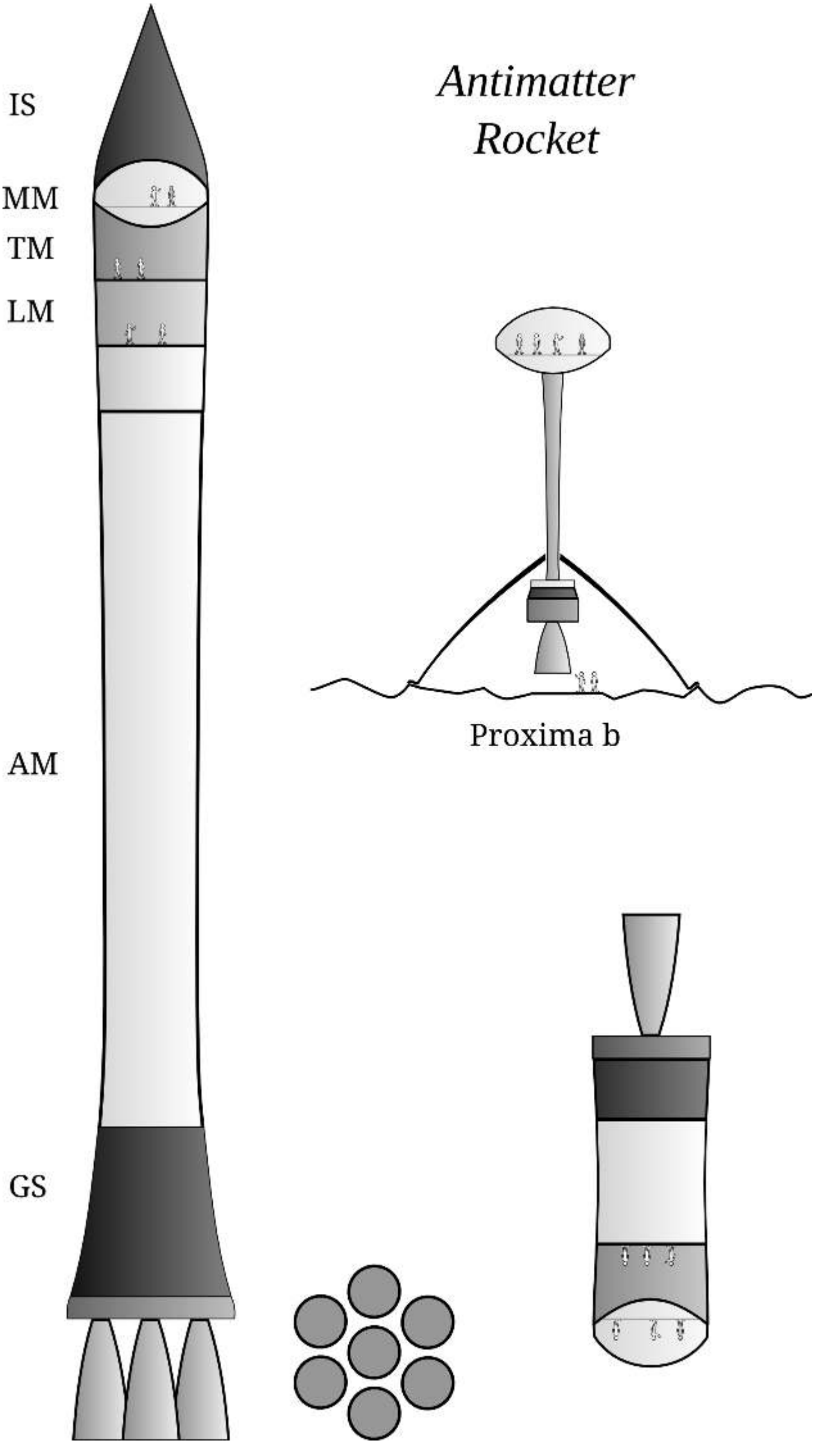
The photon rocket is close to the perfect model, we can otherwise talk about an antimatter rocket.

Annihilation reactions



Proton-antiproton annihilation is more complex and creates cascades of particles. γ photons, even more energetic than for electron-positron annihilation, are created.

11 NASA proposes a rocket propelled by a positron reactor. These are annihilated with electrons in gamma photons. The heat produced heats liquid hydrogen. *www.nasa.gov*


Antimatter
Rocket
IS
MM
TM
LM
AM
GS
Proxima b

Technical data :

Travel To Proxima Centauri / Distance 4.2 ly.
Traveler duration 3.3 years - Galactic 5.5 years.

Astronauts: 6.
Pressurized module: 3 / 10t / 6mx10m
Main Module - Technical Module - Leisure Module

Total height 126m / Diameter 15m / Total mass 2420t / Payload 20t / Antimatter mass 1200t.

Antimatter: Proximium / Density 0.2 / 200 kg/m^3.
Matter: Everything, except the payload, is progressively annihilated with the Proximium (shields, motors, etc).

Acceleration max 3 g / Speed max 89 % of c / γ_{max} 2.2 /
Periods Acceleration: a_{avg} 2 g, sleep 2.8 g
Periods Speed: a_{avg} 0.3 g, sleep zero g.

Interstellar shield: 140t / Protects from the interstellar medium 0.6 proton/cm^3 / vertex angle 38° / T_{max} 498°C.
This shield is used on the first half of the course. After turning over, the motors are forward, and the radiation pressure pushes the interstellar medium away.

Gamma shield: 860t / Protects passengers and Proximium from the rays γ emitted by the motors / Armoring Pb of 20 cm, or concrete 1.2 m, reduces the flux by a factor 10^6.

Rocket motor: efficiency 50 % / 1st phase 7 M P-2 / Thrust 10 MN / $D_{e\ 1g\ max}$ 11 g/s Proximium / 2nd 1 M P-2 / 3rd 1 M P-1 Thrust 2 MN / v_e=150 000 km/s.

Comparison :
Saturn V / M=3038t / H=111m / D=10m / $M_{propellant}$=2829t / P_{max} 34 MN / 1st stage 5 Motors F-1 v_e=2.6 km/s D_e=13.6 t/s Kerosene~O_2(l) / 2nd 5 M J-2 / 3rd 1 M J-2 v_e=4.1 km/s H_2(l)~O_2(l) / Duration 11 min 30 s from 0 to 164 km.

Exercises

1. ▲△△ Half-time

Leaving Earth, the ship reaches Proxima in a uniformly accelerated motion in two steps: the rocket cuts off its engines halfway through the journey, giving it time to turn around, and then arrives at Proxima at zero speed.
Compared to the stars considered fixed, what will be the distance traveled at half the time elapsed before the turning point? Is the result modified according to whether one considers the time of a fixed observer with respect to the stars, or that of a fixed observer with respect to the rocket? What about classical mechanics ?
We take, as usual, the following values :
$D=4$ al, $a=g=10\ m/s^2$ and $c=3\times10^8 m/s$.

Answers on page 374.

2. ▲▲△ Reality show

On January 1, 2100 at 12:00 noon, the crew of the Galaxys spaceship leaves at constant acceleration for the other end of the Milky Way.
Every day on Earth a reality show tells the adventures of the astronauts. And conversely, the astronauts also produce a daily program with the information received from the Earth during a proper

day on the spaceship. But due to time dilations, during a day on Earth we don't receive the news of a whole day lived on board the spacecraft, and vice versa. Light signals are used to transmit information.

a - Preamble: Determine the expression of position x as a function of γ, and that of γ as a function of τ.

b - Reality TV programs on Earth :

1- Let t_{obs} be the instant when the message corresponding to a proper time τ is received (the instant t is simultaneous to τ in the galactic reference frame, but the reception of the message due to the finite speed propagation of the wave is of course later). Illustrate the situation on a Minkowski diagram using the different worldlines (Earth / Ship / Photons).

2- Express t_{obs} as a function of τ, and τ as a function of t_{obs}.

3- Six months after their departure the astronauts send a message to Earth. How long after departure is the message received on Earth?

4- One year after departure, the daily reality shows will correspond to how much time spent in the spacecraft? Same question ten years after departure.

c - Reality show in the vessel:

1- Let τ_{obs} be the instant when the message corresponding to a terrestrial time t is received. Illustrate on a Minkowski diagram.

2- Express τ_{obs} as a function of t.

3- Six months after departure a message is sent to the astronauts. How long after their departure do they receive it?

4- One year after departure, the daily reality TV shows will correspond to how much time spent on Earth? Same question ten years after departure.

d - Doppler effect for an accelerating frame :
Both from the Earth and from the spacecraft a blue light signal is regularly emitted (λ=400 nm).

1- Establish the relations between the emitted frequency and the received frequency for the two reference frames, the inertial and the accelerated one.

2- After how long will the signal emitted from the Earth be perceived as red on board the vessel (λ=800 nm) ?

3- For the same time elapsed on Earth, what will be the color of the light signal received?

4- Is the Doppler effect symmetrical as in the case of inertial reference frames?

Answers on page 375.

3. ▲▲△ Head-to-head

Two vessels are traveling in opposite directions, at the same time and under the same conditions, the routes from Earth to Proxima and Proxima to Earth.

The rockets are animated with uniformly accelerated motions and complete the journey as described in this chapter.

a - Halfway, at the equidistant point, two light-years away, the ships shut down their engines to turn around. What is the galactic speed of the ships? What is their relative speed?

b - Same questions a quarter of the way.

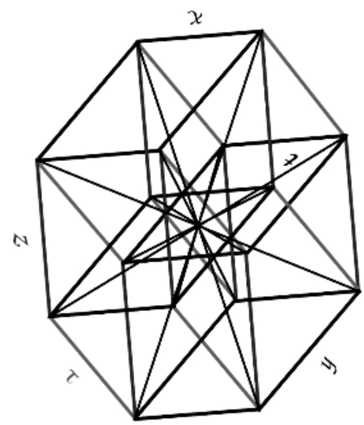
c - Propose a technical means that would allow the ships to measure their relative speed.

d - Express the galactic speed v as a function of the proper time τ.

e - Express the relative speed v_r as a function of τ.

f - Determine the acceleration a_r of the spacecraft coming from Proxima from the point of view of the reference frame of the spacecraft coming from Earth as a function of τ.
Determine this relative acceleration at the start, halfway and a quarter of the time of the spacemen's outward journey.
Is the relative motion of the spacecrafts uniformly accelerated?
What results would we find in Newtonian mechanics?

Answers on page 379.


x
t
z
τ
y

6

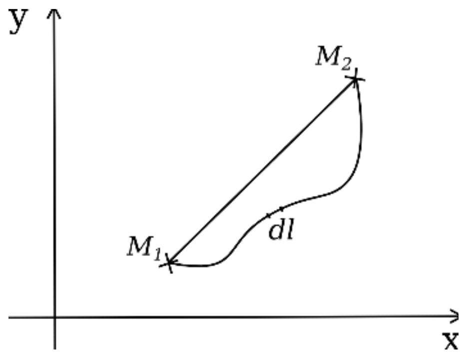
METRIC

A metric is used to measure distances. In relativity, the tool is generalized to space-time. We will give the metrics of the inertial frame of reference, of the uniformly accelerated frame in rectilinear translation, and of the uniformly rotating frame. We will then be able to determine the spacetime structure in our spaceship on its way to Proxima. What will be the geometric properties in the vessel? How does time flow at the different stages of the rocket?
Finally, we will make a parallel with the black hole metric and thus build a bridge to general relativity.
To answer these questions we will introduce the concept of metrics through various examples.

EUCLIDEAN METRIC

We measure the distance between two points. The metric can be expressed in different coordinate systems to calculate a distance, which is invariant. Let us take the case of two points M_1 and M_2 on a plane. If the coordinates of the points are Cartesian, $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$, the distance is given by:

$$L = d_{M_1 M_2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

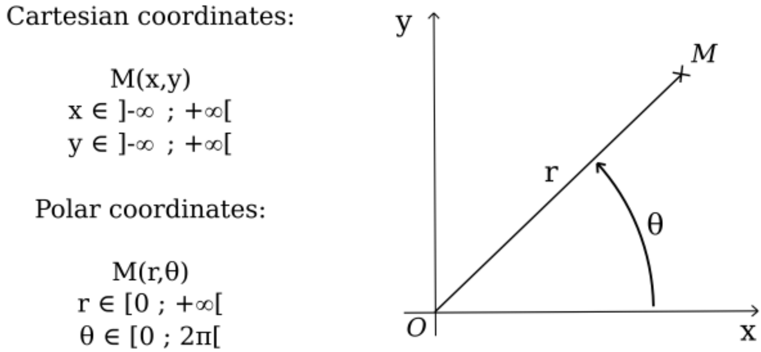


We can also determine the length of a curved path taken by a particle by integrating between the two points:

$$L = d_{M_1 M_2} = \int_{M_1}^{M_2} d\,l \quad \text{with} \quad d\,l^2 = dx^2 + dy^2$$

This element $d\,l^2$ is our metric for this example.

In the case where our physical problem has a central symmetry (common case, as for the motion of planets), the polar coordinates may be better adapted. We will have the same final result, but, in one case the computation can be very long, and in the other, very short. In polar coordinates these same points have the coordinates $M_1(r_1, \theta_1)$, $M_2(r_2, \theta_2)$ and $d\,l^2 = dr^2 + (r\,d\theta)^2$. With $x = r\cos\theta$ and $y = r\sin\theta$, we find the Cartesian metric, the steps are well equivalent .



In Euclidean geometry the length of an object (like the duration of a phenomenon) is the same for all observers. Whether one carries out a translation, a rotation, or a Galilean transformation of the coordinates, the length L is invariant (done in exercise on page 159).

More generally, the laws of Newtonian mechanics are invariant according to these transformations.

This is not the case for a dilation: if $x'=kx$, $y'=ky$ and $z'=kz$ with k the dilation factor, then, $dl'^2=dx'^2+dy'^2+dz'^2$, $dl'=k\,dl$ and $L'=kL$. The laws of physics depend on the scale, they are not the same for the infinitely small and the infinitely large.

The straight line is the shortest path between two points. We can take a rope and pull it to get a straight line. It is the path between M_1 and M_2 which minimizes L.

The Euclidean metric corresponds to a flat space: The sum of the angles of a triangle is equal to 180°, the ratio between the perimeter and the diameter of a circle is equal to π, and every straight line has a single parallel line passing through a point outside it.

ꝏ METRIC ON THE SPHERE

To better illustrate our point, let us take the case of a two-dimensional spherical space. You have to put yourself in the place of two-dimensional beings (the bidiz) who live on the surface of the sphere and are unaware of the third dimension. Euclid's postulates are no longer verified. We have simple counter-examples:

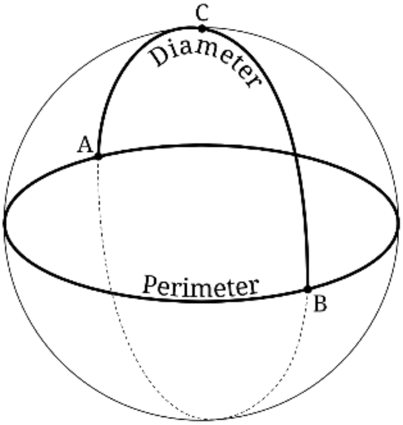
- To draw a circle, we fix a point, we attach a rope to it, and, with a tight rope, we turn around to trace it. The circle centered on the north pole and perimeter of the equator has a perimeter/diameter ratio equal to 2, a value much less than π.

- Now let's construct a particular triangle: we have a first point at the north pole, we get a second point by joining along a straight line the equator, we turn at right angles to the east and we then follow the equator for a quarter turn, we turn at right angles to the north, and we return to the north pole to finish the triangle. We have an equilateral triangle and all three angles are right. The sum of the angles of this triangle is 270°, a value much greater than 180°.

- Imagine yourself living on the surface of this sphere. You want to go on an adventure and discover unknown lands. You are unaware of the curvature of your 2D space, you go in a straight line, deviating neither to the right nor to the left, and finally you end up reaching your starting point from the opposite side! This is very disconcerting. The straight lines of the sphere are circles of the same radius as the sphere (the largest circles that can be drawn). For example, the line of the equator, a meridian, are straight lines for the sphere. You cannot draw parallel straight lines because they intersect. A latitude forms a circle with a radius less than that of the sphere, it is not a straight line: an airplane, to reach two cities at the same latitude, does not follow a latitude because it is not the shortest path .

We can clearly see, on these three examples, that the space on the surface of a sphere is not Euclidean. It is not a flat space but a curved space.

Geometry of the Sphere

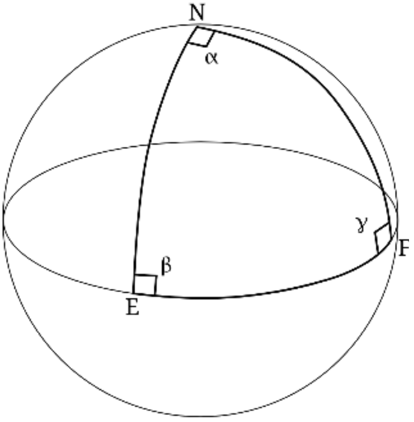


Circle centred on C :

Perimeter P.

Diameter D=$\widehat{AB}$.

P / D = 2 < π

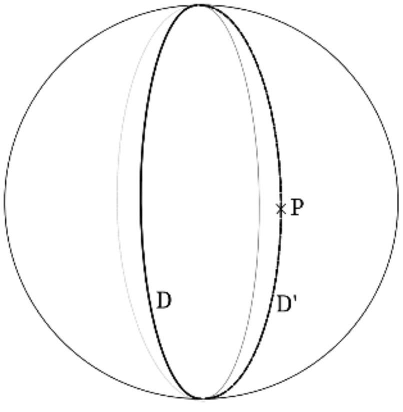


Triangle NEF :

Sum of angles:

$\alpha+\beta+\gamma=270°$

$> 180°$



Straight line D :

All straight lines D'
passing through P
intersect D.

There are no parallel
straight lines

Euclid's postulates are not verified.

The curvature can also be seen on the metric that bidizs would use, we give it for information[12]:

$$d\,l^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{x^2 + y^2}{4\,R^2}\right)^2}$$

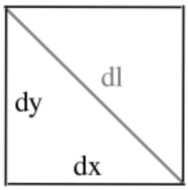
x and *y* are the two Cartesian coordinates internal to their two-dimensional space. Even if they don't "see" the third dimension, they could deduce it conceptually. It's a useful analogy for the little three-dimensional human beings that we are. Perhaps we ourselves live on the "surface" of a four-dimensional hypersphere, just as bidiz live on the surface of a hypercircle (a sphere for us!).

Here is a nice way to solve the problem of the edge of the Universe: if the Universe is not infinite, there should be a wall to define its limit, but what is behind the wall? If we live on the volume of a hypersphere, we have a Universe of finite volume, without border and without center.
An elegant vision allowed with a curved space.

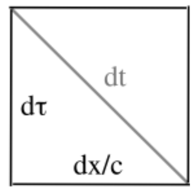
A finite Universe without edge without center.

12 *Geometry, Relativity and the Fourth Dimension*, Ruldolf v. B. Rucker, 1977.

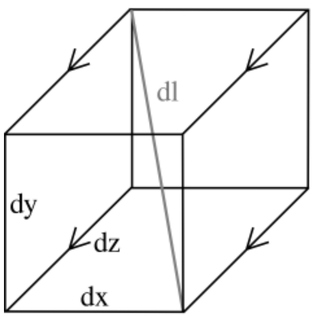
Geometries of Euclid and Minkowski



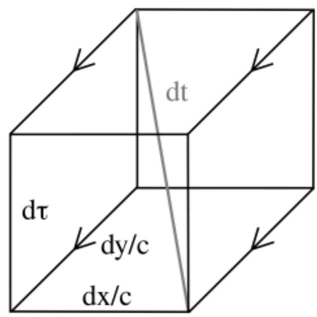
$dl^2=dx^2+dy^2$



$d\tau^2=dt^2-dx^2/c^2$

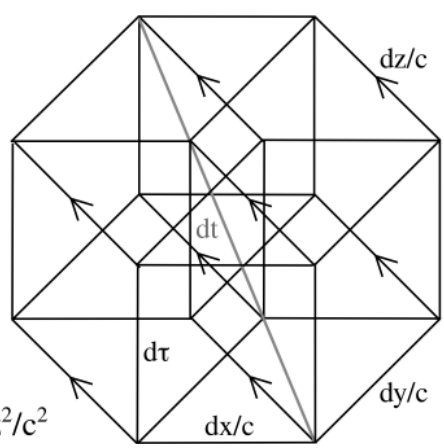


$dl^2=dx^2+dy^2+dz^2$



$d\tau^2=dt^2-dx^2/c^2-dy^2/c^2$

Spatiotemporal
Hypercube



$d\tau^2=dt^2-dx^2/c^2-dy^2/c^2-dz^2/c^2$

ꝏ MINKOWSKI METRIC

The time is now a coordinate integrated with the other three of space. It is the metric of special relativity. We have shown page 65 that the new invariant is:

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$$

We discern a temporal part and a spatial part, dt and $dl^2 = dx^2 + dy^2 + dz^2$, then $T = \int dt$ and $L = \int dl$.

But this two quantities T and L are not invariant.

Straight lines, also called geodesics, maximize the proper time τ, invariant quantity:

$$\tau = \int \sqrt{dt^2 - dl^2/c^2} \qquad (particle\ :\ ds^2 > 0)$$

Minkowski metric is invariant by translation, rotation and Lorentz transformation.

ꝏ METRIC OF AN ACCELERATING FRAME

We give the metric of the frame of reference in uniformly accelerated rectilinear translation studied in the previous chapter. This frame is not inertial and the metric is therefore necessarily different:

$$ds^2 = \left(1 + \frac{g\,x}{c^2}\right)^2 c^2\,dt^2 - dx^2 - dy^2 - dz^2$$

We recognize a Euclidean-type spatial part, so space is flat in the ship. Regarding the structure of space-time as a whole, we prove that this metric corresponds to a spacetime, also flat. For that it is shown that the components of the Riemann curvature tensor are all zero. This is very consistent with what we say about general relativity: in the absence of mass, spacetime is not curved[13].

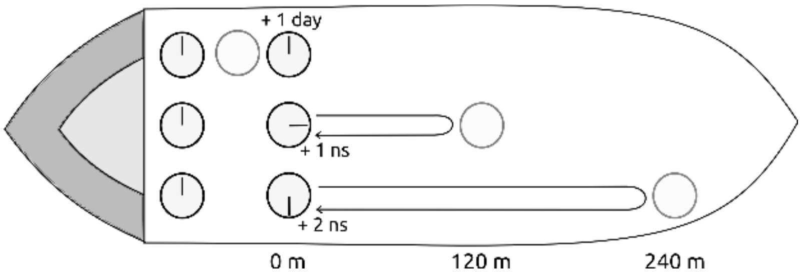
For an immobile object in the reference frame of the rocket:

$$d\tau=\left(1+\frac{g\,x}{c^2}\right)dt$$

We note, for observers motionless with respect to each other in the accelerated frame of reference, that time does not flow at the same rhythm according to where one stands in the vessel. It is a phenomenon of time dilation very different from that observed between two inertial frames of reference where the clocks are in motion relative to each other. Here, the clocks are at rest in the reference solid (the rocket), they are motionless with respect to each other, and yet they do not work at the same rate and cannot be synchronized. Let us consider, in our rocket, three clocks which we will place at three different levels spaced 120 meters apart. We start by synchronizing them on the first level at the back of the ship. We leave one clock at the stern, we place the second 120 meters forward and the third at 240

13 It's more subtle than that. For example, gravitational waves propagate a spacetime curvature that persists even in the absence of mass.

meters at the bow (we move them slowly so as not to add another source of time dilation):



After a day we take them back down to the first level to compare the elapsed times. First observation, they are no longer at the same date, moreover the clock on the second level has turned faster and is one nanosecond ahead, the third clock has turned even faster and is two nanoseconds of advance.

The advance, of the clocks placed "higher" in the vessel, is calculated using the following expression which derives directly from the metric :

$$\Delta\tau = \frac{gH}{c^2}\Delta t$$

with $\Delta t = 1\,day$, $H = 120\,m$ and $g = 10\,m/s^2$.

We will now send photons from one floor to the other. The result will be fun, and, in addition, we will find the metric, in a simple and intuitive way, without using a mathematical arsenal. You are on the second level and you send a photon down. By the time the photon moves to the bottom, the ship has gained speed. Speed measured in the inertial frame of reference which coincides with the accelerated frame of reference of the rocket at the time of the emission of the photon.

Put yourself in the place of the one receiving the photon at the bottom stage; it is now at a velocity *v* with respect to the emitter at the moment the photon was emitted. So we have a Doppler effect and as we get closer to the source, the photon "blues". The photon passes very quickly from one stage to the other and the speed of the rocket acquired over this time is very low; we will therefore only use classical formulas.

Speed acquired by the rocket : $v=gt$

and $t=\frac{x}{c}$ for the photon, then $v=\frac{gx}{c}$.

Frequency received: $f_R=(1+\beta)f_E=\left(1+\frac{gx}{c^2}\right)f_E$

We find the expected blueshift. Of course, if the photon is now sent forward, its frequency decreases, and there is a redshift:

$f_R=\left(1-\frac{gx}{c^2}\right)f_E$ and $T_R=\left(1+\frac{gx}{c^2}\right)T_E$ (small variations)

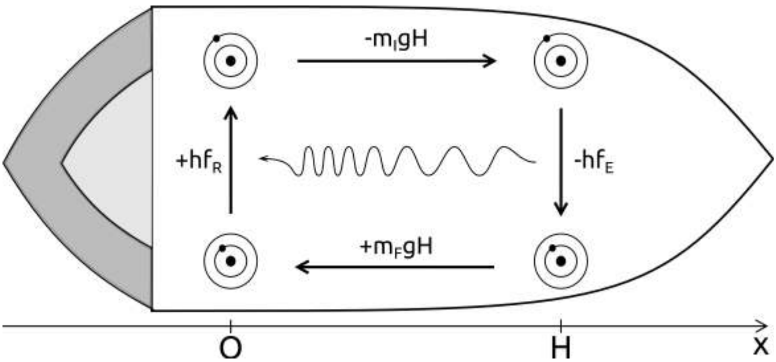
This result is directly related to the metric, because the clocks are motionless with respect to each other in the rocket's frame of reference, and each oscillation of the light wave can be considered as a mini-flash emitted by the clocks. For example, for an emission wavelength of 600 nm, the source clock emits 500,000,000,000 mini-flashes every second, and a clock placed 120 meters forward receives 7 less

mini-flashes during one of its own seconds (by Doppler effect the signal reddens as it rises and the frequency decreases).
The observer placed higher up thus deduces that the time flows slower on the floor below and faster on the floor above.

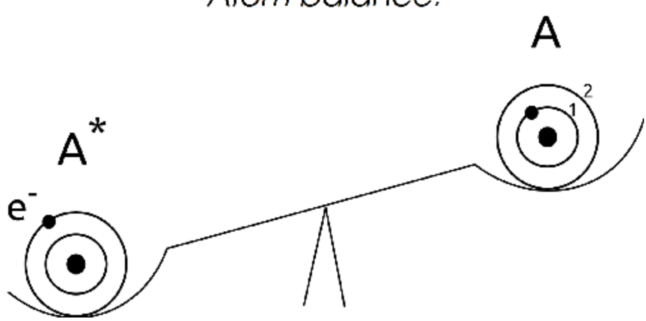
And that's not all, we can still broaden our understanding through an energetic approach. In physics we have the conservation of energy, and this fundamental law applies to special relativity by including the mass energy given by the famous formula $E=mc^2$.
We are going to move an atom from one floor to another. At the lower stage the atom is excited, we take it up in this state to the upper stage. Raising a mass requires energy from the operator. In a uniform acceleration field the energy received by an object of mass *m* is mgH. The energy of the atom increases by $m_I gH$, where m_I is the initial mass of the excited atom.
Then, the atom returns to its ground state and emits a photon of energy $e_E=hf_E$. We then go back down the atom, so the operator receives an energy $m_F gH$ where m_F is the final mass of the de-excited atom. And finally the photon of energy $e_R=hf_R$ is reabsorbed by the atom. The balance of this little game must be null because the energy must not vary:

$$-m_I gH-hf_E+m_F gH+hf_R=0$$

$$\text{then} \quad f_R - f_E = (m_I - m_F)\frac{gH}{h} = \frac{\Delta E}{c^2}\frac{gH}{h}$$



Atom balance:



An excited atom A is heavier than a de-excited atom. The difference in mass gives the energy of the emitted photon:* $A^* \rightarrow A + \gamma$

$$\Delta m\ c^2 = (m^* - m)c^2 = E_\gamma \qquad E_\gamma = \Delta E = E_2 - E_1 = hf$$

By spontaneous emission, the electron, linked to the atomic nucleus, passes from the upper level E_2 *to the fundamental level* E_1 *by emitting a photon of energy equal to the energy difference of the electronic levels. More particles are linked, more binding energy is important and more the mass of the edifice is low.*

The variation of the mass of the atom is due to the emission of the photon:

$$\text{so} \quad \Delta E = h f_E \quad \text{and} \quad f_R = f_E \left(1 + \frac{gH}{c^2}\right).$$

The received photon has a different energy than the emitted photon and we find the same expression as before. The photon gains energy when it goes down, it turns blue, and loses energy when it goes up, it reddens. The conservation of energy makes it possible to find the Doppler effect, the time dilation as a function of the position and the metric of the uniformly accelerated frame.

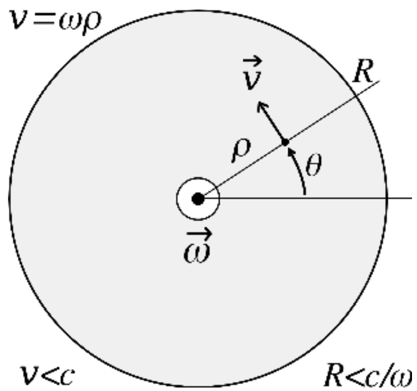
We will study the link between the uniformly accelerated reference frame and the reference frame of Schwarzschild, used for massive objects with spherical symmetry (planets, stars, black holes, etc.), in the following pages.

∞ METRIC OF A ROTATING FRAME

We are now going to approach another textbook case which can also be treated with special relativity. A case whose study opens the doors of practical applications, such as the ring laser gyroscope[14] which allows orientation much more precisely than with a mechanical gyroscope or a magnetic compass. The ring laser gyro has been used in ships, submarines, airplanes and satellites since 1963.

14 Use of the Sagnac effect conceptualized in 1913.

We have a disk of radius R rotating uniformly around a fixed axis. The disc is a rigid solid[15] whose speed increases linearly with the distance from the axis..



The speed is measured in an inertial reference frame *R* where the axis is fixed. We now place ourselves in the non-inertial frame of reference *R'* of the disc. Let us take a circle concentric with the axis of rotation, we measure the radius ρ with a stick of unit length. Then we begin to measure the circumference by transferring the stick as many times as necessary. For each report we use the inertial frame of reference coinciding at the location and given time. There is no contraction of the lengths radially, because the speed is perpendicular to the measured length, on the other hand in the orthoradial direction we are collinear with the speed and the length is contracted.

By dividing the perimeter of the circle by its

15 The rigidity criterion is verified for the disc in uniform rotation and the uniformly accelerated rocket: *L'espace-temps de Minkowski*, Nathalie Deruelle.

diameter, the value is greater than π, the space is curved[16].

Let's determine the metric by performing the following change of coordinates[17]:

$$\begin{cases} t'=t \\ \rho'=\rho \\ \theta'=\theta-\omega t \\ z'=z \end{cases}$$

The metric in the inertial frame *R* is:

$$ds^2=c^2 dt^2-dx^2-dy^2-dz^2$$

This standard expression given in Cartesian coordinates is also written in cylindrical coordinates, a coordinate system that facilitates calculations for this problem which has an axis of symmetry:

$$ds^2=c^2 dt^2-d\rho^2-\rho^2 d\theta^2-dz^2$$

The interval becomes in *R'*, removing the z coordinate for simplicity:

$$ds'^2=ds^2=c^2 dt'^2-d\rho'^2-\rho'^2(d\theta'+\omega dt')^2$$

from where, by removing the prime symbols to lighten:

$$ds^2=\left(1-\frac{\rho^2\omega^2}{c^2}\right)c^2 dt^2-2\rho^2\omega\, dt\, d\theta-d\rho^2-\rho^2 d\theta^2$$

16 It is a new pseudo-paradox of special relativity, presented in 1909 by Ehrenfest as an internal contradiction of the theory. If we accept that the space for an observer of the disk is non-Euclidean, the contradiction disappears.

17 Detailed articles: *Space geometry of rotating platforms: an operational approach*, and, *The relativistic Sagnac effect: two derivations,* Guido Rizzi and Matteo Luca Ruggiero (2008).

By calculating the components of the Riemann curvature tensor (done in the next chapter) we find that all the components are zero. The spacetime of the uniformly rotating disk is therefore flat[18]. We are well within the framework of special relativity, there is no spacetime curvature, no mass present[19], and the spacetime is well flat.

Special relativity applies in flat spacetime: a change of coordinates allows us to find the standard Minkowski metric again. In general relativity, in the presence of gravitation, this is only possible locally around an event: orders zero and one can always coincide with an inertial frame of reference (Minkowskian spacetime), on the other hand, this is no longer possible for order two, this is where the spacetime curvature is expressed.

We can create an artificial gravity with a rotating circular platform. The advantage, compared to the rocket continuously accelerated by the thrust of its reactors, is zero energy to spend. Once the disk in rotation, by conservation of energy, the disk keeps its kinetic moment, and gravity is maintained indefinitely for the occupants. On the other hand, the created gravity is not uniform, and, in addition to the centrifugal force that simulates gravity, there is the

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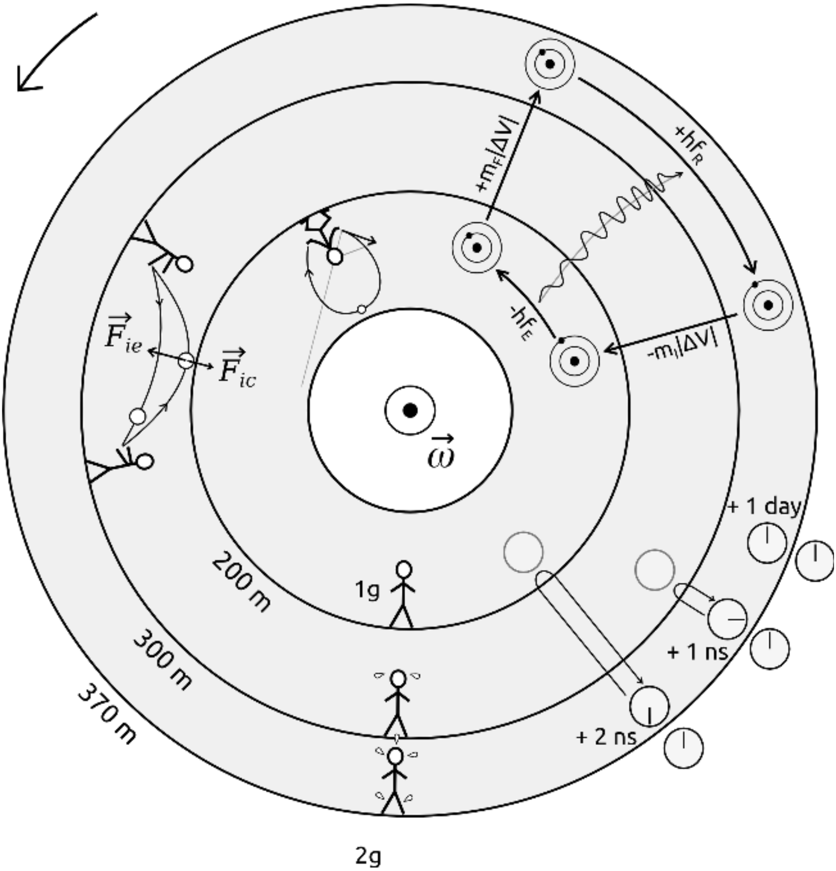
18 You will have noticed the subtlety encountered here: space is curved and spacetime is flat.

19 As with the uniformly accelerated rocket, there is no mass present which creates a gravitational field and curves spacetime. The mass of the rocket, or of the disc, is here totally negligible and does not influence the metric. We are talking about test mass.

Coriolis force that complicates the motion of the astronauts.



To minimize these two drawbacks, the radius of the centrifuge must be large enough. The centrifugal acceleration gives: $g=\omega^2\rho$ and $\Delta g/g=\Delta\rho/\rho$. For a variation in artificial gravity of less than 1% between the feet and the head, a radius of about 200 meters is required. And the corresponding angular speed of rotation is two revolutions per minute:

$$\omega = 2\pi f \quad \text{and} \quad f = \frac{1}{2\pi}\sqrt{\frac{g}{\rho}}.$$

The Coriolis acceleration is written $\vec{a}_c = 2\vec{\omega} \wedge \vec{v}_r$. When the astronauts run around the wheel, they feel heavier running in the same direction as the centrifuge and lighter running in the opposite direction, it is not very disturbing. On the other hand, if they bend up and down, they can be pushed sideways by the Coriolis force, which can be annoying.[20]. Let's calculate: $a_c/g = 2v_r/\omega\rho = 2v_r/\sqrt{g\rho}$, for a speed of 20 km/h, $a_c/g \simeq 24\%$. This is not negligible, but we can consider it reasonable.

Now let's look at the time dilation. For an observer at rest:

$$d\tau = \sqrt{1 - \frac{\rho^2\omega^2}{c^2}}\,dt \simeq \left(1 - \frac{\rho^2\omega^2}{2c^2}\right)dt$$

For observers who are immobile in respect to each other, time does not flow at the same pace. A set of rest clocks at different points on the disk cannot be synchronized. The farther away from the axis, the slower the clocks go.

We place, according to the same protocol as for the rocket, a first clock at ρ=370 m, a second at ρ=300 m, and a third at ρ=200 m.

We find: $$\Delta\tau = \frac{(\rho_2^2 - \rho_1^2)\,\omega^2}{2c^2}\,\Delta t.$$

20 Funny video: *www.voyagepourproxima.fr/ManegeTournant.mp4*

After a day we bring the clocks back down to a radius of 370 meters: the one at 300 meters advances one nanosecond and the one at 200 meters advances two nanoseconds. Here, the advances do not vary linearly with the distance. The gravity is 1.5 g at 300 m and 1.85 g at 370 m, a good exercise to build muscle and stay young!

We take back our excited atom. We count the work received by the atom at each step. We mount it from ρ_1=300 m to ρ_2=200 m. The atom then gains a potential energy:

$$w_I=-\Delta e_{p\ I}=\int m_I\ g(\rho)\ d\rho=m_I\omega^2\int\rho\ d\rho=\frac{1}{2}m_I\omega^2(\rho_2^2-\rho_1^2)$$

It emits the photon: $w_E=-e_E=-hf_E$

It goes up: $w_F=-\Delta e_{p\ F}=\frac{1}{2}m_F\omega^2(\rho_1^2-\rho_2^2)$

It receives the photon: $w_R=e_R=hf_R$

We perform the energy balance:

$$\frac{1}{2}m_I\omega^2(\rho_2^2-\rho_1^2)-hf_E-\frac{1}{2}m_F\omega^2(\rho_2^2-\rho_1^2)+hf_R=0$$

and we obtain: $f_R=f_E\left(1+\frac{\omega^2(\rho_1^2-\rho_2^2)}{2c^2}\right)$

The photon turns blue as it moves away from the axis of rotation. We always have the same phenomenon, the photon reddens as it goes up and blues as it goes down.

∞ SCHWARZSCHILD METRIC

For comparison, we give the metric of spacetime around a massive object with spherical symmetry. It is the Schwarzschild metric of general relativity which replaces Newton's force of gravity to calculate the orbits of celestial bodies. For example, it can be used for studying the motion of the space station in the gravitational field generated by the Earth. In order to respect the central symmetry, the metric is given in spherical coordinates:

$$ds^2=\left(1-\frac{2GM}{rc^2}\right)c^2dt^2-\frac{dr^2}{\left(1-\frac{2GM}{rc^2}\right)}-r^2d\theta^2-r^2\sin^2\theta\, d\varphi^2$$

M is the mass of the central body (planet, star or black hole). This mass creates a gravitational field and spacetime is curved. There is no global coordinate change that makes this metric Minkowskian. Gravitation and spacetime curvature are absent in the special relativity.

Appears in the metric a quantity with the same units as a radius, this characteristic distance of the system is called Schwarzschild radius:

we define $r_S=\frac{2GM}{c^2}$.

As for the accelerated frame in special relativity, we have an event horizon, here located in r_S.

For an object at rest we obtain the temporal part:

$$d\tau = \sqrt{1-\frac{2GM}{rc^2}}\,dt$$

The further we move away from the massive object, the lower the curvature. At great distance, space can be approximated as flat, and, according to the equivalence principle of general relativity, we must find the form of the metric of the uniformly accelerated rocket:

$$d\tau \simeq \left(1-\frac{GM}{rc^2}\right)dt \quad \text{for} \quad r \gg r_S\,.$$

For example, for the Earth, the radius r_S is about 9 millimeters. On the Earth ground, about 6370 km away, the approximation is extremely good[21].

With $r=r_0+x$ and $r_0 \gg r_S$:

$$d\tau_{r_0} = \left(1-\frac{GM}{r_0c^2}\right)dt \quad \text{and} \quad d\tau_{r_0+x} = \left(1-\frac{GM}{r_0c^2}\left(1-\frac{x}{r_0}\right)\right)dt$$

$$\text{gives} \quad d\tau_{r_0+x} = \left(1+\frac{GMx}{r_0^2c^2}\right)d\tau_{r_0}$$

The form is the same as for the uniformly accelerated rocket:

$$d\tau = \left(1+\frac{gx}{c^2}\right)dt.$$

21 Also, we can forget the Earth's rotation because the ground speed can be neglected in front of the escape velocity (geocentric reference frame).

We find the equivalence principle when:

$$g = \frac{G M}{r_0^2}.$$

Here we have also the highest clocks faster and the ascending photons that redden. On the Earth ground, over a height of 100 meters, the time lag reaches 0.9 nanoseconds in 24 hours[22]. Result close to that obtained in the rocket[23]. Locally, nothing allows astronauts to differentiate the artificial gravity field created by the acceleration of the rocket, from a natural gravity generated by a mass. On the other hand, over a sufficiently large space domain, they could differentiate the two situations: the space of the uniformly accelerated rocket is Euclidean while that of the massive celestial body is not[24].

22 In the case of the space station, even if the 110 meters beam can be maintained directed towards the Earth with a tidal stabilization, the clocks remain synchronized. At the level of the station, the gravity field is still 90% of the one on the ground, but there is no redshift, because during the rotation around the Earth, the external part goes slightly faster than the internal part and the effect is perfectly compensated. This is the principle of equivalence, for the astronauts everything happens as if there was no more gravitation (they are in weightlessness) because they are in free fall.

23 In both cases we have clocks at rest in relation to each other, which become desynchronized. For the rocket, by changing the reference frame, we can consider that it is a Doppler effect. This is not possible for gravitation and we speak of a redshift or blueshift.

24 Also in the rocket the proper acceleration is inversely proportional to the horizon distance, while for the massive object it varies with the square of the distance to the center of the body. The equivalence principle is only true very locally.

Exercises

1. ▲△△ Euclidean metric

$$d\,l^2 = dx^2 + dy^2 + dz^2$$

Show that the Euclid metric is invariant by translation, rotation and a Galilean transformation.

Answers p381.

2. ▲△△ Rapidity

1 - Show that the standard Lorentz transformation can be written:

$$\begin{cases} ct' = ct\, ch\varphi + x\, sh\varphi \\ x' = ct\, sh\varphi + x\, ch\varphi \\ y' = y \\ z' = z \end{cases}$$

We used hyperbolic trigonometry and φ is the *rapidity*.

2 - Show that, for two successive Lorentz transformations in the same direction, the rapidities are additive.

Answers p382

3. ▲△△ Rindler metric[25]

$$d\,s^2 = r^2\, d\,\tau^2 - dr^2 - dy^2 - dz^2$$

1 - What are the invariances of the Rindler

25 W. Rindler, *Relativity*, Oxford Univ. Press, 2d Ed, 2006, 430 pages.

coordinate system by rotation and Lorentz transform?

2 - Show that this coordinate system corresponds to that of a uniformly accelerating reference frame.

3 - Show that the following change of coordinates makes it possible to find a Minkowskian metric:

$$\begin{cases} ct = r\, sh\, \tau \\ x = r\, ch\, \tau \end{cases}$$

Deduce the change of coordinates between the frame of reference *(x, t)* of the uniformly accelerated rocket and the galactic frame of reference *(x', t')*.
Draw on a Minkowski diagram, in the inertial frame *R'*, the set of coordinate lines for *x* and *t*.

Answers p382

$4.^{01}_{10}$ ▲▲△ Free fall in the rocket

In our uniformly accelerated rocket, to pass the time during this trip of a few years, we have fun throwing objects at each other. Whether you drop a ball with no initial speed, or throw it to your partner, we call this motion of the object free fall, because it is not subjected to any force. We explained that the acceleration of the rocket generates artificial gravity. This is locally equivalent to a uniform gravity field, but, given the metrics of the accelerated frame, we suspect that the trajectory of an object in free fall will be modified. We will approach the question in two phases: a first qualitative approach and then a complete computation.

1 - We take two clocks initially synchronized and stationary in the same place. As in the course, one will stay in the same place, and the second one will be moved and brought back to the starting point. You play the following game: At the start both clocks indicate zero. You have the mobile clock that you can move as you wish. The only constraint is that at one minute exactly as indicated on the fixed clock, your clock will have to be back, placed very quietly next to it. The challenge is to get the greatest possible time on your clock. How do you have to move it to win?

Variation of the game: Previously the starting point was the end point. If now the finish point, while remaining at the same level, is different, how do we proceed to maximize the time on our clock?

2 - The path followed by a free particle to go from the initial event E_i to the final event E_f maximizes its proper time:

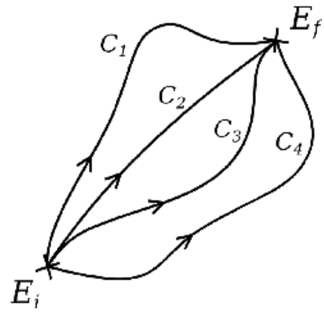
$$\tau=\int_{E_i}^{E_f} d\tau=\int_C \sqrt{g(x)-\frac{v^2}{c^2}}\,dt \quad \text{with} \quad g(x)=\left(1+\frac{a\,x}{c^2}\right)^2$$

$$\text{Lagrangian: } L(x,v)=\sqrt{g(x)-\frac{v^2}{c^2}}$$

An infinity of possible paths C links E_i to E_f.

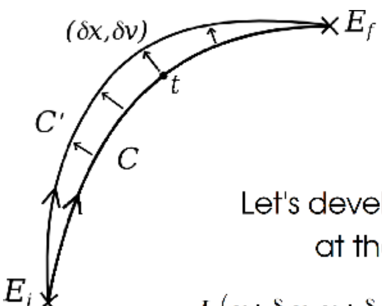
Which one extremes τ ?

We know that for the extremal path, a small variation of the parameters x and v does not modify, at order one, the proper time.



It is a simple mathematical property: at the maxima and minima of a function the slope is zero.

Suppose that C is the optimal path and consider C' infinitely close. At given t, we pass from C to C' by small variations of x and v:



$$x_{C'} = x_C + \delta x$$

and

$$v_{C'} = v_C + \delta v$$

Let's develop the Lagrangian at the first order:

$$L(x+\delta x, v+\delta v) = L(x, v) + \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial v}\delta v$$

Thus: $\int_{C'} L(x+\delta x, v+\delta v)\, dt$

$$= \int L(x, v)\, dt + \int \left(\frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial v}\delta v \right) dt = \tau + \delta\tau$$

For the searched path $\delta\tau = 0$.

a- Continue the reasoning and establish the equation of motion of an object in free fall. Show that this equation, at the start of the throw and at low speeds, gives the equation of free fall in

Newtonian mechanics.
Finally, how will you move your clock to win?

b- Demonstrate the following conservation law:

$$L-\frac{\partial L}{\partial v}v=cst$$

We consider the case of a release from rest. Find the expression of position, velocity and acceleration as a function of $g(x)$. How does g vary during the fall? Show that the falling velocity reaches a maximum and then cancels on the horizon. What is the maximum falling speed? At what distance from the horizon?

c- Perform a numerical simulation to plot position, velocity and acceleration curves as a function of time. When is the maximum speed reached? When does the object reach the horizon for an observer of the rocket?

d- *Proper time*: Give the expression of the proper time. In its proper reference frame, when does the object reach the horizon? Suppose that the object is a mini auxiliary rocket that leaves the mother ship in free fall. What will happen to the occupant of the mini-rocket when it reaches the horizon? This small rocket is very fast, the pilot decides to ignite the engine to return to the main ship, will he succeed? You can illustrate the situation on two Minkowski diagrams (galactic and rocket frames).

e- *Local Minkowskian observer*: The coordinate system of the accelerated rocket is not Minkowskian. The velocity previously determined in a non-Minkowskian metric is called the coordinate velocity. This coordinate system has been constructed in a non-inertial frame of reference and the assumptions of special relativity do not apply directly to it. This reference frame is nevertheless very useful and necessary for the occupants of the rocket, but the speed of light is not fixed at *c*. This is why we will consider a new observer, an inertial one. At each instant and position of the object in free fall, we consider the Minkowskian reference frame coinciding with that of the rocket:

$$c^2 d\tau^2 = c^2 dt_{Mink}^2 - dx^2$$

For example, imagine two rockets fixed relatively to each other and uniformly accelerated. All of a sudden, one of them cuts its engine, its reference frame becomes inertial, and for some time it coincides with the rocket still accelerated. Thus an observer in the rocket which cut its engine is minkowskien, and he can observe the fall of the object. What speed will he measure for the falling object? What will be the velocity of the falling object at the horizon for a Minkowskian observer?

3 - *Analogy with the fall into a black hole:*

a- The Schwarzschild coordinate system is that of an outside observer at the black hole. We can compare the radial fall of an object towards a black hole with the vertical fall of an object observed by the occupant of a uniformly accelerated rocket:

$$d\tau^2=g(r)dt^2-\frac{dr^2}{c^2 g(r)} \quad \text{with} \quad g(r)=1-\frac{2GM}{rc^2}$$

$$\tau=\int L(r,v)\,dt \qquad \text{and} \qquad L(r,v)=\sqrt{g(r)-\frac{1}{g(r)}\frac{v^2}{c^2}}$$

Describe the velocity profile of a falling body, dropped without initial velocity, to the horizon of the black hole $r_H=r_S=2GM/c^2$. You will draw curves for speed and acceleration as a function of *r*.
What is the maximum speed reached? At what distance from the horizon?

b- Perform a numerical simulation to plot position, speed and acceleration curves as a function of time. When is the maximum speed reached? When does the object reach the horizon for an observer outside the black hole?

c- *Proper time*: Give the expression of the proper time. In its proper reference frame, when does the object reach the horizon? Suppose the object is a spacecraft in free fall. What will happen to the occupant of the spacecraft when he reaches the horizon? This rocket is very fast and powerful, the pilot decides to start the reactor to leave the black hole, will he succeed?

d- *Local Minkowskian observer*: The Schwarzschild coordinate system is not Minkowskian. We have previously determined the coordinate velocity and *coordinate acceleration* in this coordinate system. This coordinate system is very convenient and useful but the speed of light is not fixed at *c*. That is why we will consider a new observer, him inertial. At each instant and position of the falling object, we consider the Minkowskian frame motionless with respect to the black hole and coinciding with the Schwarzschild frame of reference:

$$c^2 d\tau^2 = c^2 dt_{Mink}^2 - dr_{Mink}^2$$

Which speed is measured in this way for the object in free fall? What will be the speed of the falling object for a Minkowskian observer at the horizon?

e- *Comparison to experimental data*:

In 2018, a study[26] of the measurements made by the XMM-Newton probe, which observed a supermassive black hole of 40 million solar masses, shows a wind of matter in free fall towards the black hole that reaches relativistic speeds:

$$v \sim 0.3c \quad \text{for} \quad r \sim 20\, R_S$$

$$v \sim 0.1c \quad \text{for} \quad r \sim 200\, R_S$$

Do these results seem consistent with those found in the exercise?

Answers p384.

26 *An ultrafast inflow in the luminous Seyfert PG1211+143* , 2018, K.A.Pounds, C.J.Nixon, A.Lobban and A.R.King. University of Leicester, United-Kingdom.

5. ▲△△ Fall of a blue ball

We release from rest a blue ball into the uniformly accelerated rocket and watch it fall in free fall. What will be the color of the ball perceived during its fall by the astronauts of the rocket?

Answers p404.

6. ▲▲△ Trajectory of a ray of light in the Einstein's Elevator

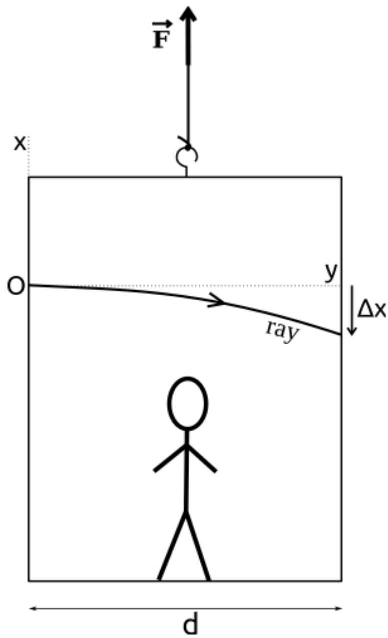
Albert Einstein proposes a thought experiment in his book *Relativity* written in 1916. We imagine a portion of empty space infinitely distant from all masses. We have at our disposal a large box in which an observer evolves in weightlessness. A hook makes it possible to exert a constant force on the box by means of a rope, which is then animated by a rectilinear translation motion uniformly accelerated. The observer thus experiments an artificial gravity.
Compared to the immobile box, constituting an inertial frame of reference, the trajectory of a light ray of speed c is rectilinear. On the other hand, in the box accelerated by the traction of the rope, a light ray, here, initially perpendicular to the direction of motion, will take a curved trajectory. Let's quote Einstein: *"It can easily be shown that the path of the same ray of light is no longer a straight line"*.

1 - *Newtonian approximation:*

We consider the speed of light constantly equal to c, and the rectilinear trajectory, in the Galilean frame of reference which initially coincides with the box.

For a constant acceleration box a, determine Δx.

Express the equation of the trajectory $y(x)$ and of the light speed $v(x)$ in the accelerated frame.



2 - *Special Relativity:*

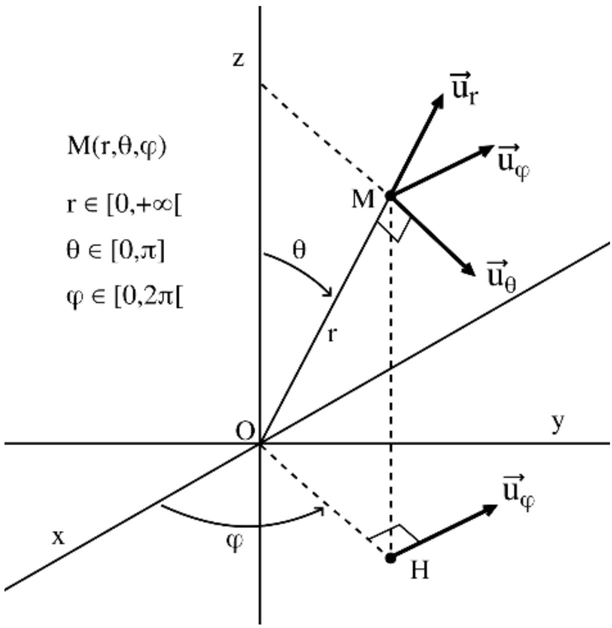
We answer the same questions as above. For that, we first consider the equation of the light ray worldline in an inertial reference frame. Then, with the appropriate change of coordinates, we obtain the equation of the worldline in the non-inertial box.

3 - Drawing of curves.

Answers p404.

7. ▲▲△ Spherical coordinate system

Spherical coordinate system definition:



1 - Conversions between spherical and rectangular coordinates.

2 - Express the position vector $\vec{r}=\overrightarrow{OM}$ and the infinitesimal element vector $\vec{dr}=\overrightarrow{MM'}$ between *M* and *M'* infinitely close.

3 - Find by integration the surface and the volume of a sphere.

4 - Definition of plane angles and solid angles: from an observation point O, we observe an object. The extensions of the periphery of the object cuts an arc on the circle unit of center O. The length of this arc gives the value of the angle in radians under which we see the object. In 3D space the circle is replaced by a sphere unit on which a surface is cut out. The area of this surface gives the solid angle in steradians under which we see the object.

a- From which solid angle do we see the whole space? The starry sky on a clear night? A room from one of its corners?

b- Calculate the solid angle of an angle cone α.

c- What is the probability that a star is in the plane of the ecliptic within ten degrees?

Answers p407.

$\widetilde{u}$

7

Four-vectors

We have introduced special relativity through the Minkowski spacetime: events space with its metric[27]. We can extend this points space to build more complex elements such as vectors or tensors.

The following presentation is a bit formal but necessary for a full understanding of relativity. We will continue to rely on a geometrical vision as soon as possible.

The elements of a vector space E are vectors, noted in this book with bold letters : **v**.

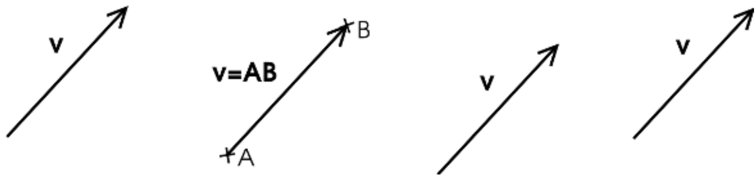
If we need to specify that we are in a Euclidean vector space, we will use the classic notation with arrows : $\vec{v}$.

In the case of the Minkowski space, we can clarify the context by talking about four-vectors noted with tildes : $\tilde{v}$.

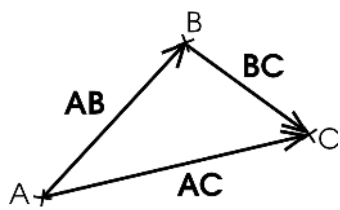
---

27 We considered the standard Minkowski metric of an inertial frame $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ in an orthonormal Cartesian coordinate system. While keeping an inertial reference frame, the form of the metric can be different. For example, in cases where the metric is expressed in a non-orthonormal or non-Cartesian coordinate system. We then speak of Minkowskian metric. When the change of coordinates gives a non-inertial frame of reference (as for our rocket and the rotating disk) the special relativity is applied by adding metric effects (page 229).

In general, a vector can be uniquely defined from two points (or events) in our space (or spacetime):

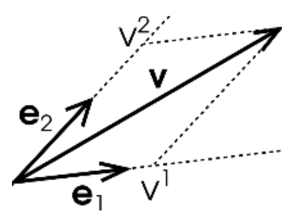


Vector space is affine and with a third point we have the relation **AC** = **AB** + **BC** :



By multiplying by a real we have a new vector k **AB** and the vector is directed **BA** = - **AB**. Any linear combination of E vectors is a new E vector.

We express a **v** vector in a basis. The basis vectors are denoted $\boldsymbol{e}_i$ and form a spanning and generating set of E.



For a vector space of dimension *n*:

$$\boldsymbol{v}=v^1\boldsymbol{e}_1+v^2\boldsymbol{e}_2+\ldots+v^n\boldsymbol{e}_n=\sum_{i=1}^{n}v^i\boldsymbol{e}_i=v^i\boldsymbol{e}_i$$

We use Einstein summation convention, the summation is implied for a repeated index up and down. The v^i are the components of **v** expressed with the basis vectors ($\mathbf{e}_1$, $\mathbf{e}_2$, ... , $\mathbf{e}_n$).

Scalar product of two vectors **a** and **b** :

$$\boldsymbol{a}\cdot\boldsymbol{b}=(a^i\boldsymbol{e}_i)\cdot(b^j\boldsymbol{e}_j)=\boldsymbol{e}_i\cdot\boldsymbol{e}_j a^i b^j$$

We define the components of the metric tensor **g** such as: $g_{ij}=\boldsymbol{e}_i\cdot\boldsymbol{e}_j$.

$$\text{so}\qquad \boldsymbol{a}\cdot\boldsymbol{b}=g_{ij}a^i b^j.$$

For example, for *n=2*, we have:

$$\boldsymbol{a}\cdot\boldsymbol{b}=g_{11}a^1b^1+g_{12}a^1b^2+g_{21}a^2b^1+g_{22}a^2b^2$$

The scalar product[28] is commutative and the components of the metric tensor are symmetrical:

$$g_{ij}=g_{ji}$$

We can write the components of the metric tensor in a matrix.
For example, for *n=3* in the basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$) :

$$\boldsymbol{g}=\begin{pmatrix} g_{11} & g_{12} & g_{13}\\ g_{21} & g_{22} & g_{23}\\ g_{31} & g_{32} & g_{33}\end{pmatrix}$$

We have a second way to project a vector. The first

28 In math, we talk about bilinear form, it associates to two vectors a number, called scalar.

components, given above, are obtained parallel to the basis vectors. We can obtain a new set of v_i components with orthogonal projections:

$$\boldsymbol{v}\cdot\boldsymbol{e}_i=(v^j\boldsymbol{e}_j)\cdot\boldsymbol{e}_i=g_{ij}v^j=v_i$$

We then have a new basis associated with these new components: $\boldsymbol{e}^i=g^{ij}\boldsymbol{e}_j$. The g^{ij} are calculated from the g_{ij} with: $g_{ik}g^{kj}=\delta_i^j$ where δ_i^j is the Kronecker delta, null, if the indices are different, and, equal to one, if they are equal.
We then have a new writing :

$$\boldsymbol{v}=v_i\boldsymbol{e}^i$$

Lower-index objects are covariant quantities, while upper-index objects are contravariant quantities.
For example, the components v_i are covariants and the basis vectors $\boldsymbol{e}^i$ are contravariants. The components g_{ij} are two times covariants and the tensor g^{ij} is two times contravariants. We will see the precise meaning and importance of this vocabulary at the moment of the change of basis.
The metric tensor allows us to switch between these two types of quantities.
In the end, we can have four different writings for the scalar product:

$$\boldsymbol{a}\cdot\boldsymbol{b}=g_{ij}a^ib^j=a^ib_i=a_ib^i=g^{ij}a_ib_j$$

Orthogonal vectors: $\boldsymbol{a} \cdot \boldsymbol{b} = 0$.

In the case of orthogonal bases:

if $i \neq j$ then $g_{ij} = 0$.

For example, for $n=2$: $\boldsymbol{a} \cdot \boldsymbol{b} = g_{11} a^1 b^1 + g_{22} a^2 b^2$

and $\boldsymbol{g} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}$.

Vectors, tensors and scalars are essential mathematical objects for physics. The laws of nature are expressed using equations constructed from these three types of objects, because if we change the basis, the laws keep the same form. The new basis is associated with new coordinates used to realize a translation, a rotation or a change of Galilean or inertial reference frame. We will study the change of coordinates later.

Following this somewhat abstract interlude, let us approach different practical cases.

∞ Euclidean vector space

Newton's laws and all classical mechanics is built with vectors, scalars and tensors.

Newton's second law:

$$\vec{F} = m\vec{a} \ ,$$

Kinetic power:

$$P_k = \frac{dE_k}{dt} = \frac{d}{dt}\left(\frac{1}{2} m\vec{v}\cdot\vec{v}\right) = P = \vec{F}\cdot\vec{v} \ ,$$

Angular momentum:

$$\frac{d\vec{\sigma}}{dt} = \frac{d}{dt}(m\vec{r}\wedge\vec{v}) = \vec{r}\wedge\vec{F} \ .$$

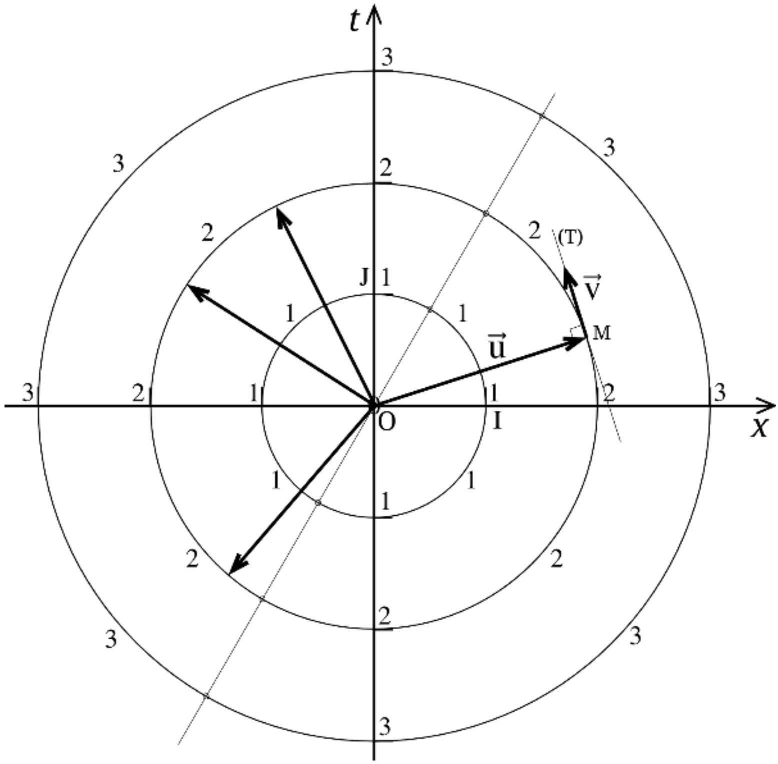
All these laws keep the same form by translation, rotation and Galilean transformation. The use of vectors assures us that.

In Euclidean geometry the scalar product of a vector with itself can only be positive or zero, we can then define a **norm**:

$$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$$

The norm is positive definite:

- $\vec{v}\cdot\vec{v} \geqslant 0$.
- $\vec{v}\cdot\vec{v} = 0$ if and only if $\vec{v} = \vec{0}$.



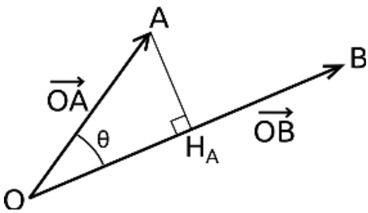
In Euclidean geometry, the norm of a vector is represented by its length and this length is independent of the chosen basis. Starting from O, all the ends of vectors of the same norm are placed on the same circle (we have represented four vectors of norm 2).
A property of the circle: if we draw a radius OM, the tangent (T) is always perpendicular to (OM). We thus obtain a pair of orthogonal vectors:

$$\vec{u}\cdot\vec{v}=0.$$

For a set of concentric circles of radii multiple of unity, a line through O intersects the circles at a set of equidistant points.

Geometric determination of the scalar product:

$$\vec{a}\cdot\vec{b}=\|\vec{a}\|\|\vec{b}\|\cos\left(\widehat{\vec{a},\vec{b}}\right)$$



$$\overrightarrow{OA}\cdot\overrightarrow{OB}=OA\times OB\times\cos\theta$$

$$=\pm OH_A\times OB$$

$$=\pm OH_B\times OA$$

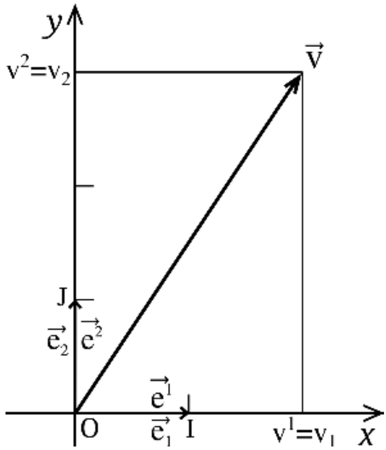
$$\vec{a}\cdot\vec{b}=\overrightarrow{OA}\cdot\overrightarrow{OB}=\left(\overrightarrow{OH}+\overrightarrow{HA}\right)\cdot\overrightarrow{OB}=\left(\vec{c}+\vec{n}\right)\cdot\vec{b}=\vec{c}\cdot\vec{b}+\vec{n}\cdot\vec{b}$$

In the end, if we find an orthogonal vector $\vec{n}$, the dot product comes down to that of two collinear vectors and the value is the product of their radii:

$$\vec{a}\cdot\vec{b}=\vec{c}\cdot\vec{b}=\pm R_c\times R_b$$

The sign is positive if the two collinear vectors are in the same direction, and negative if they are in opposite directions. We have two equivalent options, find a vector orthogonal to $\vec{a}$ or to $\vec{b}$.

- **Orthonormal Cartesian bases:**



We can always go back to an orthonormal Cartesian basis:

$$\vec{e}_i\cdot\vec{e}_j=\delta_{ij}.$$

For example, for *n=2*, we have in this case:

$$\boldsymbol{g}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

$$\vec{a}\cdot\vec{b}=a^1 b^1+a^2 b^2$$

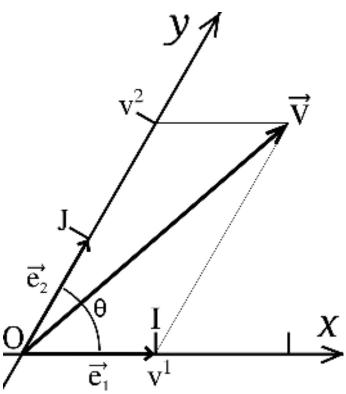
and for the norm:

$$v=\sqrt{\|\vec{v}\|}=\sqrt{(v^x)^2+(v^y)^2}$$

The covariant and contravariant components are then identical. The same applies to the bases.

- **Normal and non-orthogonal Cartesian bases:**

Case for a vector of the plane (2-vector) :



We know the contravariants components of $\vec{v}$ in the covariant base:

$$\vec{v}=v^1\vec{e}_1+v^2\vec{e}_2=\vec{e}_1+2\vec{e}_2$$

with $\quad g_{ij}=\begin{pmatrix}1 & \cos\theta\\ \cos\theta & 1\end{pmatrix}$

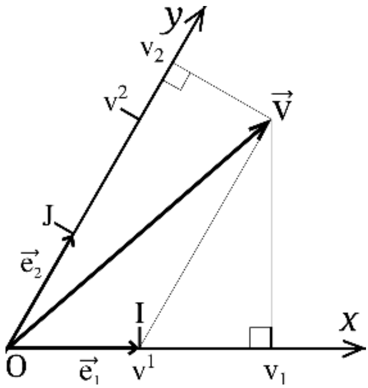
and $\quad \theta=\dfrac{\pi}{3}$.

Let's determine the covariant components of $\vec{v}$:

$$v_i=g_{ij}v^j=g_{i1}v^1+g_{i2}v^2$$

$$v_1=g_{11}v^1+g_{12}v^2=v^1+\cos\theta\, v^2=2=\vec{v}\cdot\vec{e}_1$$

$$v_2 = g_{21} v^1 + g_{22} v^2 = \cos\theta\, v^1 + v^2 = \frac{5}{2} = \vec{v} \cdot \vec{e}_2$$



We now have two possible decompositions for $\vec{v}$:

$$\vec{v} = \vec{e}_1 + 2\vec{e}_2 = 2\vec{e}^1 + \frac{5}{2}\vec{e}^2$$

Let us determine the metric tensor components in the contravariant base:

$$g_{i1} g^{1j} + g_{i2} g^{2j} = \delta_i^j$$

$g_{11} g^{11} + g_{12} g^{21} = 1$ then $g^{11} + \cos\theta\, g^{21} = 1$

$g_{11} g^{12} + g_{12} g^{22} = 0$ and $g^{12} = -\cos\theta\, g^{22}$

$g_{21} g^{12} + g_{22} g^{22} = 1$ and $\cos\theta\, g^{12} + g^{22} = 1$

$g_{21} g^{11} + g_{22} g^{21} = 0$ and $g^{21} = -\cos\theta\, g^{11}$

so: $g^{11} = g^{22} = 1/\sin^2\theta$

and $g^{12} = g^{21} = -\cos\theta/\sin^2\theta$

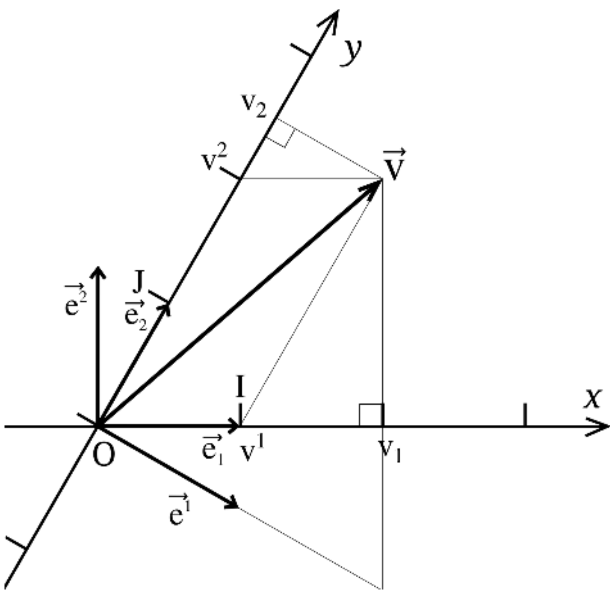
Metric : $g^{ij} = \frac{1}{\sin^2\theta} \begin{pmatrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{pmatrix}$

Let's find the contravariant basis:

$$\vec{e}^i = g^{ij}\vec{e}_j = g^{i1}\vec{e}_1 + g^{i2}\vec{e}_2$$

so $\vec{e}^1 = g^{11}\vec{e}_1 + g^{12}\vec{e}_2 = \dfrac{\vec{e}_1 - \cos\theta\,\vec{e}_2}{\sin^2\theta} = \dfrac{4}{3}\left(\vec{e}_1 - \dfrac{1}{2}\vec{e}_2\right)$

$$\vec{e}^2 = g^{21}\vec{e}_1 + g^{22}\vec{e}_2 = \frac{-\cos\theta\,\vec{e}_1 + \vec{e}_2}{\sin^2\theta} = \frac{4}{3}\left(-\frac{1}{2}\vec{e}_1 + \vec{e}_2\right)$$



Now, if you are a math teacher in middle school and when studying non-orthogonal coordinate systems a pupil asks you, "*Why do we project along parallels and not perpendiculars?*" you will know what to answer. The pupil is absolutely right, both types of projections are possible and even complementary.

ꝏ MINKOWSKI VECTOR SPACE

We will establish the new physical laws of special relativity based on four-vectors. For the formulas, we will be inspired by Newton's mechanics via the low speed limit.

We note the components of an event E with indices from 0 to 3 :

$$\tilde{x} = x^{\mu}\left(x^0, x^1, x^2, x^3\right)$$

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad \text{and} \quad x^3 = z$$

$$\tilde{v} = \widetilde{OE} = x^{\mu}(E) - x^{\mu}(O)$$ [29]

For the scalar product: $\tilde{a} \cdot \tilde{b} = g_{\mu\nu} a^{\mu} b^{\nu}$.

With the Minkowski metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We will show that this metric gives back the triangle of times.

We have: $\tilde{a} \cdot \tilde{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3$.

29 Vectors, or tensors, are regularly misidentified with their components. In general, this does not lead to confusion.

For the spatial part, we recognize a Euclidean scalar product, we can then write:

$$\tilde{a}\cdot\tilde{b}=a^0 b^0-\vec{a}\cdot\vec{b}.$$

The scalar product of a vector $\tilde{v}$ with itself can be positive, zero or negative:

$$\tilde{v}\cdot\tilde{v}=(v^0)^2-\|\vec{v}\|^2.$$

Contrary to the Euclidean case, the Minkowskian scalar product of a vector with itself is not always positive. Moreover, $\tilde{v}\cdot\tilde{v}=0$ does not imply $\tilde{v}=\tilde{0}$. There is no norm for a vector in Minkowski space. The quantity $\tilde{v}\cdot\tilde{v}$ is sometimes called pseudo-norm[30].

In Euclidean space the length of a vector, represented on an orthonormal coordinate system, corresponds to its norm, and the vectors of the same norm, starting from the same point, are distributed on the same circle. This is no longer the case on a Minkowski diagram: two vectors can have the same pseudo-norm and not appear with the same length.[31]. The 4-vectors of the same pseudo-norm are distributed on hyperbolas.

---

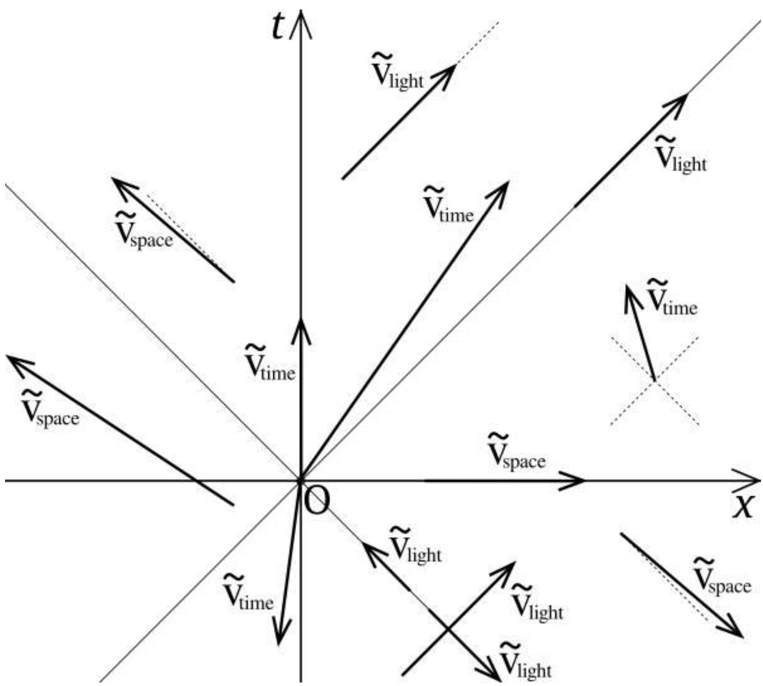
30 Term used and debatable: this term refers to the Euclidean norm without taking up all its principles. Contrary to the norm, the pseudo-norm does not have the same units as the vector (the square root is missing). We could consider the quantity: $k=\sqrt{|\tilde{v}\cdot\tilde{v}|}$ where k is the parameter of the hyperbola associated with the 4-vector. We could name k, the timelike or spacelike norm depending on the case (as in Euclidean where R is the parameter of the circle and the norm of the vector). We will use the term *intensity* for the k of a four-vector.

31 We represent the two-dimensional Euclidean space on a sheet of paper which is itself a 2D Euclidean physical object. On the other hand, using a Euclidean sheet to represent Minkowski's plane requires an effort of abstraction.

We have three kinds of 4-vectors :

- timelike : $\tilde{v} \cdot \tilde{v} > 0$
- lightlike : $\tilde{v} \cdot \tilde{v} = 0$
- spacelike : $\tilde{v} \cdot \tilde{v} < 0$

The light-like vectors are on the light cones associated with the world-lines of photons. The time-like vectors are in the cone (towards the vertical), and the space-like vectors towards the outside of the cone.



Depending on the sign of the time component, a four-vector can point towards the future or the past. This property and that of the time, light or space-like

kind do not depend on the inertial frame of reference considered.
When the scalar product of two vectors is null we have orthogonal vectors:

$$\tilde{a} \cdot \tilde{b} = 0$$

This property of orthogonality is also valid in all inertial frames of reference.

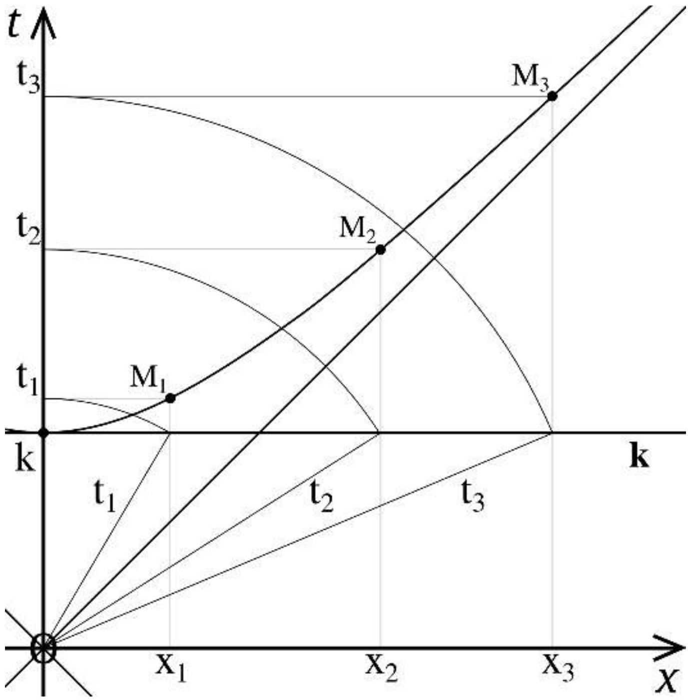
Again, the situation is not as intuitive as in Euclidean, it is not because two vectors are orthogonal that they appear perpendicular on a diagram.

We have two types of hyperbolas, those time-like, internal to the light cone, of equations $t^2 - x^2 = k^2$ (to simplify we have set c=1), and the external ones, space-like, of equations $t^2 - x^2 = -k^2$ [32].

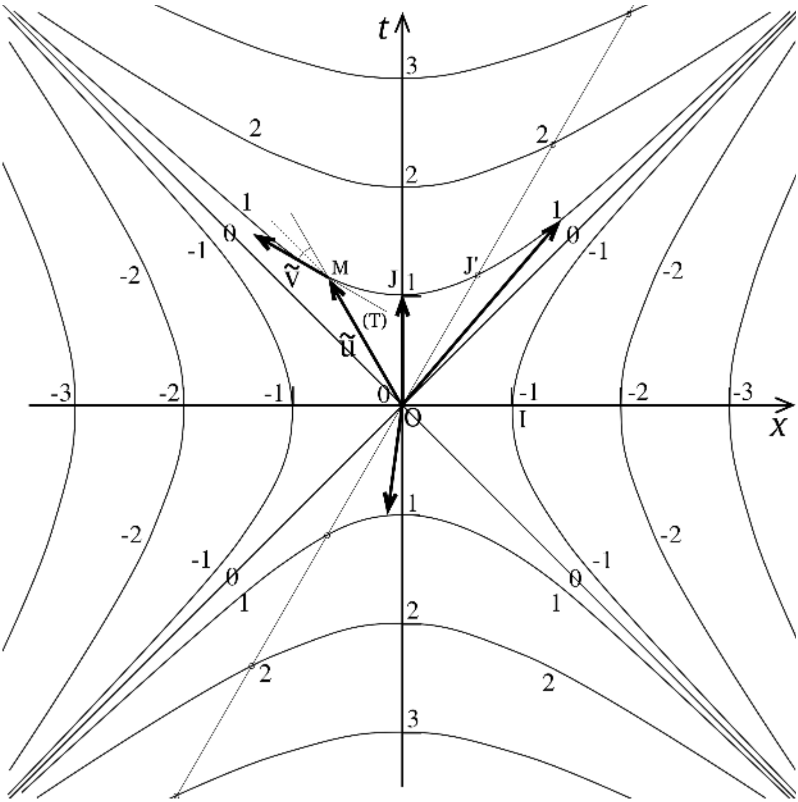
k defined as positive.

32 "S*pace and Time*", Hermann Minkowski, lecture delivered at Cologne on 21st September 1908.

We easily find again the hyperbolas by a construction with *the triangle of times*:



Plot of an internal hyperbola of parameter k*. For a given* x *it corresponds to a value of* t *which forms a right-angled triangle with* k*:* $t^2=k^2+x^2$*. For a 4-vector position* x^μ*, time-like,* k *corresponds to a proper time* τ*. For an external hyperbola,* k *is represented by a vertical line and it is* x *which is placed at the hypotenuse:* $x^2=k^2+t^2$*.*



A hyperbolic geometry: vectors of the same pseudo-norm, that start in O, end on the same pair of hyperbolas. We have represented four 4-vectors which have the same pseudo-norm 1, they join the unit hyperbola on one or the other of these two branches. The time-like hyperbolas are indexed by k *and the space-like hyperbola by* -k*.*

A property of the hyperbola: if we plot a radius OM, the tangent (T) is always symmetrical, with respect to the bisectors, at (OM). We thus obtain a pair of orthogonal vectors: $\tilde{u}\cdot\tilde{v}=0$.

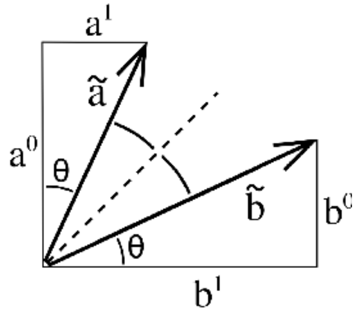
For a set of hyperbolas with the same center O, the same orthogonal axes, and parameters multiple of the unit, a straight line passing through O cuts the hyperbolas into a set of equidistant points.

In **2D**, in Minkowski's plane:

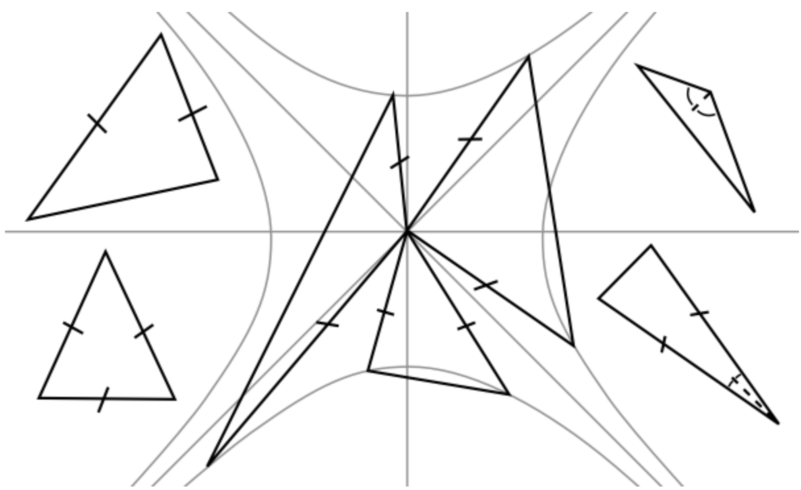
$$\tilde{a}\cdot\tilde{b}=0 \quad \Rightarrow \quad a^0 b^0 = a^1 b^1$$

Two orthogonal 4-vectors are symmetrical with respect to the photon worldlines:

$$\tan\theta = \frac{a^1}{a^0} = \frac{b^0}{b^1}$$



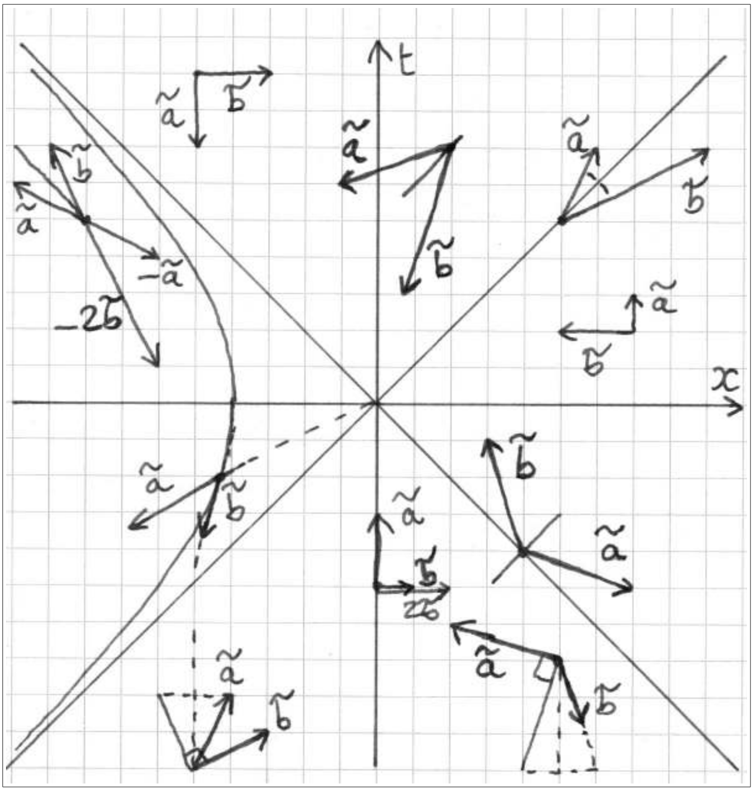
Triangles



Four isosceles triangles, one equilateral triangle, one right-angled triangle and one isosceles right triangle. All these triangles keep their properties by 90° rotation and change of scale.

Examples of 4-vectors orthogonal

For all pairs represented: $\tilde{a} \cdot \tilde{b} = 0$



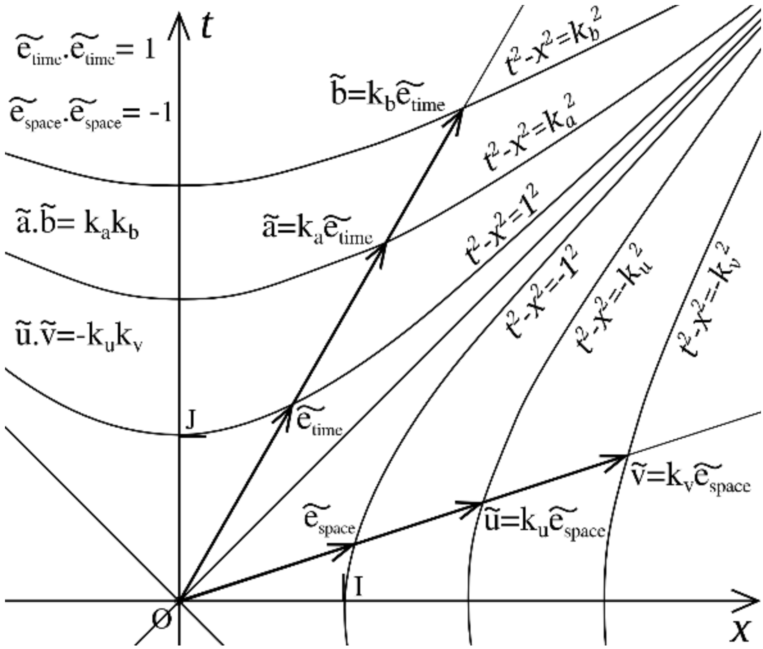
By taking the opposite of one of the vectors of the pair, or by multiplying it by a constant, the pair remains orthogonal.

Geometrical methods:

- *Use of the hyperbola.*
- *Symmetry with respect to the photon worldlines.*
- *Passage through the Euclidean: two perpendicular vectors and we take the symmetry with respect to the vertical of one of them.*

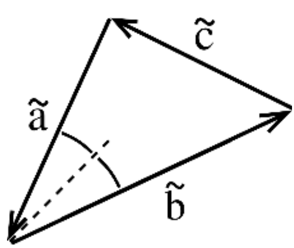
Case of 4-vectors collinear :

Two examples, the pair ($\tilde{a}$, $\tilde{b}$) and the pair ($\tilde{u}$, $\tilde{v}$)



Pythagorean theorem in Minkowski space:

$\tilde{a}+\tilde{b}=\tilde{c}$ with $\tilde{a}$ and $\tilde{b}$ orthogonal.



$$\boxed{k_a^2-k_b^2=\pm k_c^2}$$

k : parameter of the hyperbola / magnitude / intensity of the 4-vectors.

∘ Geometric determination of the scalar product

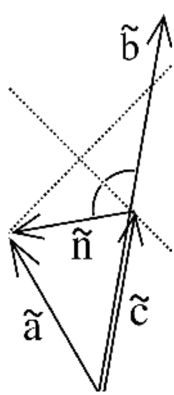
To evaluate $\tilde{a}\cdot\tilde{b}$ in the space of Minkowski:

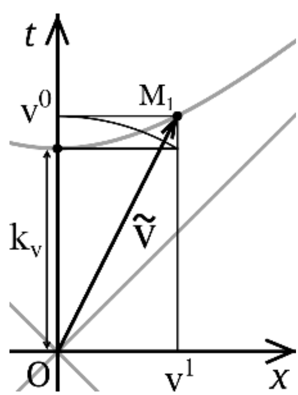
- We break down one of the two four-vectors as the sum of an orthogonal vector and a collinear vector to the second one.

$$\tilde{a}\cdot\tilde{b}=(\tilde{c}+\tilde{n})\cdot\tilde{b}=\tilde{c}\cdot\tilde{b}+\tilde{n}\cdot\tilde{b}$$

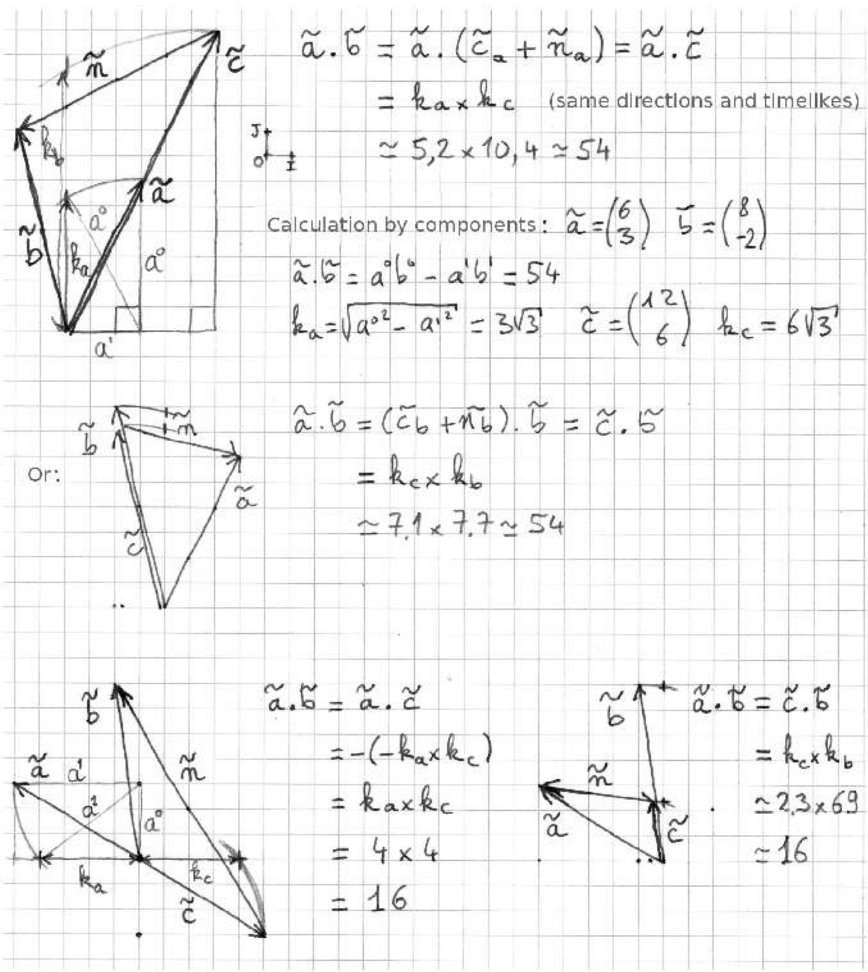
- We determine with a compass the parameters of the hyperbolas of the two collinear vectors obtained.

- The scalar product is the product of the two parameters: $\tilde{a}\cdot\tilde{b}=\tilde{c}\cdot\tilde{b}=\pm k_c\times k_b$.
  The sign is positive if the two collinear vectors are timelike and in the same direction, or, if they are spacelike and in opposite directions. In other cases the sign is negative.





Examples of geometric determination of the scalar product :

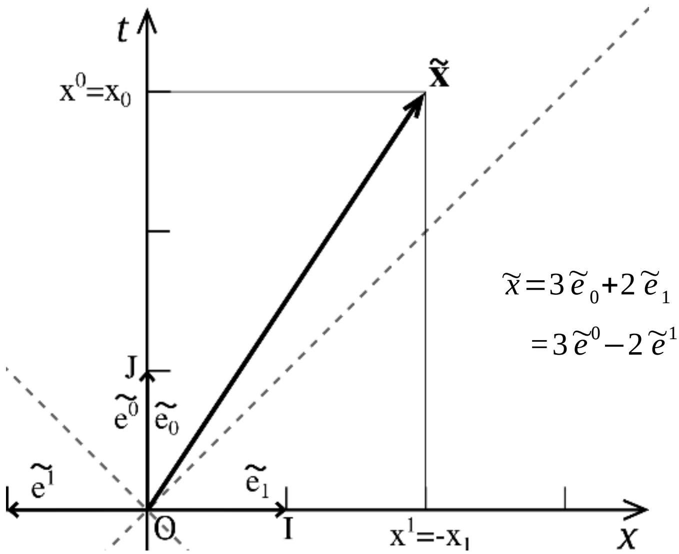


◦ Orthogonal bases

We can always go back to an orthogonal base such as $\tilde{e}_\mu \cdot \tilde{e}_\nu = 0$ for $\mu \neq \nu$.

• Reference frame *R*

Let's look at the case of the contravariant and covariant components on a Minkowski diagram:



Let's check, on this particular case, the general formulas:

$$g_{\mu\nu} = \tilde{e}_\mu \cdot \tilde{e}_\nu, \quad \tilde{x} = x^\mu \tilde{e}_\mu,$$

$$x_\mu = g_{\mu\nu} x^\nu, \quad \tilde{e}^\mu = g^{\mu\nu} \tilde{e}_\nu \quad \text{and} \quad \tilde{x} = x_\mu \tilde{e}^\mu.$$

We have well, by graphically calculating scalar products : $\tilde{e}_0 \cdot \tilde{e}_1 = \left(\vec{e}_0 \cdot s\left(\vec{e}_1\right)\right)_{Euclid} = 0 = g_{10}$.

Also $\tilde{e}_0 \cdot \tilde{e}_0 = \vec{e}_0 \cdot \vec{e}_0 = 1$, $\tilde{e}_1 \cdot \tilde{e}_1 = -\vec{e}_1 \cdot \vec{e}_1 = -1$ then

$g_{00}=1$ and $g_{11}=-1$. $\tilde{e}_0$ pseudo-norm worth 1 and $\tilde{e}_1$ pseudo-norm worth -1.

2D metric : $g_{\mu\nu}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$.

For the covariant components :

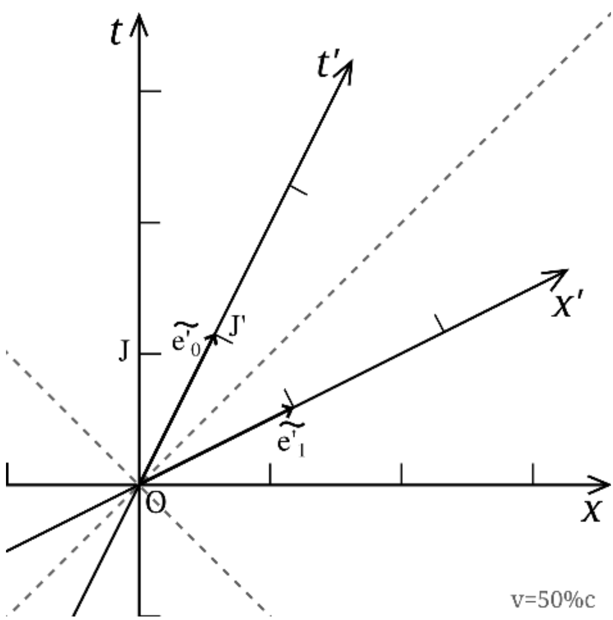
$$x_0=g_{00}x^0+g_{01}x^1=x^0 \quad \text{and} \quad x_1=g_{10}x^0+g_{11}x^1=-x^1$$

$$\tilde{e}^0=g^{00}\tilde{e}_0+g^{01}\tilde{e}_1=\tilde{e}_0 \quad \text{and} \quad \tilde{e}^1=g^{10}\tilde{e}_0+g^{11}\tilde{e}_1=-\tilde{e}_1$$

$$\tilde{x}=x_0\tilde{e}^0+x_1\tilde{e}^1=x^0\tilde{e}_0+x^1\tilde{e}_1=x_0\tilde{e}_0-x_1\tilde{e}_1$$

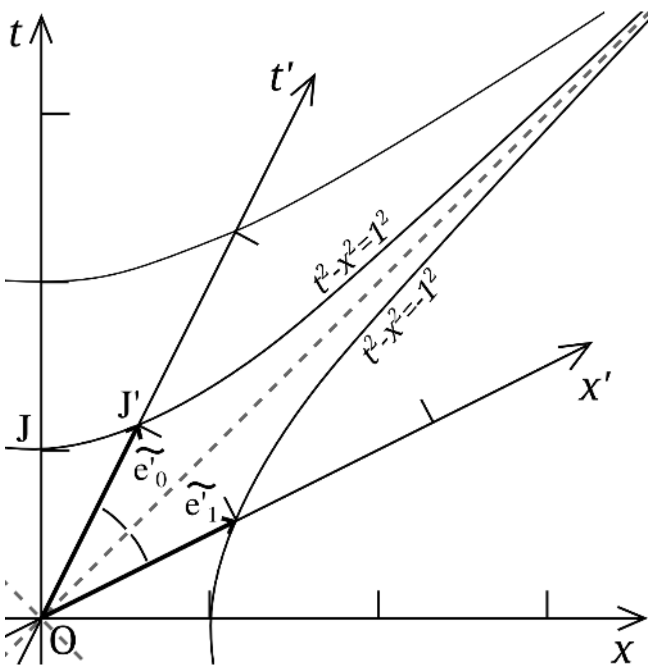
- **Reference frame *R'***

Let's now take the case of the inertial frame *R'* seen from *R* :



An unwarned Euclidean glance would naively see a non-orthogonal coordinate system, and, basis vectors longer than one. It is not so, the basis vectors are well orthogonal because they are symmetrical with respect to the worldline of a photon, and, besides, the time vector of the bases of *R'* is along the unit hyperbola and, therefore, of pseudo-norm 1, the space vector is along the hyperbola corresponding to a pseudo-norm -1. The metric is thus the same as for *R*, which is to be expected because there is no privileged inertial frame of reference:

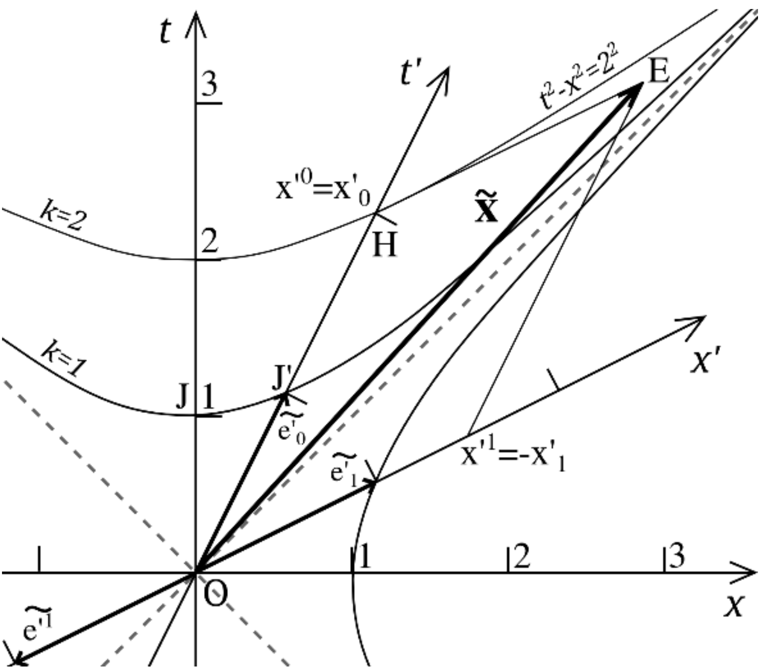
$$g'_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



For the covariant components and the contravariant basis, we necessarily have the same relationships as for R :

$$x'_0 = x'^0, \quad x'_1 = -x'^1, \quad \tilde{e}'^0 = \tilde{e}'_0 \quad \text{and} \quad \tilde{e}'^1 = -\tilde{e}'_1.$$

$$\tilde{x} = x'^0 \tilde{e}'_0 + x'^1 \tilde{e}'_1 = x'_0 \tilde{e}'^0 + x'_1 \tilde{e}'^1$$



$$\tilde{x} = 2\tilde{e}'_0 + \frac{3}{2}\tilde{e}'_1 = 2\tilde{e}'^0 - \frac{3}{2}\tilde{e}'^1 = \frac{11}{2\sqrt{3}}\tilde{e}_0 + \frac{5}{\sqrt{3}}\tilde{e}_1$$

Change of coordinates

We can switch from a system of *n* coordinates x^i to a new system of *n* coordinates x'^i, where each of the *n* coordinates x'^i depend on the *n* coordinates x^i:

$$x'^i\left(x^1,\dots,x^2,\dots,x^n\right)$$

We have a function with *n* variables. For a function *f* with two variables, we add the variations in both directions :

$$df\,(x,y)=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy$$

When we move from $M(x,y)$ to $M'(x+dx,y+dy)$, infinitely close, the function *f* varies by df.

The generalization gives : $df\,(x^i)=\sum_{i=1}^{n}\frac{\partial f}{\partial x^i}dx^i$.

Then $d\,x'^i=\frac{\partial x'^i}{\partial x^j}dx^j$ and $d\,x^i=\frac{\partial x^i}{\partial x'^j}d\,x'^j$.

We note : $\Lambda^i{}_j=\frac{\partial x'^i}{\partial x^j}$ and $\Lambda_j{}^i=\frac{\partial x^i}{\partial x'^j}$.

These two tensors are used to switch from one coordinate system to the other, they are the change of basis matrices. The superscript indices correspond to the rows and the subscript indices to the columns.

Let's do the product of the two matrices[33] :

$$\Lambda^i{}_k \Lambda_j{}^k = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i_j \,.$$

The matrices are inverse to each other :

$$\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda} = \boldsymbol{I}$$

The covariant components of a vector are transformed according to $\boldsymbol{\Lambda}$, and the contravariant components according to $\boldsymbol{\Lambda}^{-1}$. This is where the famous name comes from. The same is true for the base vectors :

$$v'_i = \Lambda_i{}^j v_j \qquad v'^i = \Lambda^i{}_j v^j \qquad v_i = \Lambda^j{}_i v'_j \qquad v^i = \Lambda_j{}^i v'^j$$

$$\boldsymbol{e}'_i = \Lambda_i{}^j \boldsymbol{e}_j \qquad \boldsymbol{e}'^i = \Lambda^i{}_j \boldsymbol{e}^j \qquad \boldsymbol{e}_i = \Lambda^j{}_i \boldsymbol{e}'_j \qquad \boldsymbol{e}^i = \Lambda_j{}^i \boldsymbol{e}'^j$$

We can easily verify that the scalar product of two vectors is invariant by basis change :

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_i B^i = \Lambda^j{}_i A'_j \Lambda_k{}^i B'^k = \delta^j_k A'_j B'^k = A'_j B'^j$$

Also if two n-vectors are equal, they are still equal after changing the coordinate system:

$$A^i = B^i \;\Rightarrow\; \Lambda^i{}_k A^k = \Lambda^i{}_k B^k \;\Rightarrow\; A'^i = B'^i \;\Rightarrow\; \boldsymbol{A} = \boldsymbol{B}$$

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33 Some additional mathematical tools :

If $x'(x,y)$ and $y'(x,y)$ then $\dfrac{\partial x'}{\partial y'} = \dfrac{\partial x'}{\partial x}\dfrac{\partial x}{\partial y'} + \dfrac{\partial x'}{\partial y}\dfrac{\partial y}{\partial y'}$.

Generalized : $\dfrac{\partial x'^i}{\partial x'^j} = \dfrac{\partial x'^i}{\partial x^k}\dfrac{\partial x^k}{\partial x'^j}$ and $\dfrac{\partial f}{\partial x'^j} = \dfrac{\partial f}{\partial x^k}\dfrac{\partial x^k}{\partial x'^j}$.

Fundamental properties for constructing physical laws, whether in classical mechanics, special relativity or general relativity.

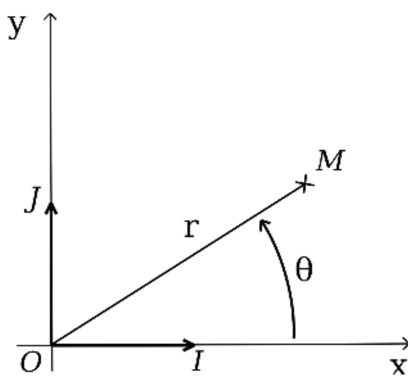
Let's look for the new metric:

$$g'_{ij} = \boldsymbol{e}'_i \cdot \boldsymbol{e}'_j = \Lambda_i^{\ k} \boldsymbol{e}_k \cdot \Lambda_j^{\ l} \boldsymbol{e}_l = \Lambda_i^{\ k} \Lambda_j^{\ l} g_{kl}$$

In general, the change of basis matrix is applied as many times as there are indices on a tensor. For example, on the Riemann curvature tensor :

$$R'^{\alpha}_{\ \beta\gamma\delta} = \Lambda^{\alpha}_{\ \mu} \Lambda_{\beta}^{\ \nu} \Lambda_{\gamma}^{\ \rho} \Lambda_{\delta}^{\ \lambda} R^{\mu}_{\ \nu\rho\lambda}$$

- **Rotation in Euclidean geometry :**



$$\begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta \end{cases}$$

$$x'^1 = x(x^1 = r\,;\, x^2 = \theta)$$

$$x'^2 = y$$

$$\Lambda^1_{\ 1} = \frac{\partial x}{\partial r} = \cos\theta \qquad \Lambda^1_{\ 2} = \frac{\partial x}{\partial \theta} = -r\sin\theta$$

$$\Lambda^2_{\ 1} = \frac{\partial y}{\partial r} = \sin\theta \qquad \Lambda^2_{\ 2} = \frac{\partial y}{\partial \theta} = r\cos\theta$$

then $\boldsymbol{\Lambda} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$

$$\Lambda_1{}^1 = \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta \quad \text{because} \quad r = \sqrt{x^2 + y^2}$$

$$\Lambda_1{}^2 = \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r} \quad \text{as} \quad \frac{\partial \theta}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial \theta}{\partial y}\frac{\partial y}{\partial \theta} = 1$$

$$\Lambda_2{}^2 = \frac{\partial \theta}{\partial y} = \frac{1/x}{1 + y^2/x^2} = \frac{\cos\theta}{r} \qquad \Lambda_2{}^1 = \frac{\partial r}{\partial y} = \sin\theta$$

$$\text{finally :} \quad \boldsymbol{\Lambda}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

we well have $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda} = \boldsymbol{I}$.

$$\boldsymbol{e}_1 = \vec{e}_r = \Lambda^1{}_1 \boldsymbol{e}'_1 + \Lambda^2{}_1 \boldsymbol{e}'_2 = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$$

$$\boldsymbol{e}_2 = \vec{e}_\theta = \Lambda^1{}_2 \boldsymbol{e}'_1 + \Lambda^2{}_2 \boldsymbol{e}'_2 = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$$

The basis $(\vec{e}_r, \vec{e}_\theta)$ is orthogonal and not normalized. For an orthonormal basis we have the unit vectors as follows $\vec{e}_r = \vec{u}_r$ and $\vec{e}_\theta = r\vec{u}_\theta$.

$$\text{Metrics :} \quad g'_{ij} = \begin{pmatrix} \vec{i}\cdot\vec{i} & \vec{i}\cdot\vec{j} \\ \vec{j}\cdot\vec{i} & \vec{j}\cdot\vec{j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and} \quad g_{ij} = \Lambda^k{}_i \Lambda^l{}_j g'_{kl} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

for example $g_{22} = \Lambda^1{}_2 \Lambda^1{}_2 g'_{11} + \Lambda^2{}_2 \Lambda^2{}_2 g'_{22} + 0 + 0 = r^2$

Invariant length element :

$$dl^2 = \vec{dl} \cdot \vec{dl} = d\,x'_i\, d\,x'^{\,i} = g'_{ij}\, d\,x'^i\, d\,x'^j = dx^2 + dy^2$$
$$dl^2 = d\,x_i\, d\,x^i = g_{ij}\, d\,x^i\, d\,x^j = dr^2 + r^2 d\,\theta^2$$

Vector components : $\vec{v}(v^x, v^y)$

$$v^1 = v^r = \Lambda_1{}^1 v'^1 + \Lambda_2{}^1 v'^2 = \cos\theta\, v^x + \sin\theta\, v^y$$

$$v^2 = v^\theta = \Lambda_1{}^2 v'^1 + \Lambda_2{}^2 v'^2 = -\frac{\sin\theta}{r} v^x + \frac{\cos\theta}{r} v^y$$

we well have $\vec{v}\cdot\vec{v} = g_{ij} v^i v^j = (v^x)^2 + (v^y)^2 = g'_{ij} v'^i v'^j$

- **Lorentz transformation :** $\begin{cases} ct'(ct, x) = \gamma(ct - \beta x) \\ x'(ct, x) = \gamma(x - \beta ct) \end{cases}$

$$x'^0 = ct' \left(x^0 = ct \,;\, x^1 = x\right) \qquad x'^1 = x'$$

$$\Lambda^0{}_0 = \frac{\partial\, ct'}{\partial\, ct} = \gamma \qquad \Lambda^0{}_1 = \frac{\partial\, ct'}{\partial\, x} = -\gamma\beta$$

$$\Lambda^1{}_0 = \frac{\partial\, x'}{\partial\, ct} = -\gamma\beta \qquad \Lambda^1{}_1 = \frac{\partial\, x'}{\partial\, x} = \gamma$$

then $\quad \mathbf{\Lambda} = \Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$

Inverse standard Lorentz boost : $\begin{cases} ct = \gamma(ct' + \beta x') \\ x = \gamma(x' + \beta ct') \end{cases}$

Then : $\mathbf{\Lambda}^{-1} = \Lambda_\nu{}^\mu = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$ and $\mathbf{\Lambda}\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1}\mathbf{\Lambda} = \boldsymbol{I}$.

Basis vectors :

$$\tilde{e}_0=\tilde{e}_t=\Lambda^0{}_0\tilde{e}'_0+\Lambda^1{}_0\tilde{e}'_1 \quad \text{and} \quad \tilde{e}_t=\gamma(\tilde{e}_{t'}-\beta\tilde{e}_{x'})$$

$$\tilde{e}_0=\tilde{e}_t=\Lambda^0{}_0\tilde{e}'_0+\Lambda^1{}_0\tilde{e}'_1 \quad \text{and} \quad \tilde{e}_x=\gamma(-\beta\tilde{e}_{t'}+\tilde{e}_{x'})$$

also $\tilde{e}_{t'}=\gamma(\tilde{e}_t+\beta\tilde{e}_x)$ and $\tilde{e}_{x'}=\gamma(\beta\tilde{e}_t+\tilde{e}_x)$

For the Minkowski diagrams, we find the results given on page 42 and following. On a Euclidean sheet of paper the vector $\tilde{e}_{t'}$ appears longer than $\tilde{e}_t$: $\|\tilde{e}_{t'}\|_{Euclid}=\gamma\sqrt{1+\beta^2}$.

Apparent angle : $\widehat{(\tilde{e}_t,\tilde{e}_{t'})}_{Euclid}=\arctan\beta$.

$$\text{Metrics : } g_{\mu\nu}=\begin{pmatrix}\tilde{e}_t\cdot\tilde{e}_t & \tilde{e}_t\cdot\tilde{e}_x\\ \tilde{e}_x\cdot\tilde{e}_t & \tilde{e}_x\cdot\tilde{e}_x\end{pmatrix}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\tilde{e}_{t'}\cdot\tilde{e}_{t'}=\gamma^2(\tilde{e}_t\cdot\tilde{e}_t+2\beta\tilde{e}_t\cdot\tilde{e}_x+\beta^2\tilde{e}_x\cdot\tilde{e}_x)=1$$

and so on, hence $g'_{\mu\nu}=\Lambda_\mu{}^\alpha\Lambda_\nu{}^\beta g_{\alpha\beta}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$

The metric remains the same.

The invariant ds^2 : $\quad ds^2=g_{\mu\nu}dx^\mu dx^\nu=c^2dt^2-dx^2$

$$=g'_{\mu\nu}dx'^\mu dx'^\nu=c^2dt'^2-dx'^2$$

Vector components : $\tilde{v}(v^t,v^x)$

$$v'^0=v^{t'}=\Lambda^0{}_0v^0+\Lambda^0{}_1v^1=\gamma(v^t-\beta v^x)$$

$$v'^1=v^{x'}=\Lambda^1{}_0v^0+\Lambda^1{}_1v^1=\gamma(-\beta v^t+v^x)$$

We find the Lorentz transformation that applies to any four-vector.

Also : $\widetilde{v}\cdot\widetilde{v}=g_{\mu\nu}v^{\mu}v^{\nu}=(v^{t})^{2}-(v^{x})^{2}=(v^{t'})^{2}-(v^{x'})^{2}$

And the scalar product is well invariant :

$$\begin{aligned}
&\widetilde{u}\cdot\widetilde{v}=g_{\mu\nu}u'^{\mu}v'^{\nu}=u'^{0}v'^{0}-u'^{1}v'^{1}-u'^{2}v'^{2}-u'^{3}v'^{3}\\
&=\gamma^{2}(u^{0}-\beta u^{1})(v^{0}-\beta v^{1})-\gamma^{2}(u^{1}-\beta u^{0})(v^{1}-\beta v^{0})-u^{2}v^{2}-u^{3}v^{3}\\
&=\gamma^{2}(1-\beta^{2})u^{0}v^{0}+0+0-\gamma^{2}(1-\beta^{2})u^{1}v^{1}-u^{2}v^{2}-u^{3}v^{3}\\
&=u^{0}v^{0}-u^{1}v^{1}-u^{2}v^{2}-u^{3}v^{3}=g_{\mu\nu}u^{\mu}v^{\nu}
\end{aligned}$$

For all 4-vectors we have the standard Lorentz transformation :

$$\begin{cases}
v^{t'}=\gamma\left(v^{t}-\beta v^{x}\right)\\
v^{x'}=\gamma\left(v^{x}-\beta v^{t}\right)\\
v^{y'}=v^{y}\\
v^{z'}=v^{z}
\end{cases}$$

The change of basis lambda matrices :

$$\mathbf{\Lambda}=\Lambda^{\mu}{}_{\nu}=\begin{pmatrix}\gamma & -\gamma\beta & 0 & 0\\ -\gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

$$\mathbf{\Lambda^{-1}}=\Lambda_{\nu}{}^{\mu}=\begin{pmatrix}\gamma & \gamma\beta & 0 & 0\\ \gamma\beta & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}$$

∞ Four-velocity

After building a new geometry of space and time, let us build the new physics associated with it. The position vector and universal time have been replaced by the four-vector $\tilde{x}$. What about the other physical quantities introduced by Newton: velocity, acceleration, momentum, energy, force, etc?

First of all, we are looking for quantities that transform according to Lorentz's transformation, then we will establish laws that give back the classical mechanics at low speeds, and of course, the supreme criterion, the experimental verification will finalize the selection.

We will construct the covariant velocity from the four-vector x^{μ}. We resume the classical approach which allows to build a vector tangent to the trajectory of an object. For two infinitely close events on a worldline, we have the infinitesimal 4-vector :

$$d\tilde{x}=\widetilde{EE'}=\tilde{x}(E')-\tilde{x}(E).$$

To define the velocity, simply divide by the duration, just as infinitesimal, which separates these two events. Of course, in Newton's mechanics, there is no hesitation to have, on the other hand, in special relativity, we have the duration dt measured in the same frame of reference as the dx^{μ}, or, the duration $d\tau$ measured in the proper reference frame of the moving object. No hesitation because $d\tau$ is the only duration invariant by the Lorentz transformation[34], hence the expression of the four-vector velocity :

$$\widetilde{u}=\frac{d\widetilde{x}}{d\tau} \quad \text{and} \quad u^{\mu}=\frac{dx^{\mu}}{d\tau}$$

For the three spatial components, we find well the classical velocity $\vec{v}$ at low speeds :

$$\widetilde{u}=(\gamma c, \gamma\vec{v})$$

$$\text{with} \quad \gamma=\frac{1}{\sqrt{1-\beta^2}}, \quad \beta=\frac{v}{c}, \quad v=\|\vec{v}\|,$$

$$\gamma(v)=\frac{dt}{d\tau}, \quad v^i=\frac{dx^i}{dt} \quad \text{and} \quad \vec{v}=(v^1, v^2, v^3).$$

This four-velocity transforms well according to the Lorentz transformation given on page 205, which was not the case for the classical velocity (easy to convince oneself by looking at the relations on page 362).

For example, along the *x* axis : $u^x=\frac{dx}{d\tau}=\gamma v^x$.

To think about relativity, it seems logical to reason with the velocity provided by this same theory, and not with that of Newton. But as with the notion of absolute space and absolute time, habits are tenacious, and it must be noted that Newton's velocity makes resistance.

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34 $d\tau$ is obtained by doing the scalar product of two four-vectors, it is therefore invariant by the Lorentz transformation : $d\widetilde{x}\cdot d\widetilde{x}=g_{\mu\nu}dx^{\mu}dx^{\nu}=c^2dt^2-dl^2=c^2d\tau^2$

"You can't go faster than the speed of light" we hear. Everything would then happen as if there were a forbidden zone from *c* to infinity. We don't like the prohibitions, and neither does nature, it seems to realize everything that is possible. So, not supporting limits, in this supposedly inaccessible zone, we put strange particles, *tachions*, particles that would always have been faster than light... except that these tachions violate causality, a basic principle in physics.

Let's think differently, let's use the right definition for velocity, the one that respects the symmetries of spacetime. When you give each time more energy to a particle to accelerate it, it gains speed and its velocity tends towards infinity :

$$v_{Newton} = \frac{dx}{dt} \rightarrow c, \quad \gamma \rightarrow \infty \quad \text{and} \quad v_{Einstein} = \frac{dx}{d\tau} \rightarrow \infty.$$

The prohibited zone no longer exists!

Let's take again the example of the journey for Proxima. From the Earth the astronaut travels 4 ly, his journey lasts 3 years, and 5 years for the Earthlings. Sometimes I hear "but he goes faster than light!". He is going well, slower than light, he arrives after a ray of light, and in the ship's frame of reference he has traveled a distance of only 2.4 ly. But it is interesting to note that the person finally refers to the covariant velocity $u = \Delta x / \Delta \tau = 4/3\ c$, and, in terms of covariant velocity, that of light is infinite. Finally, we are not so limited as that, at speeds close to *c* we find ourselves on the other side of the galaxy very quickly. For example, an ultra-relativistic electron can travel 100,000 ly in one year (in its own frame of reference!).

The temporal component of $\widetilde{u}$ is always positive, the four-velocity is always directed towards the future.

Let's calculate the pseudo-norm :

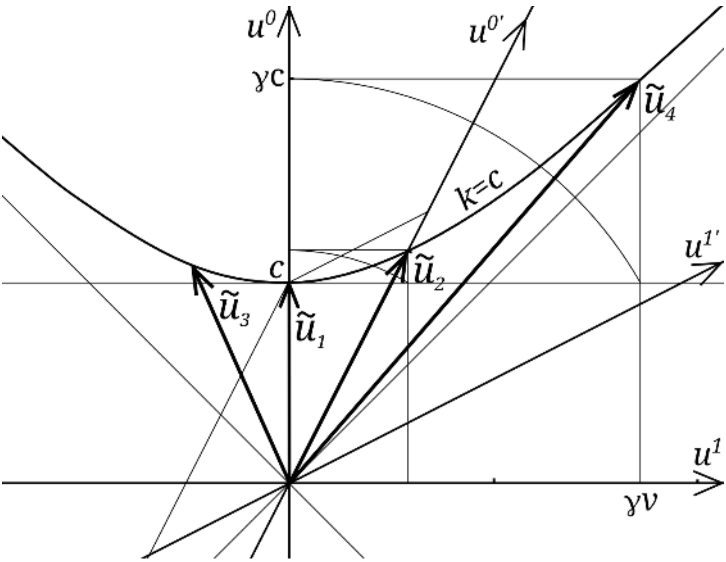
$$\widetilde{u} \cdot \widetilde{u} = \gamma^2 c^2 - \gamma^2 v^2 = c^2 > 0$$

The 4-velocity is a time-like vector whose end is located on the upper branch of the *c* parameter hyperbola. The 4-velocity cannot be null. For a particle at rest there is only the time component which corresponds, in a way, to the speed of the flow of time.

Particle at rest : $\widetilde{u} = (c, \vec{0})$.

Particle in motion : $\widetilde{u} = \gamma c (1, \vec{\beta})$.

Minkowski Diagram for the 4-velocity :



$\tilde{u}_1$: *relativistic velocity of an object at rest in R. The vector is vertical.*

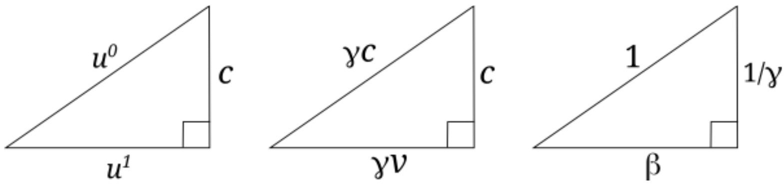
$\tilde{u}_2$: *4-velocity of an object moving to the right. The tip is on the hyperbola of parameter c. The corresponding gamma is 1.15 and v=50%c.*

$\tilde{u}_3$: *4- velocity of an object moving to the left.*

$\tilde{u}_4$: *The more gamma increases, the closer the velocity vector gets to the asymptote and the light cone.*

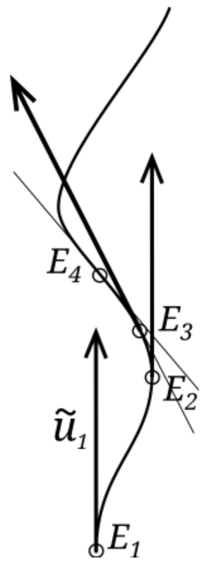
We have built the frame where the particle 2 is motionless. By projecting the tip of $\tilde{u}_1$ *into R', we obtain a particle 1 that moves to the left at 50 % of c.*

The velocity triangle : $\tilde{u}\cdot\tilde{u}=(u^t)^2-(u^x)^2=c^2$



(Triangles for $\gamma=2$ *and* $\beta=\sqrt{3}/2$*)*

Here is the worldline of a particle. The velocity is always tangent to the worldline and contained in the future light cone. In E_1 *the tangent is vertical, the particle is at rest, then it starts moving to the right, slows down and stops further to the right in* E_2*. It resumes its motion to the left, accelerates and reaches its maximum speed at the point of inflection in* E_4*.*



∞ FOUR-ACCELERATION

The approach is of course quite similar:

$$\tilde{w}=\frac{d\tilde{u}}{d\tau} \quad \text{and} \quad w^{\mu}=\frac{du^{\mu}}{d\tau}$$

As for the 4-velocity, we do not use the classical notations so that the differences appear without ambiguity : $\tilde{w}$ for the 4-acceleration and $\vec{a}$ for the Newton acceleration.
To begin with, we have a nice property, 4-velocity and 4-acceleration are orthogonal vectors :

$$\frac{d}{d\tau}(\tilde{u}\cdot\tilde{u})=0=\frac{d\tilde{u}}{d\tau}\cdot\tilde{u}+\tilde{u}\cdot\frac{d\tilde{u}}{d\tau} \quad \text{then} \quad \tilde{u}\cdot\tilde{w}=0.$$

As we have established the link between $\tilde{u}$ and $\vec{v}$, we are going to make the link between $\tilde{w}$ and $\vec{a}$. There, however, the link will be much less immediate and the calculations are longer:

$$\tilde{w}=\frac{d\tilde{u}}{d\tau}=\left(\frac{d\gamma}{d\tau}c\,,\frac{d\gamma}{d\tau}\vec{v}+\gamma\frac{d\vec{v}}{d\tau}\right)$$

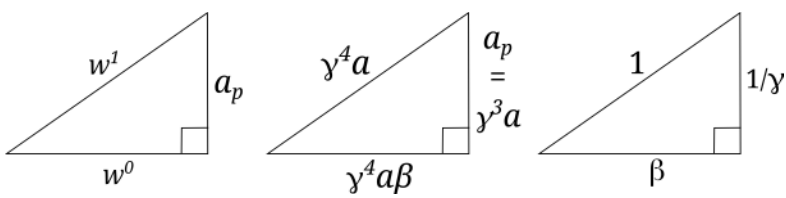
after calculation $\frac{d\gamma}{dt}=\frac{\gamma^3}{c^2}\vec{a}\cdot\vec{v}$ with $\vec{\beta}=\frac{\vec{v}}{c}$

we have $\tilde{w}=(\gamma^4\vec{a}\cdot\vec{\beta}\ ,\ \gamma^4(\vec{a}\cdot\vec{\beta})\vec{\beta}+\gamma^2\vec{a})$

Now let's determine the pseudo-norm of $\tilde{w}$. The scalar product is the same in all inertial frames of reference. We then place ourselves in the inertial frame of reference which coincides at a given moment with the proper frame of reference. In this coinciding reference frame, by definition, $\vec{v}=\vec{0}$ at $t=0$. Thus $\widetilde{w}=(0,\vec{a}(0))$ and $\widetilde{w}\cdot\widetilde{w}=-a_p^2$, where a_p is the acceleration felt in the proper frame of reference. All inertial observers will agree on the value of the proper acceleration a_p. The 4-acceleration is a space-like vector, in accordance with the orthogonality with the 4-velocity.

In the Minkowski plane $(w^0)^2-(w^1)^2=-a_p^2$ and $\widetilde{w}$ is placed on a space-like hyperbola of parameter a_p.

The acceleration triangle:



For one-dimensional motion : $\widetilde{w}=\gamma\, a_p(\pm\beta,\pm 1)$

Generally speaking, one can always place oneself locally in an inertial reference frame that contains the worldline in a Minkowski plane coinciding on a portion. We then have an osculating hyperbola that allows us to determine the proper acceleration.

∘ A look back on the trip to Proxima

We are on a particular case of rectilinear motion at constant proper acceleration, where the worldline of the rocket corresponds with the hyperbola of parameter *g*.

We will elegantly retrieve the expressions of the page 116.

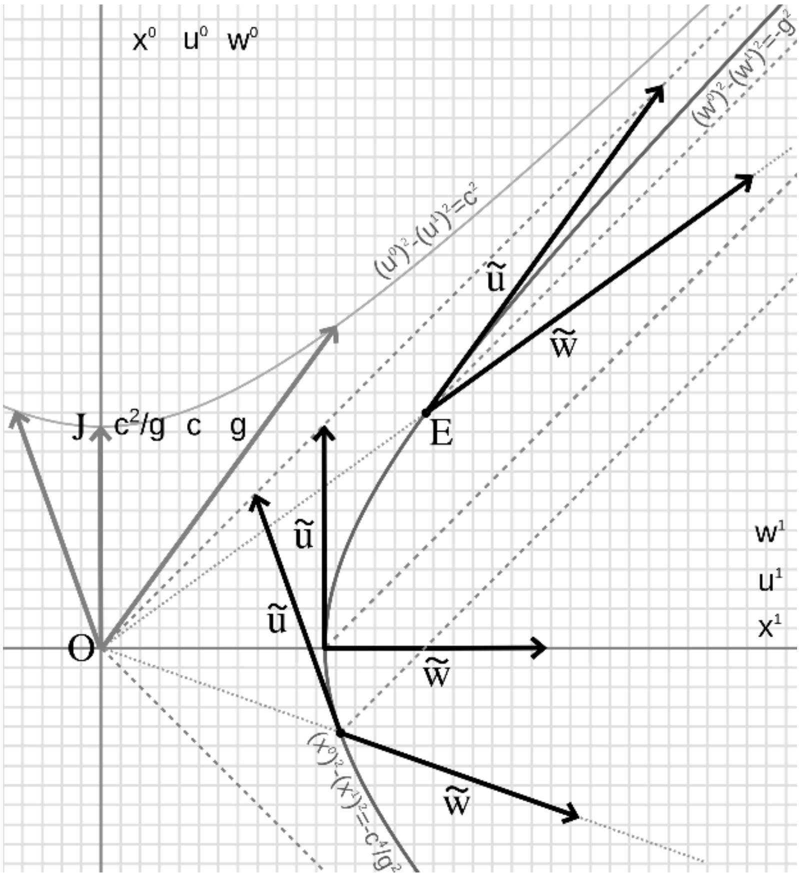
In the coinciding inertial reference frame $\tilde{w}=(0,g)$.

We perform a Lorentz transformation to obtain the coordinates of this same acceleration in the terrestrial frame of reference:

$$\tilde{w}=(\gamma\beta g,\gamma g),$$

as $\gamma\beta g=\gamma^4\vec{a}\cdot\vec{\beta}$ we have $a(t)=\frac{dv}{dt}=\frac{g}{\gamma^3}$

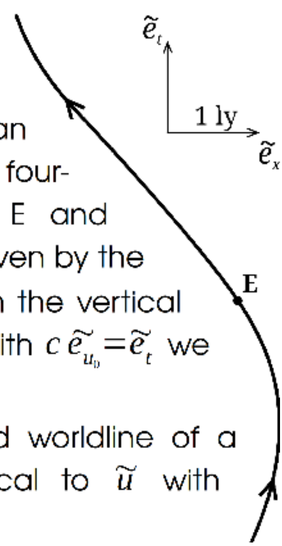
after integration we find the expressions for $v(t)$ and $x(t)$.



We have represented the Minkowski diagrams for the three four-vectors $\widetilde{x}$, $\widetilde{u}$ and $\widetilde{w}$. We have made an appropriate choice of units so that the hyperbolas correspond: OJ worth c^2/g for the 4-position, c for the 4-velocity and g for the 4-acceleration. We study the uniformly accelerated motion in its generality, both for positive and negative t : in the latter case $\vec{v}$ and $\vec{a}$ are in opposite directions, the rocket decelerates, and $\widetilde{w}=(-\gamma\beta g,\gamma g)$. For this motion, the rocket worldline is a *hyperbole branch of equation* $c^2 t^2 - x^2 = -c^4 / g^2$ *which coincides here with the space-like hyperbole branch of* $\tilde{w}$ *The hyperbola branch of* $\tilde{u}$ *is simply rotated by 90°. For any event E of our worldline,* $\tilde{u}$ *and* $\tilde{w}$ *are as it should be symmetrical with respect to the bisectors, but, in this particular situation, they appear, moreover, of the same length on our Euclidean sheet. Indeed we have in this case* $\tilde{u}/c = \gamma(1,\beta)$ *and* $\tilde{w}/g = \gamma(\pm\beta, 1)$*. The drawing is very simple, for any event E, you draw the line (OE),* $\tilde{w}$ *corresponds with* $\widetilde{OE}$*, and* $\tilde{u}$ *is the symmetrical with respect to the photon worldline. Although the 4-acceleration remains constantly on the spacelike hyperbola of parameter g, on the diagram, the Euclid's length of the relativistic acceleration* $\tilde{w}$ *increases with* γ*, while that of the classical acceleration* $\vec{a}$ *decreases in* γ^3*.*

○ Geometric determination of 4-acceleration

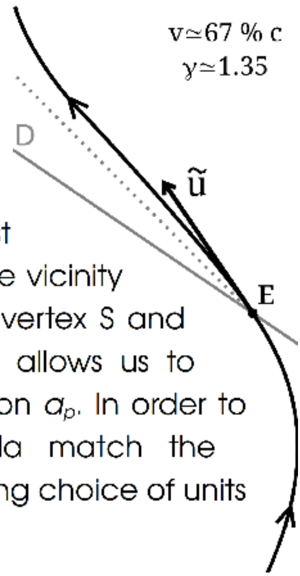
• From the worldline :



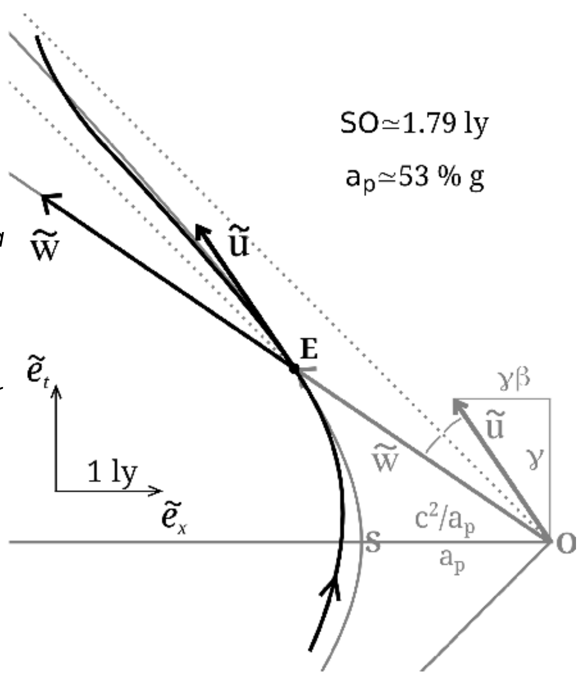
All the information is available in this line. For any E event we can determine the four-velocity and four-acceleration. $\tilde{u}$ is tangent at E and directed towards the future. β is given by the arctangent of the angle between the vertical and $\tilde{u}$. By adapting the scales with $c\,\tilde{e}_{u_0} = \tilde{e}_t$ we can carry out the plot.

We then placed in E the dotted worldline of a photon. The line D is symmetrical to $\tilde{u}$ with respect to the photon.

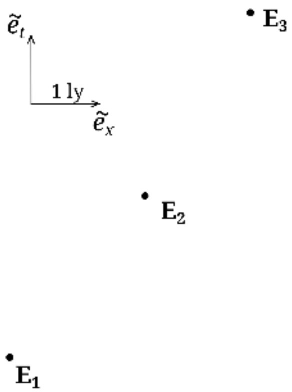
As $\widetilde{w}$ is orthogonal to $\widetilde{u}$ its end is necessary on D. As the worldline continues below $\widetilde{u}$, there is acceleration and $\widetilde{w}$ is upwards. We place the osculating hyperbola that best coincides with the worldline in the vicinity of E. The distance between the vertex S and the center O of the hyperbola allows us to determine the proper acceleration a_p. In order to make the osculating hyperbola match the acceleration we have the following choice of units $a_p \widetilde{e_{w_1}} = c^2 / a_p \widetilde{e_x} = \widetilde{SO}$.



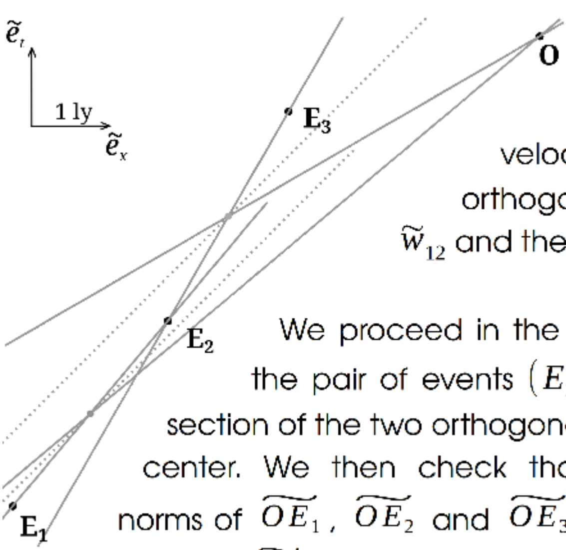
For any event of a world line, there is always a tangent hyperbola unique that gives the proper acceleration.



- **From three close events :**

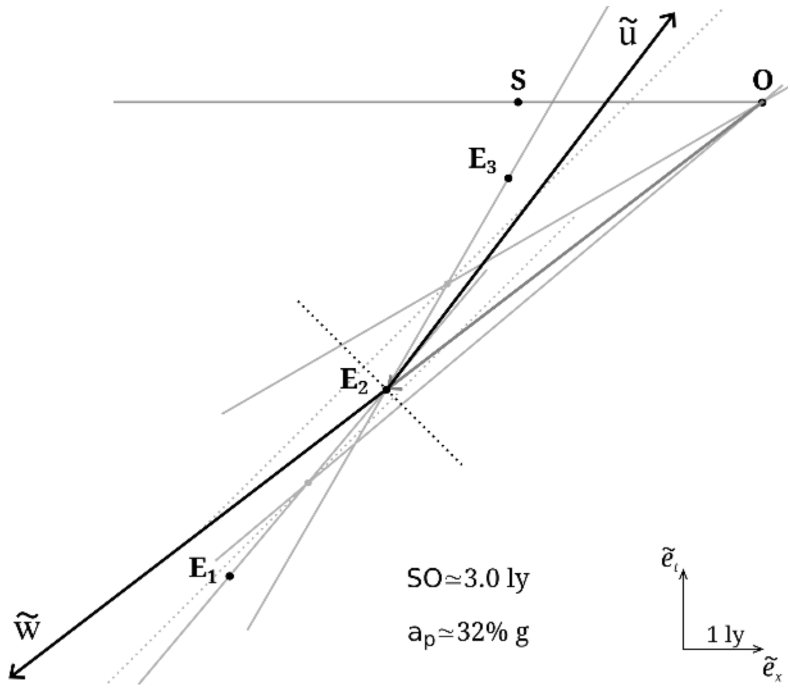


Previously, it was not easy to determine the tangent hyperbole. Here, from three events we will find the osculating hyperbola optimized for the midpoint E_2. We know that three points determine a single hyperbola. The approach is the same in Euclidean geometry, if you have three points of a circle you find the osculating circle using two perpendicular bisectors whose intersection provides the center of the circle. Here again the tangents are orthogonal to the radii.



$\widetilde{E_1 E_2}$ is collinear to the average velocity $\tilde{u}_{12}$, and the orthogonal line contain $\tilde{w}_{12}$ and the center O of the hyperbola.

We proceed in the same way with the pair of events (E_2, E_3). The intersection of the two orthogonal lines gives the center. We then check that the pseudo-norms of $\widetilde{OE_1}$, $\widetilde{OE_2}$ and $\widetilde{OE_3}$ are equal. We then have $\widetilde{OS}$, the parameter k_x of the hyperbola, and the proper acceleration $a_p = -c^2 / \overline{SO}$.



∞ Mass-energy equivalence

Let us look for the relativistic equivalent of the Newton's second law. In classical mechanics :

$$m\vec{a}=\vec{F} \quad \text{or} \quad \frac{d\vec{p}}{dt}=\vec{F}$$

with the momentum $\vec{p}=m\vec{v}$

We will also need the kinetic power theorem :

$$P_k=\frac{d\,E_k}{dt}=\vec{F}\cdot\vec{v}$$

◦ Four-momentum

The mass is a property specific to a particle, it does not depend on the frame of reference. It thus seems natural to consider the four-vector $\tilde{p}=m\,\tilde{u}$.

For the 4-momentum we keep the letter *p* because contrary to the 4-velocity or the 4-acceleration, this one has been directly adopted in the scientific mores. Its spatial part is commonly called *momentum* and the 4-vector as a whole can be called the 4-momentum or more precisely the 4-vector energy-momentum: $\tilde{p}=(m\gamma c, m\gamma\vec{v})$.

$\boxed{\tilde{p}=(E/c,\vec{p})}$ with $\boxed{E=m\gamma c^2}$ and $\boxed{\vec{p}=m\gamma\vec{v}}$

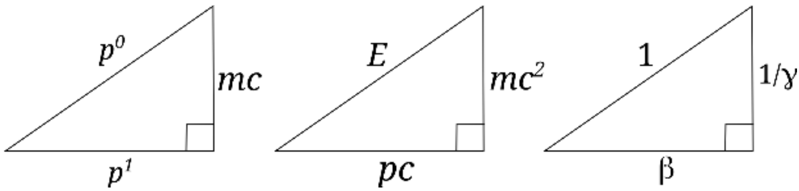
The temporal component shows a quantity with the units of an energy. Let's find out what this energy corresponds to. In the coinciding reference frame $\tilde{p}=(mc,\vec{0})$ and $\tilde{p}\cdot\tilde{p}=m^2c^2$. In the observational frame $\tilde{p}\cdot\tilde{p}=E^2/c^2-\vec{p}^2$. In the proper frame, where the particle is at rest, $\tilde{p}\cdot\tilde{p}=E_r^2/c^2$, then $E_r=mc^2$.

A completely new notion, absent in classical mechanics, appears, an energy is associated with the mass of an object. Even at rest, a particle has an energy, it is an energy of mass.

When the particle is in motion :

$$m^2c^2=E^2/c^2-\vec{p}^2 \quad \text{and} \quad E^2=(mc^2)^2+(pc)^2.$$

The Energy-Momentum Triangle :



E corresponds to the total energy of the particle, which includes its mass energy and its kinetic energy:

$$E^2 = m^2 c^4 + p^2 c^2 = m^2 c^4 + m^2 \gamma^2 v^2 c^2 = m^2 \gamma^2 c^4$$

and we find: $E = m \gamma c^2$

For the kinetic energy: $E_k = E - E_r$.

At low speeds:

$$E = m(1-\beta^2)^{-1/2} c^2 \simeq mc^2 + \frac{1}{2} m v^2$$

We find again the classical expression of kinetic energy.

For a massless particle, like a photon, $E = pc$, $\tilde{p} = (p, \vec{p})$ and $\tilde{p} \cdot \tilde{p} = 0$.

- **Four-force**

For the 4-force $\tilde{g}$ we suggest :

$$\frac{d\tilde{p}}{d\tau} = \tilde{g}$$

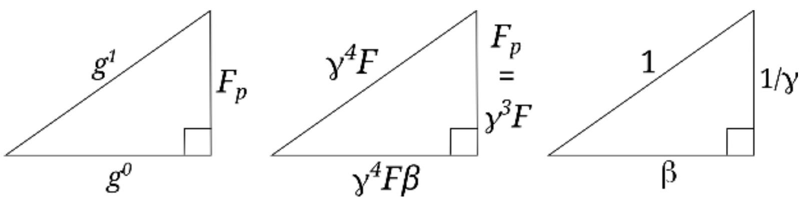
Equation covariant with respect to the Lorentz transformation. In the classical limit, the temporal part gives back the kinetic power theorem, and the spatial part gives the Newton's second law:

$$\frac{d\widetilde{p}}{d\tau}=m\widetilde{w}=\left(\gamma^4\vec{F}\cdot\vec{\beta}\ ,\ \gamma^4(\vec{F}\cdot\vec{\beta})\vec{\beta}+\gamma^2\vec{F}\right)=\widetilde{g}$$

The link between 4-force and Newton's force is not obvious. Classically, the force $\vec{F}$ is collinear and has the same direction as acceleration $\vec{a}$, in relativity it is the case for $\widetilde{g}$ and $\widetilde{w}$.

Pseudo-norm : $\widetilde{g}\cdot\widetilde{g}=-F_p^2$ with $F_p=ma_p$.

Force Triangle :



For one-dimensional motion : $\widetilde{g}=\gamma F_p(\pm\beta,\pm1)$.

For the spatial part : $\dfrac{d\vec{p}}{d\tau}=\vec{g}$ and $\dfrac{d\vec{p}}{dt}=\dfrac{\vec{g}}{\gamma}$.

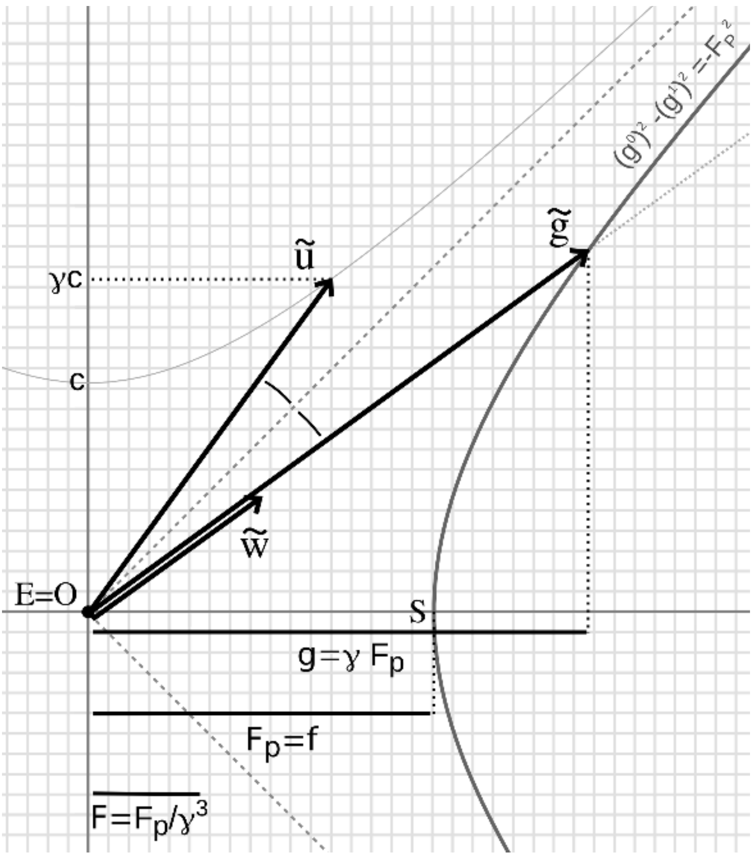
We have the spatial part $\vec{g}$ of the 4-force, and on the other hand the classical force $\vec{F}$, the Newton's second law then takes the following form:

$$\frac{d\vec{p}}{dt}=\frac{\vec{g}}{\gamma}=\gamma^3(\vec{F}\cdot\vec{\beta})\vec{\beta}+\gamma\vec{F}=\vec{f}$$

The relationship between $\vec{g}$ and $\vec{F}$ is not simple and we find that they are not collinear. Within the limit of low speeds, we find Newton's second law $m\vec{a}=\vec{F}$.

Most often, to build relativity, the third force $\vec{f}$ is used. When one injects, in Newton's law, the relativistic momentum instead of the classical one, it is the force that appears. This force $\vec{f}$ is commonly used as an equivalent of the classical force at the relativistic level. This standard force has a definition similar to that of classical mechanics, but it is not the spatial part of a covariant four-vector.

In Newtonian mechanics the force is independent of the inertial frame of reference $\vec{F}'=\vec{F}$, in relativity it is also the case for the four-force $\tilde{g}'=\tilde{g}$. On the other hand, we have in general $\vec{f}'\neq\vec{f}$ and $\vec{g}'\neq\vec{g}$.



- **Power**

$$\tilde{w}\cdot\tilde{u}=0 \quad \Rightarrow \quad \tilde{g}\cdot\tilde{p}=0$$

$$\frac{d\tilde{p}}{d\tau}\cdot\tilde{p}=\frac{dE/c}{d\tau}E/c-\frac{d\vec{p}}{d\tau}\cdot\vec{p}=0$$

$$\gamma\frac{dE}{d\tau}=\vec{g}\cdot\vec{u} \quad \text{and} \quad \frac{dE}{dt}=\vec{f}\cdot\vec{v}$$

○ Conservation of momentum and energy

For an *isolated* system, $\tilde{g}=\tilde{0}$ and the momentum-energy four-vector is constant. For a set of particles, the total momentum is the sum of the individual momenta, and the same applies to the energy :

$$\tilde{p}=\sum \tilde{p}_i, \quad E=\sum E_i \quad \text{and} \quad \vec{p}=\sum \vec{p}_i$$

This quantities are then conserved:

$$\tilde{p}=\widetilde{cst}\,, \quad E=cste \quad \text{and} \quad \vec{p}=\overrightarrow{cst}$$

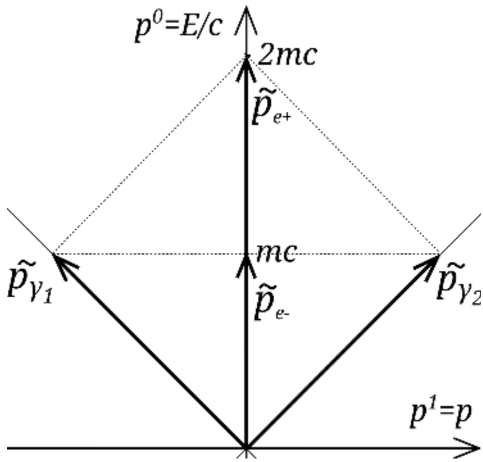
For example, during a collision, the particles may change in nature and number, but whatever happens there will always be conservation of these three quantities: they will have the same values before and after the impact. We can consider an isolated system in three situations: no force is exerted on the system, the sum of the forces is zero, or, as in a collision, the interaction being very brief, the 4-momentum of the system has no time to vary significantly. The forces internal to the system do not intervene in these balances.

- **Annihilation of an electron with a positron**

Two gamma photons are produced :

$$e^{-}+e^{+}\rightarrow 2\gamma \quad \text{with} \quad \tilde{p}_{e^-}+\tilde{p}_{e^+}=\tilde{p}_{\gamma_1}+\tilde{p}_{\gamma_2}$$

We take the case where the electron and the positron have the same velocities (opposite directions). In the frame of reference where the particle and the antiparticle are at rest, we have the following Minkowski diagram of momentums-energies:



We have at least two photons produced by annihilation. It is not possible that only one photon is produced because a photon cannot be at rest and its momentum cannot be annulled to respect the conservation of the momentum in the considered frame of reference. If two photons are created, they necessarily have the same energy and they go in opposite directions. The energy of a photon corresponds to the mass energy of an electron (or what is the same of a positron). Photons thus have energies of 511 keV. They are very energetic photons, as a comparison the visible photons have an energy of the order of eV.

We study in exercise the collision of two protons with the creation at the threshold of a proton-antiproton pair.

Summary

Quantity	Classical Physics	Links / Standards	Special Relativity
position	$\vec{r}=(x,y,z)$		$\tilde{x}=(ct,\vec{r})$ $\tilde{x}\cdot\tilde{x}=c^2\tau^2$
velocity	$\vec{v}=\dfrac{d\vec{r}}{dt}$	$\vec{u}=\gamma\vec{v}$ $\gamma=\dfrac{dt}{d\tau}$	$\tilde{u}=\dfrac{d\tilde{x}}{d\tau}$ $\tilde{u}=(\gamma c,\vec{u})$ $\tilde{u}\cdot\tilde{u}=c^2$
momentum	$\vec{p}=m\vec{v}$	$\vec{p}=m\gamma\vec{v}$	$\tilde{p}=m\tilde{u}$ $\tilde{p}=(E/c,\vec{p})$ $\vec{p}=m\vec{u}$
acceleration	$\vec{a}=\dfrac{d\vec{v}}{dt}$	$w^0=\gamma^4\vec{a}\cdot\vec{\beta}$ $\vec{w}=\gamma^4(\vec{a}\cdot\vec{\beta})\vec{\beta}$ $+\gamma^2\vec{a}$	$\tilde{w}=\dfrac{d\tilde{u}}{d\tau}$ $\tilde{w}=(w^0,\vec{w})$ $\tilde{w}\cdot\tilde{w}=-a_p^2$ $\tilde{u}\cdot\tilde{w}=0$
force	$\vec{F}=m\vec{a}$	$\vec{f}=\dfrac{d\vec{p}}{dt}$ $\vec{g}=\gamma\vec{f}$	$\vec{g}=\dfrac{d\vec{p}}{d\tau}$ $\tilde{g}=m\tilde{w}$ $\tilde{g}=(g^0,\vec{g})$
energy	$\dfrac{dE_k}{dt}=\vec{F}\cdot\vec{v}$ $E_k=\dfrac{1}{2}mv^2$	$\dfrac{dE}{dt}=\vec{f}\cdot\vec{v}$	$\gamma\dfrac{dE}{d\tau}=\vec{g}\cdot\vec{u}$ $E=\gamma mc^2$ $E_k=E-mc^2$

electro -magnetic field	$\vec{F}_E = q\vec{E}$ $\vec{F}_B = q\vec{v}\wedge\vec{B}$	Lorentz force : $\vec{f} = q(\vec{E}+\vec{v}\wedge\vec{B})$ $\vec{g} = \gamma\vec{f}$	$\tilde{g} = \boldsymbol{F}\tilde{j}$ $\tilde{j} = q\tilde{u}$ $\vec{g} = q(\gamma\vec{E}+\vec{u}\wedge\vec{B})$

The standard definition $\vec{f}$ for force is widely used by the scientific community, which summarizes relativity in a few equations:

$$\vec{p} = m\gamma\vec{v} \quad \vec{f} = \frac{d\vec{p}}{dt} \quad \vec{f}_L = q(\vec{E}+\vec{v}\wedge\vec{B})$$

$$\frac{dE}{dt} = \vec{f}\cdot\vec{v} \qquad E = \gamma m c^2 = T + m c^2.$$

Taught directly in this way it is fast and effective, but at the same time, if the student wants to deepen the concepts it will be necessary for him to enlarge his view in order to have a clear vision and avoid confusion. Moreover, in our book we put forward a geometrical perspective which is mainly based on the approach of Hermann Minkowski. These are of course the covariant quantities that are naturally represented in a diagram and are simply transformed with the Lorentz boost.

For the electromagnetic field, the quantities are detailed in exercise on page 252.

ꝏ NON-INERTIAL REFERENCE FRAMES

As we know how to do in Newtonian mechanics, we must also learn to apply special relativity in non-inertial frames.
Let us recall the approach in classical mechanics. Newton's laws are verified in Galilean frames and by a change of frame of reference we find their new expressions in any moving frame:

$$m\vec{a}_r=\vec{F}+\vec{F}_{ie}+\vec{F}_{ic}$$

Everything happens as if we had new forces, called inertial or fictitious. One may wonder if these forces really exist. Indeed, these forces are not related to fundamental interactions but to the change in the frame of reference. Nevertheless, the driver and passengers of a car experience these different dynamic effects as real during the acceleration phases, such as a sudden start, more or less tight bends and braking strokes.

Classical mechanics give an interpretation of these effects in terms of forces: coincident forces and Coriolis forces.

It goes without saying that special relativity must allow all these effects to be found. At low speeds, they must be equivalent. We will have new effects that will appear with increasing speed. But also at low speeds, for precise measurements and for the behavior of light which is now included in the theoretical framework. The interpretation is however very different.

In special relativity, there are no inertial forces but *metric effects*. By a non-inertial change of frame, we deviate from the Minkowskian metric and a free particle follows a geodesic which modifies its initially rectilinear and uniform motion to follow a curved and accelerated trajectory.

For example, when the car accelerates at a green traffic light, it is not an inertial force that puts you against the seat, but a metric modification that puts you in free fall towards the back of the car (as in the uniformly accelerated rocket). At the same time, the watches of the passengers in the back of the car are slow with respect to those in the front. Quite the opposite when you brake, the metric modification makes you plunge in free fall towards the windshield. In a turn, the metric change causes you to fall towards the outside of the bend, the watches will also go out of sync and Euclid's postulates will no longer be verified.

In special relativity, the notion of inertial force is replaced by that of metric effect. We have previously studied the two particular cases of the uniformly accelerated reference frame and the uniformly rotating frame and we will now focus on the general case[35].

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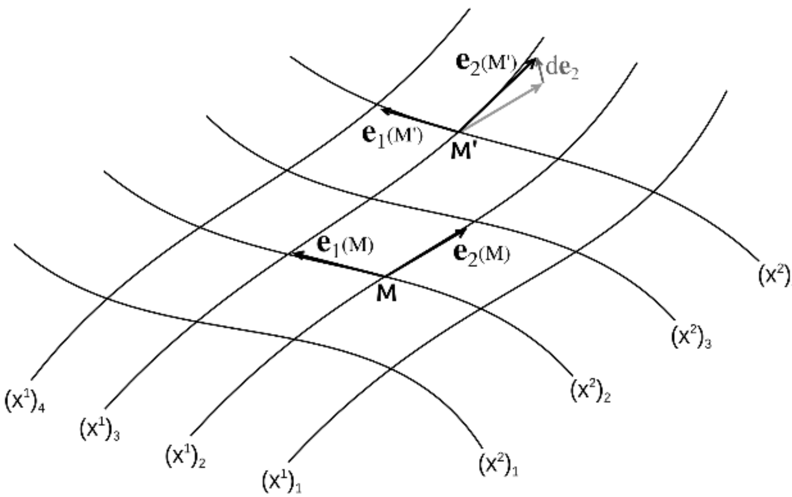
35 Here we make the analogy between classical mechanics and special relativity, but historically we are rather used to the analogy made with general relativity. In this analogical framework, during a brake

- **Coordinate lines, local basis and connections**

Here we complete our description of a vector space. These are very general mathematical concepts that can be used in all scientific fields.



Coordinate lines are obtained when one coordinate varies and all others are fixed.

At each point of this network we have a local basis with the basis vectors tangent to the lines. When we go from M to M' infinitely close, we have a small variation of the basis vectors :

---

stroke, we say that everything happens as if a gravitational field was pulling you forward. This gravitational field is of course fictitious. If it were real, at the same time as you brake, a gigantic massive wall of infinite size would have to appear in front of the car to justify such a gravitational field! In general relativity, the gravitational field creates an additional metric effect, spacetime is then curved, and the gravitational field is very real (it exists in all observation frames of reference).

$$d\,\boldsymbol{e}_i = \frac{\partial \boldsymbol{e}_i}{\partial x^j} dx^j = \Gamma^k{}_{ij} \boldsymbol{e}_k\, dx^j$$

This variation can be projected on the starting basis. The quantities $\Gamma^k{}_{ij}$ allow to encode the variation of the local basis at this point. We will call *connection* the object $\Gamma^k{}_{ij}$. For a global basis, which does not depend on the point, all the components of the connection are null.

The connection is symmetrical on the last two indices:

$$\Gamma^k{}_{ij} \boldsymbol{e}_k = \frac{\partial \boldsymbol{e}_i}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{\partial \boldsymbol{MM'}}{\partial x^i}\right) = \frac{\partial^2 \boldsymbol{MM'}}{\partial x^j \partial x^i} = \frac{\partial^2 \boldsymbol{MM'}}{\partial x^i \partial x^j} = \Gamma^k{}_{ji} \boldsymbol{e}_k$$

The metric contains all the information about space. We can establish the expression of the connection coefficients according to the metric:

$$g_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{e}_j \qquad d\,g_{ij} = \partial_k g_{ij}\, dx^k = (d\,\boldsymbol{e}_i) \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \cdot (d\,\boldsymbol{e}_j)$$

$$g_{ij,k}\, dx^k = (\Gamma^l{}_{ir} \boldsymbol{e}_l\, dx^r) \cdot \boldsymbol{e}_j + \boldsymbol{e}_i \cdot (\Gamma^m{}_{jn} \boldsymbol{e}_m\, dx^n)$$

$$g_{ij,k} = g_{lj} \Gamma^l{}_{ik} + g_{im} \Gamma^m{}_{jk}$$

$$g_{ij,k} + g_{ki,j} - g_{jk,i}$$
$$= g_{lj} \Gamma^l{}_{ik} + g_{im} \Gamma^m{}_{jk} + g_{li} \Gamma^l{}_{kj} + g_{km} \Gamma^m{}_{ij} - g_{lk} \Gamma^l{}_{ji} - g_{jm} \Gamma^m{}_{ki}$$

$$g_{ij,k} + g_{ki,j} - g_{jk,i} = 2 g_{im} \Gamma^m{}_{jk}$$

$$g^{ni}(g_{ij,k} + g_{ki,j} - g_{jk,i}) = 2 g^{ni} g_{im} \Gamma^m{}_{jk}$$

Finally : $\Gamma^i{}_{jk} = \frac{1}{2} g^{il}(g_{lj,k} + g_{kl,j} - g_{jk,l})$

- **Covariant derivative**

Variation of a vector **A** when moving from M to M' : d**A**=**A**(M')-**A**(M). In the Minkowski basis, or in a Cartesian basis, we are in particular cases where the basis is global, the basis does not depend on the point and only the variations on the components are to be taken into account.
In the general case: $d\boldsymbol{A}=d(A^i\boldsymbol{e}_i)=d(A^i)\boldsymbol{e}_i+A^i d\boldsymbol{e}_i$.

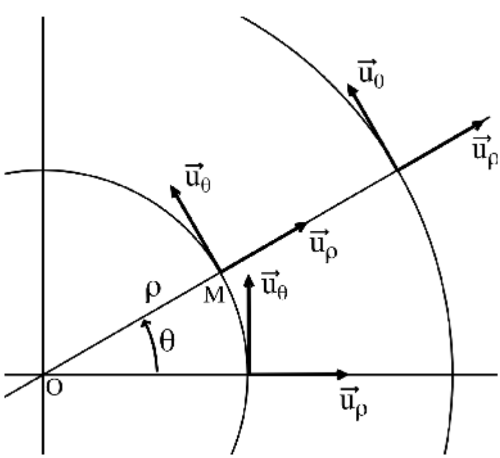
$$d\boldsymbol{A}=\partial_j A^i dx^j \boldsymbol{e}_i+\Gamma^k{}_{ij}A^i dx^j \boldsymbol{e}_k=(\partial_j A^i+\Gamma^i{}_{kj}A^k)dx^j \boldsymbol{e}_i$$

Notations: $D_j A^i=A^i{}_{;j}=\partial_j A^i+\Gamma^i{}_{kj}A^k$, $DA^i=A^i{}_{;j}dx^j$

The capital D makes it clear that all variations have been taken into account. For inertial frames of reference, the connections are null in the Minkowski basis, and ∂_μ was our covariant derivative. In non-inertial frames D_μ is the covariant derivative.

- **Illustration on an example**

Let's take the case of polar coordinates.



The basis depends on the point.
It rotates with the angle θ and remains unchanged when ρ varies:

$\vec{u}_\rho(\theta)$ and $\vec{u}_\theta(\theta)$

In classical mechanics, we

usually take unit vectors.

Basis Variations : $\dfrac{d\vec{u}_\rho}{d\theta}=\vec{u}_\theta$ and $\dfrac{d\vec{u}_\theta}{d\theta}=-\vec{u}_\rho$.

Then : $\overrightarrow{OM}=\vec{r}=\rho\vec{u}_\rho$ gives $\vec{v}=\dot{\rho}\vec{u}_\rho+\rho\dot{\theta}\vec{u}_\theta$ and

$$\vec{a}=(\ddot{\rho}-\rho\dot{\theta}^2)\vec{u}_\rho+(\rho\ddot{\theta}+2\dot{\rho}\dot{\theta})\vec{u}_\theta$$

We can retrieve this result with the metric and the connections :

$$ds^2=g_{ij}dx^i dx^j=dl^2=d\rho^2+\rho^2 d\theta^2 \qquad dl^2/dt^2=g_{ij}v^i v^j$$

$$\vec{e}_\rho=\vec{u}_\rho \quad \vec{e}_\theta=\rho\vec{u}_\theta \quad \overrightarrow{OM}=\rho\vec{e}_\rho+\theta\vec{e}_\theta \qquad \vec{v}=\frac{d\vec{l}}{dt}=(\dot{\rho},\dot{\theta})$$

$$g_{22,1}=2\rho \qquad \Gamma^1{}_{11}=0 \qquad \Gamma^2{}_{22}=0 \qquad \Gamma^2{}_{11}=0$$

$$\Gamma^1{}_{22}=-\frac{1}{2}g^{11}g_{22,1}=-\rho \qquad \Gamma^2{}_{12}=\frac{1}{2}g^{22}g_{22,1}=\frac{1}{\rho} \qquad \Gamma^1{}_{12}=0$$

$$d\vec{v}=(\partial_j v^i+\Gamma^i{}_{kj}v^k)dx^j\vec{e}_i$$

$$\vec{a}=(\partial_j v^i+\Gamma^i{}_{kj}v^k)\dot{x}^j\vec{e}_i=\partial_t v^i\vec{e}_i+\Gamma^i{}_{kj}v^k\dot{x}^j\vec{e}_i$$

$$\vec{a}=\dot{v}^1\vec{e}_1+\dot{v}^2\vec{e}_2+\Gamma^1{}_{22}v^2\dot{x}^2\vec{e}_1+\Gamma^2{}_{12}v^1\dot{x}^2\vec{e}_2+\Gamma^2{}_{21}v^2\dot{x}^1\vec{e}_2$$

then $\vec{a}=\ddot{\rho}\vec{e}_\rho+\ddot{\theta}\vec{e}_\theta-\rho\dot{\theta}\dot{\theta}\vec{e}_\rho+\frac{1}{\rho}\dot{\rho}\dot{\theta}\vec{e}_\theta+\frac{1}{\rho}\dot{\theta}\dot{\rho}\vec{e}_\theta$.

We have a new method that uses the metric to account for local basis variations using connections.

◦ Geodesics

Geodesics are the worldlines followed by free particles. These curves, the equivalent of Euclid's straight lines, maximize proper time.

On a geodesic, the proper acceleration is zero.
Let us take up again the building of special relativity for non-inertial frames of reference:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} u^\mu u^\nu d\tau^2, \quad u^\mu = \frac{d x^\mu}{d\tau} \text{ and } p^\mu = m u^\mu.$$

With the covariant derivative, we can generalize the Newton's second law :

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{and} \quad \vec{a} = \frac{\vec{F}}{m} - \vec{a}_e - \vec{a}_c \quad \text{becomes} \quad \frac{D\tilde{p}}{D\tau} = \tilde{g}$$

Equations of Motion: $\dfrac{du^\mu}{d\tau} = \dfrac{g^\mu}{m} - \Gamma^\mu{}_{\alpha\beta} u^\alpha u^\beta.$

For the geodesics equation: $g^\mu = 0.$

The metric effects, equivalent to the classical forces of inertia, are expressed through the connections, which themselves reflect the variations of the metric in a non-inertial frame.

In classical mechanics : $\dfrac{dv^i}{dt} = \dfrac{F^i}{m} - \Gamma^i{}_{jk} v^j v^k.$

- **Classical limit**

In the classical case we already noticed that the mass of the particle did not play a role: $\vec{a} = -\vec{a}_e - \vec{a}_c$.
For the calculation of the acceleration $\vec{a}$ from the velocity $\vec{v}$, we have two kinds of terms, those which involve the variation of the coordinates only, and the others for the variations of the basis:

$$\vec{a} = \vec{a}_{coord} + \vec{a}_{base} \quad \text{and} \quad \vec{a}_{coord} = -\vec{a}_e - \vec{a}_c - \vec{a}_{base}$$

These are the three terms on the right that are expressed using connections.

Uniformly accelerated frame :

→ *Mechanics of Newton* :

$$\vec{a}_r = -\vec{a}_e = \frac{d^2\overrightarrow{OM}}{dt^2} = -\vec{a}_{R'}(O)$$

R : rocket, $\vec{a}_{R'}(O) = \dfrac{d^2\overrightarrow{O'O}}{dt^2} = a\,\vec{i}$ and $\ddot{x} = -a$.

→ *Special relativity* : as demonstrated in the exercise on page 243, the non-zero connection components are $\Gamma^1{}_{00} = \dfrac{g'}{2}$ and $\Gamma^0{}_{10} = \Gamma^0{}_{01} = \dfrac{g'}{2g}$ with $g(x) = \left(1 + \dfrac{ax}{c^2}\right)^2$.

Then:

$$\frac{du^1}{d\tau} = \frac{d^2x}{d\tau^2} = -\Gamma^1{}_{00}u^0u^0 = -\frac{a}{c^2}\left(1+\frac{ax}{c^2}\right)\gamma^2c^2 = -\gamma^2 a\left(1+\frac{ax}{c^2}\right)$$

We find the classical limit: $\ddot{x} = -a$.

Rotating frame :

→ *Mechanics of Newton* : $\vec{a}_r = -\vec{a}_e - \vec{a}_c$

$$\vec{a} = \omega^2\overrightarrow{HM} - 2\vec{\omega}\wedge\vec{v} = -\omega^2\rho\,\vec{u}_\rho - 2\omega\,\vec{u}_z\wedge(\dot{\rho}\,\vec{u}_\rho + \rho\,\dot{\theta}\,\vec{u}_\theta)$$

$$\vec{a} = (\ddot{\rho} - \rho\,\dot{\theta}^2)\vec{u}_\rho + (\rho\,\ddot{\theta} + 2\dot{\rho}\,\dot{\theta})\vec{u}_\theta = \omega^2\rho\,\vec{u}_\rho - 2\,\omega\dot{\rho}\,\vec{u}_\theta + 2\,\omega\rho\,\dot{\theta}\,\vec{u}_\rho$$

→ *Special relativity* : $\tilde{u} = \gamma(c, \dot{\rho}, \dot{\theta}, \dot{z})$

Only non-zero connections :

$$\Gamma^1{}_{00} = -\frac{\rho\,\omega^2}{c^2} \qquad \Gamma^1{}_{02} = \Gamma^1{}_{20} = -\frac{\rho\,\omega}{c} \qquad \Gamma^1{}_{22} = -\rho$$

$$\Gamma^2{}_{10}=\Gamma^2{}_{01}=\frac{\omega}{\rho c} \qquad \Gamma^2{}_{12}=\Gamma^2{}_{21}=\frac{1}{\rho}$$

Then :

$$\frac{d u^1}{d\tau}\tilde{e}_1+\frac{d u^2}{d\tau}\tilde{e}_2$$

$$=\left(-\Gamma^1{}_{00}u^0u^0-2\Gamma^1{}_{02}u^0u^2-\Gamma^1{}_{22}u^2u^2\right)\tilde{e}_1$$
$$+\left(-2\Gamma^2{}_{10}u^1u^0-2\Gamma^2{}_{12}u^1u^2\right)\tilde{e}_2$$

$$\frac{d\gamma\dot{\rho}}{d\tau}\tilde{e}_\rho+\frac{d\gamma\dot{\theta}}{d\tau}\tilde{e}_\theta$$

$$=\left(-\rho\omega^2\gamma^2+2\rho\omega\gamma^2v^\theta+\rho\gamma^2(v^\theta)^2\right)\tilde{e}_\rho$$
$$+\left(-2\omega\gamma^2v^\rho-2\gamma^2v^\rho v^\theta\right)\tilde{e}_\theta/\rho$$

We find the classical limit :

$$\ddot{\rho}\vec{u}_\rho+\ddot{\theta}\rho\vec{u}_\theta=\left(-\rho\omega^2+2\rho\omega\dot{\theta}+\rho\dot{\theta}^2\right)\vec{u}_\rho+\left(-2\omega\dot{\rho}-2\dot{\rho}\dot{\theta}\right)\vec{u}_\theta$$

We now understand how particles move in a non-inertial frame of reference. Special relativity gives us a new interpretative and experimental framework where metric effects take the place of the inertial forces of the old Newtonian framework.

In a flat space-time and a non-inertial frame of reference, a free particle maximizes its proper time by following a curved trajectory.

This is not simply a new point of view, but a generalization to massless particles, and, as a correction, with modified experimental measurements.

The classical notion of force is abandoned in favor of a relativistic description in terms of space-time geometry. Here, it is the concept of force of inertia that becomes useless, we follow the same kind of approach in general relativity, where geometry makes the concept of gravitational force disappear.

No need for the space to be curved, for a free particle to have a curved trajectory.

- **Lagrangian approach**

The geodesic equations are found with the Lagrange equations. The approach is explained in the exercise on page 160. We are looking for geodesics that extremes proper time :

$c^2\tau=\int g_{\mu\nu}u^{\mu}u^{\nu}d\tau,\ L=g_{\mu\nu}u^{\mu}u^{\nu}$ and $\dfrac{\partial L}{\partial x^{\mu}}-\dfrac{d}{d\tau}\dfrac{\partial L}{\partial u^{\mu}}=0$

$$\frac{\partial L}{\partial x^{\mu}}=g_{\alpha\beta,\mu}u^{\alpha}u^{\beta} \quad \text{and} \quad \frac{\partial L}{\partial u^{\mu}}=g_{\alpha\mu}u^{\alpha}+g_{\mu\beta}u^{\beta}$$

$$\frac{d}{d\tau}\frac{\partial L}{\partial u^{\mu}}=g_{\alpha\mu,\nu}u^{\nu}u^{\alpha}+g_{\alpha\mu}\frac{du^{\alpha}}{d\tau}+g_{\mu\beta,\rho}u^{\rho}u^{\beta}+g_{\mu\beta}\frac{du^{\beta}}{d\tau}$$

$$g_{\alpha\beta,\mu}u^{\alpha}u^{\beta}-g_{\alpha\mu,\nu}u^{\nu}u^{\alpha}-g_{\mu\beta,\rho}u^{\rho}u^{\beta}-2g_{\mu\beta}\frac{du^{\beta}}{d\tau}=0$$

Hence the geodesic equation : $\Gamma^{\mu}{}_{\alpha\beta}u^{\alpha}u^{\beta}+\dfrac{du^{\mu}}{d\tau}=0$

Conclusion and synthesis

Let's come back to the notion of inertial frame of reference.
We have a circular definition: the postulates are true in inertial frames of reference, and a reference frame is inertial if the postulates are verified.
If a particle in a reference frame has a curved trajectory, is it due to a force or to the non-inertial nature of the frame?

In Newtonian mechanics, if we know beforehand the nature of the forces, we can determine whether a reference frame is Galilean. Let's take the electromagnetic and gravitational forces: if there are no charges and masses present, and the trajectory is nevertheless curved, you can deduce that the reference frame is non-Galilean. You have to imagine such a region of empty space, far enough away from all matter that the remote action of the forces is negligible.

Do you know the Olbers' paradox?
In cosmology, the universe is like a fluid homogeneous and isotropic of galaxies. You see the stars in the dark night, the resulting brightness is low, but logically the night should be white. Indeed, the further away you look, the weaker is the light received by the observer from each luminous object, but at the same time their number increases in the same proportions. The night finally is dark because the Universe is expanding.

But back to the reference frames, if we apply the Olbers' Paradox to gravitation, we have the same result, the gravitational field would tend towards infinity at all points in the Universe... Here we want to illustrate how the foundations of classical mechanics are not trivial. Moreover, can we determine the nature of forces without the help of Newton's laws?

In relativity, the situation is much simpler, we use geometry. The behavior of spacetime alone makes it possible to determine if the frame of reference is inertial — without using the notion of force.
Beforehand, it is sufficient to have a set of clocks at rest and synchronized on the region being studied. If, during the experiment, the clocks do not go out of sync, the reference frame is inertial.

Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Inertial frame of reference

Minkowskian metric

$$ds^2 = g_{\mu\nu} \, dx^{\mu} \, dx^{\nu} \quad \text{with} \quad v_{light} = c$$

Inertial frame/ Maxwell's equations

Flat spacetime

Non-inertial frame / Metric effects

Accelerated rocket / Rotating disk

Zero curvature tensor

Back to the Minkowski metric by a change of coordinates

SPECIAL RELATIVITY

GENERAL RELATIVITY

Equivalence principle / Einstein's equations

Curved spacetime in vacuum

Gravitation / Spatiotemporal waves

Energy-momentum and Ricci tensors are zero

Non-zero curvature tensor

Curved spacetime

Matter / Sources of the gravitational field

Energy-momentum and Ricci tensors are non-zero

Non-zero curvature tensor

Exercises

1. ▲△△ Change of basis

Let consider the basis $\tilde{e}'_\mu$ of the inertial frame.

1 - Determine the basis $\tilde{e}_\mu$ of the uniformly accelerated reference frame of the rocket as the function of $\tilde{e}'_\mu$.
Place some examples of vectors from this base on a Minkowski diagram.

2 - Determine the basis $\tilde{e}_\mu$ of the uniformly rotating reference frame of the disk as the function of $\tilde{e}'_\mu$.
Represent this base on a Minkowski diagram.

Answers p409

2. ▲▲▲ Riemann curvature tensor

We give here the curvature tensor without justification. We will apply the formulas to show that for the accelerated rocket, as for the rotating disk, we are in flat space-time despite the non-inertial nature of the reference frames. If all the components of the tensor are zero the spacetime is flat, if even one of the components is non-zero the spacetime is curved.
Riemann tensor as a function of the connections:

$$R^{\alpha}{}_{\beta\gamma\delta}=\Gamma^{\alpha}{}_{\beta\delta,\gamma}-\Gamma^{\alpha}{}_{\beta\gamma,\delta}+\Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta}-\Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}$$

Connection coefficients[36] :

$$\Gamma^{\alpha}{}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}\left(\partial_{\mu}g_{\beta\nu}+\partial_{\nu}g_{\beta\mu}-\partial_{\beta}g_{\mu\nu}\right)$$

Notation: $\frac{\partial}{\partial x^{\mu}}=\partial_{\mu} = {}_{,\mu}$ so $\Gamma^{\alpha}{}_{\beta\delta,\gamma}=\partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta}$.

The curvature tensor is antisymmetric in the last two indices. The connection coefficient is symmetric in the last two indices.

1 - Rocket: uniformly accelerated reference frame.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

b- Determine all the connection coefficients. You must identify the non-zero coefficients for the calculation of the curvature.

Helps: you can set $g(x)=\left(1+\frac{ax}{c^2}\right)^2$.

Help yourself as much as possible with the symmetries. Identify the non-zero terms of $g_{\mu\nu}$ *and* $g^{\mu\nu}$*. Are they constant? Which coordinates do they depend on? Which terms* $\partial_{\mu}g_{\beta\nu}$ *are non-zero?*

c- Show that all the components of the curvature tensor are zero.

Help: what is the consequence of antisymmetry?

2 - Disk : uniformly rotating reference frame.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

36 Also called Christoffel symbols.

b- Determine all the connection coefficients.

c- Demonstrate that all the components of the curvature tensor are zero.

3 - Spherical body: reference frame studied with Schwarzschild coordinate system. To compare with a situation where spacetime is curved.

We invite you to set $g=1-\frac{r_S}{r}=e^f$.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

b- Determine all the non-zero connection components.

c- To show that the spacetime is curved calculate the component $R^0{}_{101}$.

Prove that $R_{0101}=\frac{r_S}{r^3}$. Answers *p412*

3. ▲▲▲ A non-uniformly rotating Disk

In the previous exercise we demonstrated that the curvature tensor was null in the uniformly rotating frame of the disc. We will continue the demonstration in the case of any rotational motion of the disk. We had for the inertial observer as a function of the coordinates of the observer at rest with respect to the disc: $\theta'=\theta+\omega t$. We now take the general expression: $\theta'=\theta+\lambda(t)$, where $\lambda(t)$ is any function of time. Thus are included the possible phases of acceleration, deceleration, oscillation, etc.

1 - Determine the connection coefficients.

2 - Calculate the Riemann curvature tensor.

3 - Was the result expected?

Answers p419

4. ▲▲▲ Spatial curvatures

The Riemann curvature tensor applies to any space, space-time and sub-space regardless of the number of dimensions. We have calculated the curvature of 4-dimensional space-time and we will calculate the curvatures for the spatial parts. We take the three examples of the uniformly accelerated, the Schwarzschild and the uniformly rotating frames.

Let us detail the method and explain the general approach to measure times and distances[37].

For the time, we determine the proper time interval $d\tau$ by setting the $dx^i=0$ (i=1, 2 or 3) :

$$d\tau=\frac{1}{c}\sqrt{g_{00}}\,dx^0 \quad \text{and} \quad \tau=\frac{1}{c}\int\sqrt{g_{00}}\,dx^0 \quad (x^0=ct)$$

For the space, if the reference system is synchronous $g_{0i}=0$ and: $ds^2=g_{00}\,c^2dt^2-dl^2$

with $dl^2=-g_{ij}\,dx^i\,dx^j=\gamma_{ij}\,dx^i\,dx^j$

The curvature tensor is then calculated with the three-dimensional metric tensor γ_{ij} as before. Here, we run the indices from 1 to 3.

37 Landau / Lifchitz, *The Classical Theory of Field*, § *Distances and time intervals*.

If the reference system is not synchronous, the temporal coordinate is not directly separated from the spatial coordinates, and, we show that:

$$\gamma_{ij}=-g_{ij}+\frac{g_{0i}g_{0j}}{g_{00}} \quad \text{and} \quad dl^2=\gamma_{ij}dx^i dx^j$$

We can then calculate *dl* with the three-dimensional metric tensor. On the other hand, we cannot, in general, determine the distance between two bodies. Also, the curvature tensor cannot be directly calculated in the form previously given[38]. Nevertheless, in the particular case where the reference frame is stationary, metric coefficients $g_{\mu\nu}$ independent of time, we can integrate the element *dl* and the curvature tensor is in the usual form :

$$\text{Stationary frame: } \frac{\partial g_{\mu\nu}}{\partial t}=0, \quad l=\int dl \quad \text{and} \quad R^i{}_{jkl}.$$

1 - Rocket: Is the reference system synchronous?
Is the space curved?

2 - Spherical body:
Is the reference system synchronous?
Is the space curved?

3 - Disk:

a- Is the reference system synchronous?

b- Determine γ_{ij}.

c- Is the reference frame stationary? What is the ratio of the perimeter of a circle to its diameter?

38 Cattaneo's projection technique.
Rizzi / Ruggiero, *Space geometry of rotating platforms*, 2008.

(circle centered on the axis of rotation)
Does the observer attached to the rotating disc experience a curvature?

d- Calculate $R^{i}{}_{jkl}$.

e- It is shown that, for a two-dimensional space, there is only one independent component of the curvature tensor R_{ijkl} (i=1, 2)[39].

Calculate the Gaussian curvature K of the surface:

$$K=\frac{1}{R_1 R_2}=\frac{R_{1212}}{\gamma_{11}\gamma_{22}-\gamma_{12}^2}$$

where R_1 and R_2 are the radii of curvature at a point of the disk. You can compare it to the Gaussian curvature of a sphere.

Answers p419

5.┼ ▲△△ Pair production

A high-energy particle can under certain conditions create a particle-antiparticle pair. Let's take the example of the collision of two protons. In the barycentric reference frame they arrive face to face with the same velocity. When their kinetic energy is just sufficient, we say at the *threshold*, they create four particles at rest:

$$p+p \rightarrow p+p+p+\bar{p}$$

39 Landau, § *Properties of the curvature tensor*.

Draw the Minkowski diagram at the threshold in the barycentric frame where $\sum \vec{p}_i = \vec{0}$.

Answers p422.

6. ▲▲▲ Wave equation

The wave equation describes the behavior of a multitude of waves: waves on water, sound waves, seismic waves, electromagnetic waves, etc. These waves, although of different physical natures, all obey the same equation. The amplitude of the wave $\varphi(\vec{r}, t)$ is the solution to the following differential equation:

$$\Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad \text{so} \quad \Box \varphi = 0$$

c is the celerity of the wave which depends on the type of wave and the medium.

Definition of the Laplacian in Cartesian coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

d'Alembert operator : $$\Box = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$$

1 - Demonstrate that the wave equation is not invariant under the Galilean transformation.

Help: In classical mechanics, the amplitude of the wave is a physical quantity that should not depend on the chosen coordinate system. At a point M and at a given time: $\varphi'(x',t')=\varphi(x,t)$*. Such as, for example, the wave height, or the sound pressure. By identifying* $d\varphi$ *and* $d\varphi'$ *deduce the relations between the partial derivatives.*

2 - Show that the electromagnetic wave equation in vacuum is invariant under the Lorentz transformation: $\Box\vec{E}=0$ and $\Box\vec{B}=0$. In this case the amplitude of the wave depends on the reference frame, the transformation formulas are given on page 427.

Answers p422.

7. ▲▲△ Schrödinger equation

In quantum physics, the wave function obeys the following equation of evolution:

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\Psi+V\Psi$$

The probability density of presence of a particle is obtained by multiplying the wave function by its complex conjugate:

$$\rho=\frac{dP}{dV}=\Psi\Psi^*$$

We can limit the study to the motion in one dimension of a free particle of mass *m* :

$$i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2},$$

and a standard Galilean transformation: $\vec{v}_{R'/R}=v\,\vec{i}$

1 - The probability of presence of a particle in a given volume should not depend on the reference frame. On the other hand, the wave function is not unique and the probability density is not modified if we multiply the wave function by a complex number of modulus one.

Show that the Schrödinger equation is invariant under a Galilean transformation with:

$$\Psi'=e^{\frac{i}{\hbar}(Et-px)}\Psi \quad \text{where} \quad E=\frac{1}{2}mv^2 \quad \text{and} \quad p=mv.$$

2 - Show why the Schrödinger equation cannot be invariant under the Lorentz transformation.

Answers p424.

8. ▲▲▲ The electromagnetic field

Electric and magnetic fields are not written as fourvectors but as components of a rank-2 tensor:

$$\boldsymbol{F}=F^{\mu\nu}=\begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

The $\vec{E}$ and $\vec{B}$ fields are in fact one and only one physical entity and their components depend on the observational inertial frame of reference. We are here in the inertial frame *R*, and we will also consider the frame *R'* in uniform rectilinear translation along *x*: $\vec{v}_{R'/R}=\vec{v}=v\,\vec{u}_x$.

The tensor of the electromagnetic field is antisymmetric: $F^{\mu\nu}=-F^{\nu\mu}$.

1 - Like mass, electric charge is an attribute of the particle that does not depend on the reference frame. We can simply build a four-vector for the charge and its motion:

$$\tilde{j}=q\,\tilde{u} \qquad \text{(4-vector current)}$$

We will demonstrate that the 4-vector $\boldsymbol{F}\,\tilde{j}$ is identified with the electromagnetic 4-force:

$$\frac{d\tilde{p}}{d\tau}=\boldsymbol{F}\,\tilde{j} \quad \text{and for the components} \quad \frac{dp^{\mu}}{d\tau}=F^{\mu\nu}\,j_{\nu}$$

By developing the components, temporal then spatial, show that we find the electromagnetic power, as well as the expression of the Lorentz force.

2 - Give the expression of the components of $\vec{E}\,'$ and $\vec{B}\,'$ in *R'* as a function of those of $\vec{E}$ and $\vec{B}$ in *R*.

3 - Determine the components of the tensor $F_{\mu\nu}$.

4 - Find the expressions of the two Lorentz invariants of electromagnetic fields. They are scalar invariants functions of $\vec{E}$ and $\vec{B}$. The first one is obtained by contracting all components of the electromagnetic tensor with itself: $F^{\mu\nu}F_{\mu\nu}$. The second use the *completely antisymmetric unit tensor* of fourth rank: $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. $\epsilon^{\mu\nu\alpha\beta}$ components are zero if two indices are the same and ±1 else. The tensor alternates sign under interchange of any pair of indices. We set: $\epsilon^{0123}=1$.

5 - In the reference frame of the laboratory *R*, we have two planar metallic plates separated by a distance *e* and respective plate charge densities σ and -σ. The capacitor plates are assumed to be infinite and we will take the z-axis from the negative plate to the positive plate.

We will use the Gauss's and Ampère's circuital laws:

$$\oiint_S \vec{E}\cdot\vec{dS}=\frac{Q_{\text{in}}}{\epsilon_0} \quad \oint_\Gamma \vec{B}\cdot\vec{dl}=\mu_0 I_{enc} \quad (\epsilon_0\mu_0 c^2=1)$$

The use of these tools is not explained here. A book in itself on this subject would be necessary. Refer to a undergraduate level course on electrostatics and magnetostatics.

a- Determine the electric field at any point in the space. Write the matrix $F^{\mu\nu}$ in *R*.

b- We are now in the frame of reference *R'* in uniform rectilinear translation along the *x*-axis at the velocity $\vec{v}$. For a classical observer of this frame of reference the charge density remains the same on the plates and the electric field $\vec{E}\,'=\vec{E}$. On the other hand, as the charges are in motion, a surface current density appears: determine the magnetic field at any point. Write the matrix $F\,'^{\mu\nu}$ in *R'*.

c- Starting from the tensor $F^{\mu\nu}$ do you find $F\,'^{\mu\nu}$ with the change of basis lambda matrices? Do we well have the invariance of the two Lorentz invariants?

6 - In the reference frame of the laboratory *R*, we have a homokinetic beam of protons of velocity $\vec{v}$, radius *r* and density *n*. We call *R'* the proper referential of protons.

a- Determine the electric field outside the beam in *R'*.

b- By general considerations, determine the structure of this same field in *R* with few calculations.

Answers on page 425.

9. ▲▲▲ Maxwell's equations

James Clerk Maxwell established in 1864 the theory of electromagnetism which unifies Michael Faraday's theory of electricity and André-Marie Ampère's theory of magnetism through the following equations:

In vacuum:

$$\vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t} \qquad \vec{\nabla}\cdot\vec{B}=0$$

With sources:

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \qquad \vec{\nabla}\wedge\vec{B}=\mu_0\vec{j}+\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

The fields are derived from a potential V and a vector potential $\vec{A}$ according to:

$$\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t} \quad \text{and} \quad \vec{B}=\vec{\nabla}\wedge\vec{A}$$

Lorentz gauge condition: $\frac{1}{c^2}\frac{\partial V}{\partial t}+\vec{\nabla}\cdot\vec{A}=0$

Charge conservation: $\vec{\nabla}\cdot\vec{j}+\frac{\partial\rho}{\partial t}=0$

Definition of operators in the Cartesian coordinate system:

Gradient of f : $\vec{\nabla} f=\frac{\partial f}{\partial x}\vec{i}+\frac{\partial f}{\partial y}\vec{j}+\frac{\partial f}{\partial z}\vec{k}$

Divergence of $\vec{C}$: $\vec{\nabla}\cdot\vec{C}=\frac{\partial C_x}{\partial x}+\frac{\partial C_y}{\partial y}+\frac{\partial C_z}{\partial z}$

Curl of $\vec{C}$:

$$\vec{\nabla}\wedge\vec{V}=\left(\frac{\partial V_z}{\partial y}-\frac{\partial V_y}{\partial z}\right)\vec{i}+\left(\frac{\partial V_x}{\partial z}-\frac{\partial V_z}{\partial x}\right)\vec{j}+\left(\frac{\partial V_y}{\partial x}-\frac{\partial V_x}{\partial y}\right)\vec{k}$$

1 - Galilean transformation:

a- Show that Newton's second law is invariant under the Galilean transformation.

b- Lorentz's force is considered invariant under this same transformation. From this, deduce the Galilean transformation laws of $\vec{E}$ and $\vec{B}$ as a function of $\vec{v}_e=\vec{v}_{R'/R}$. Check that they well correspond to the non-relativistic limit of the Lorentz transformation of these same fields.

c- Show that the first two Maxwell's equations $\vec{\nabla}\cdot\vec{B}=0$ and $\vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}$ remain invariant under a Galilean transformation.

Help to do the calculations in vector form:

Partial derivatives: $\vec{\nabla}=\vec{\nabla}'$ *and* $\frac{\partial}{\partial t}=\frac{\partial}{\partial t'}-\vec{v}_e\cdot\vec{\nabla}'$

Useful formula:

$$\vec{\nabla}\wedge(\vec{A}\wedge\vec{B})=\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}.$$

d- Show that the following two Maxwell's equations are not invariant under a Galilean transformation (to simplify the calculations, we can consider the case without the sources ρ and $\vec{j}$).

Useful formula: $\vec{\nabla}\cdot(\vec{A}\wedge\vec{B})=\vec{B}\cdot(\vec{\nabla}\wedge\vec{A})-\vec{A}\cdot(\vec{\nabla}\wedge\vec{B})$.

2 - <u>Lorentz transformation</u>: Let us show that from 1905 the Maxwell equations could incorporate their natural relativistic framework.

a- Show that Maxwell's equations are invariant under the Lorentz transformation.

b- We introduce the 4-vector current density $\tilde{j}=\rho_p\tilde{u}$ where ρ_p is the charge volume density in the proper frame of reference. Show that by using the 4-vector gradient $\partial_\mu=\tilde{\nabla}=\left(\frac{\partial}{\partial ct},\vec{\nabla}\right)$ we obtain a charge conservation equation in covariant form.

c- We propose to introduce the potential 4-vector $\tilde{A}=(V/c,\vec{A})$. Show that the Lorentz gauge condition is simply written in tensor form with A^μ and the 4-vector gradient ∂_μ. Show that by judiciously combining the four-vectors A^α and ∂^β, we obtain the tensor $F^{\mu\nu}$.

d- Show that the covariant equation $\partial_\mu F^{\mu\nu}=\mu_0 j^\nu$ gives back the Maxwell equations with sources.

e- Show that the equation $\partial^\alpha F^{\mu\nu}+\partial^\mu F^{\nu\alpha}+\partial^\nu F^{\alpha\mu}=0$ gives back the first two Maxwell equations.

f- Find the expression of the propagation wave equations of V and $\vec{A}$.

3 - Show that the fields are not modified by the following gauge change:

$$\forall f \quad \begin{cases} V' = V - \dfrac{\partial f}{\partial t} \\ \vec{A}' = \vec{A} + \vec{\nabla} f \end{cases}$$

This is called gauge invariance. The Lorentz gauge condition corresponds to a particular gauge choice that gives the potential propagation equations a simpler form. Above all, A^μ then behaves like a 4-vector, and the invariance of Maxwell's equations becomes immediate.

Answers p431.

Tome XVI. Samedi 2 Juillet 1898 5e Année. — N° 27.

L'Éclairage Électrique

REVUE HEBDOMADAIRE D'ÉLECTRICITÉ

CHAMP ÉLECTRIQUE ET MAGNÉTIQUE

PRODUIT PAR UNE CHARGE ÉLECTRIQUE CONCENTRÉE EN UN POINT ET ANIMÉE D'UN MOUVEMENT QUELCONQUE

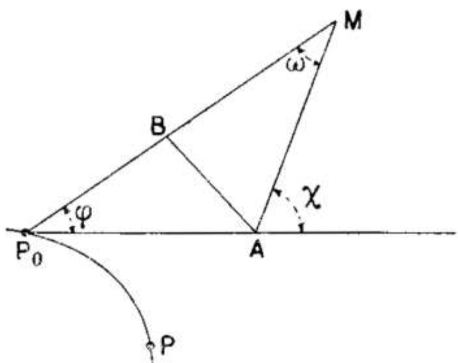


Fig. 2.

A. Liénard,
Professeur à l'École des Mines de Saint-Etienne.

INTERACTIONS

We study the interaction between two charged particles. We want to draw the Minkowski diagram of two electrons that repel each other. To do this, we will place ourselves in the barycentric frame of reference. The elements of electromagnetism treated in the exercises *The electromagnetic field* page 252 and *Maxwell's equations* page 255 are assumed to be acquired.

ꝏ FIELD CREATED BY A PUNCTUAL CHARGE

A particle of charge q is at rest at P, origin of the reference frame R'. We observe the static field created at a given point M:

$$\vec{r}\,'=\left(\overrightarrow{PM}\right)_{R'} \qquad \vec{E}\,'=\frac{q}{4\pi\epsilon_0}\frac{\vec{r}\,'}{r'^3} \qquad \vec{B}\,'=\vec{0}$$

We now place ourselves in a inertial reference frame R, of origin O and in rectilinear and uniform translation with respect to R': $\vec{v}_{R'/R}=\vec{v}$. The particle is in motion in R and passes through O at t=0. We want to obtain the expression of the fields in M in this new frame of reference R. We apply the Lorentz transformation and the transformation of the fields[40] :

$$\left(\overrightarrow{PM}\right)_{R'}=(x',y',z')=(\gamma(x-\beta ct),y,z) \quad \text{with} \quad \vec{v}=v\,\vec{i}$$

$$\vec{r}=\left(\overrightarrow{PM}\right)_R=\left(\overrightarrow{PO}\right)_R+\left(\overrightarrow{OM}\right)_R=\left(\overrightarrow{OM}\right)_R-\vec{v}\,t=(x-vt,y,z)$$

$$r'^2=\gamma^2(x-vt)^2+y^2+z^2=\gamma^2[r^2-(y^2+z^2)\beta^2]=r^2\gamma^2(1-\beta^2\sin^2\theta)$$

with $\theta=(\vec{v},\vec{r})$. Moreover $\vec{E}=\left(E'_{x'},\gamma E'_{y'},\gamma E'_{z'}\right)$ then:

40 H. Lumbroso, *Relativité*, *Interaction de deux particules chargées*.

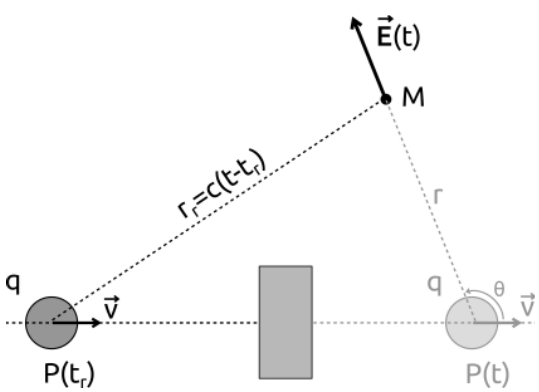
$$\vec{E}=\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$

and $$\vec{B}=\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\frac{\mu_0}{4\pi}q\frac{\vec{v}\wedge\vec{r}}{r^3}=\frac{\vec{v}\wedge\vec{E}}{c^2}$$

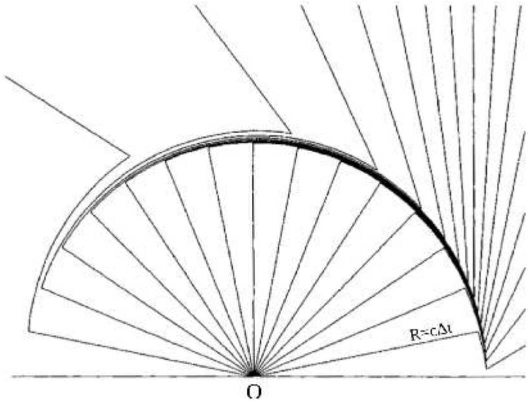
We obtain the relativistic expressions of the Coulomb's law and the Biot-Savart law for non-accelerated charged particles.

The electric field always seems radial, but with a non-isotropic angular distribution. In fact the situation is more complex, because the signal now propagates at finite speed, and this field was not generated by the particle at t=0 in O, but at an earlier position. The corresponding event is at the intersection of the past cone of *M(t=0)* with the worldline of the particle.

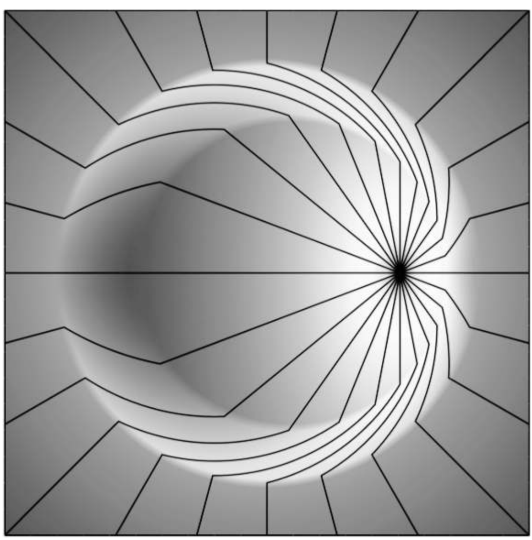
In the previous formulas the fields at the instant t are expressed using quantities themselves function of t, whereas it would be judicious that they are expressed according to the retarded time t_r.



Electric field of a positive charge in rectilinear and uniform motion. Even if an obstacle is interposed on the trajectory between P_r and P, a radial field with respect to P will be present at time t at point M. However, the charge will never be at P. Everything happens as if the field anticipated a rectilinear and uniform motion of the charge (Boratav and Kerner's book)



Electric field lines of a charge which, first comes from the left at 95% of c, then stops abruptly at O at t=0. Picture of field lines in the observation frame of reference at t=Δt [41].



Lines and amplitudes of the field. The particle first at rest, then begins to accelerate uniformly, then continues at constant speed c/√2 (longitudinal factor 1-β^2). The electric field decreases along the direction of motion and increases in the transverse direction[42].

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41 *Picture of Dynamic Electric Fields*, Tsien, American J. of P., 1972.

42 *Electric field lines of relativistically moving point charges*, Daja Ruhlandt, Steffen Mühle and Jörg Enderlein, 2019.

We give the general expression of electromagnetic fields, as a function of t_r, for any motion of the charge[43]. This formula was first established in 1898 by Alfred-Marie Liénard[44]. We might be surprised that this relativistic expression of fields was expressed even before the special relativity was revealed in 1905. In fact, there is nothing anachronistic about it, since Lienard relies on Maxwell's equations, which, as will be seen later, are purely relativistic.

$$\vec{E}(M,t)=\frac{q}{4\pi\epsilon_0}\left[\frac{1-\beta^2}{r^2}\frac{\vec{e}-\vec{\beta}}{(1-\vec{e}\cdot\vec{\beta})^3}+\frac{\vec{e}\wedge[(\vec{e}-\vec{\beta})\wedge\dot{\vec{\beta}}]}{r(1-\vec{e}\cdot\vec{\beta})^3}\right]_{P_r}$$

$$\vec{B}(M,t)=\frac{\vec{e}_{P_r}\wedge\vec{E}}{c}\qquad \vec{r}=(\overrightarrow{PM})_R\qquad \vec{e}=\frac{\vec{r}}{r}$$

$\vec{\beta}=\vec{v}_R/c$ and $\dot{\vec{\beta}}$ are the instantaneous velocity and acceleration. $\vec{e}$ is the unit vector directed from the charge at P to the observation point M.

The date t_r verifies $c(t-t_r)=r_{P_r}$.

The first term $1/r^2$ depends only on the velocity of the particle and corresponds to that found for the static charge. Here, the proper frame *R'* of the charged particle is no longer inertial and the proper acceleration is non-zero. A second term which depends on the acceleration appears, it is a term *1/r*, radiative: An accelerated charge emits electromagnetic radiation.

The fields $\vec{E}$ and $\vec{B}$ are still orthogonal.

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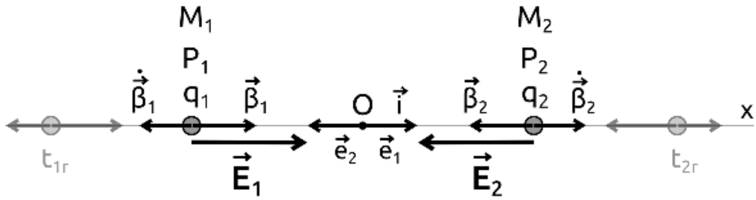
43 Landau, *The Lienard-Wiechert potentials*. Also: Jackson, *Classical Electrodynamics*, 1962, 641 pages, equation (14.14).

44 A. Liénard, *Champ électrique et magnétique produit par une charge électrique concentrée en un point et animée d'un mouvement quelconque*, L'Éclairage Électrique, July 2, 1898. Expressions also established, independently and two years later, by the geophysicist Emil Johann Wiechert.

For the two electrons arriving head to head:



One-dimensional motion and central symmetry simplify the resolution. To lighten, all retarded quantities have not been reproduced in gray, only the quantities at time t are placed in black. All vectors are along the axis and the radiative term is therefore null on the axis: $(\vec{e}-\vec{\beta})\wedge\dot{\vec{\beta}}=\vec{0}$. Here, the unit vectors $\vec{e}_1$ and $\vec{e}_2$ do not depend on time: $\vec{e}_1=\vec{i}=-\vec{e}_2$. Also, at a given instant, the velocities and accelerations according to indices 1 or 2 are in opposite directions and have the same values in norm : $\beta_i=\dot{x}_i/c$, before the collision $\beta_2<0$.

For M=M$_1$ and P=M$_2$: $\quad \vec{r}_r=\left(\overrightarrow{P_r M}\right)_R$

$$E_{xM}(x_M,t)=\frac{e}{4\pi\epsilon_0}\frac{1}{\left(x_M-x_{P_r}\right)^2}\frac{1-\beta_{P_r}}{1+\beta_{P_r}}$$

$$E_{y1}=0 \qquad E_{z1}=0 \qquad \vec{B}_{1axis}=\vec{0}$$

We had an instantaneous longitudinal factor $\left(1-\beta^2\right)$ that did not depend on the direction of the velocity, as for the Lorentz contraction. Here, this is no longer the case with the retarded longitudinal factor $(1\pm\beta)/(1\mp\beta)$, as in the Doppler effect, which takes into account the direction and propagation delay.

∘ **Forces between two charges**

Two charges move in *R* with velocities $\vec{v}_1$ and $\vec{v}_2$. The electromagnetic force $\vec{f}_1$ exerted by P_2 on M_1 is expressed using the Lorentz force:

$$\vec{f}_1 = q_1 (\vec{E}_2 + \vec{v}_1 \wedge \vec{B}_2)$$

For rectilinear and uniform translation:

$$\vec{f}_1 = q_1 \left[\vec{E}_2 \left(1 - \frac{\vec{v}_1 \cdot \vec{v}_2}{c^2} \right) + \vec{v}_2 \frac{\vec{v}_1 \cdot \vec{E}_2}{c^2} \right]_t$$

As it should be, the classical principle of action and reaction is no longer verified. This principle presupposed simultaneity and instantaneous action of interactions.

For our two electrons in frontal collision:

$$\vec{f}_1(x_{1,} t) = q_1 E_{x1}(x_{1,} t)\, \vec{i} = -\frac{e^2}{4\pi\epsilon_0} \frac{1}{(x_1 - x_{2r})^2} \frac{1-\beta_{2r}}{1+\beta_{2r}} \vec{i}$$

ↀ Radiation damping

An accelerated charge emits electromagnetic radiation.

∘ **Radiated energy**

Energy emitted by units of time in the proper frame of the particle:

$$P = \frac{1}{4\pi\epsilon_0} \frac{2e^2}{3c^3} a^2$$

When the proper frame of reference is non-inertial, the particle, immobile and with proper acceleration a_p, radiates. The radiation is here integrated on all directions and frequencies. For non-relativistic particles, it is the Larmor formula established in 1897, valid then for any moving particle in the proper reference frame. This radiation makes the atom unstable in the Rutherford planetary model. Indeed, in this model, the electron radiates until it crashes on the nucleus[45].

For any relativistic motion of a charged particle in an inertial reference frame [46]:

$$P=\frac{1}{4\pi\epsilon_0}\,\frac{2e^2}{3c^3}\,\frac{a^2-\frac{(\vec{v}\wedge\vec{a})^2}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^3}$$

This radiated power slows down the particle. However, it is not simply equal to the kinetic power, because the particle is not an isolated system and it interacts with other charges. The particle is subjected to the damping force and the Lorentz force. On the other hand, in order not to have to take into account the interaction energy, we can calculate the energy of the system when all its constituents are at a large distance from each other. For example, for the scattering of two electrons, the difference in kinetic energy before and after the impact corresponds to the radiated energy.

◦ Damping force

We need to complete the equation of motion of a charge by adding emission damping $\tilde{g}$:

$$\frac{d\tilde{p}}{d\tau}=\boldsymbol{F}\,\tilde{j}+\tilde{g}$$

45 Hence the emergence of quantum physics, which has replaced the classical approach. The term classical can mean non-relativistic or non-quantum.

46 Landau,§ *Radiation from a rapidly moving charge*.

Covariant expression of the damping 4-force[47] :

$$g^{\mu}=\frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\left(\frac{dw^{\mu}}{d\tau}-\frac{u^{\mu}u^{\nu}}{c^2}\frac{dw_{\nu}}{d\tau}\right)$$

ꝏ RETARDED POTENTIALS

The formulas for the fields are deduced from the potentials:

$$V=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\frac{1}{(1-\vec{e}\cdot\vec{\beta})}\right]_{ret} \quad \text{and} \quad \vec{A}=\frac{q}{4\pi\epsilon_0}\left[\frac{1}{r}\frac{\vec{\beta}}{(1-\vec{e}\cdot\vec{\beta})}\right]_{ret}$$

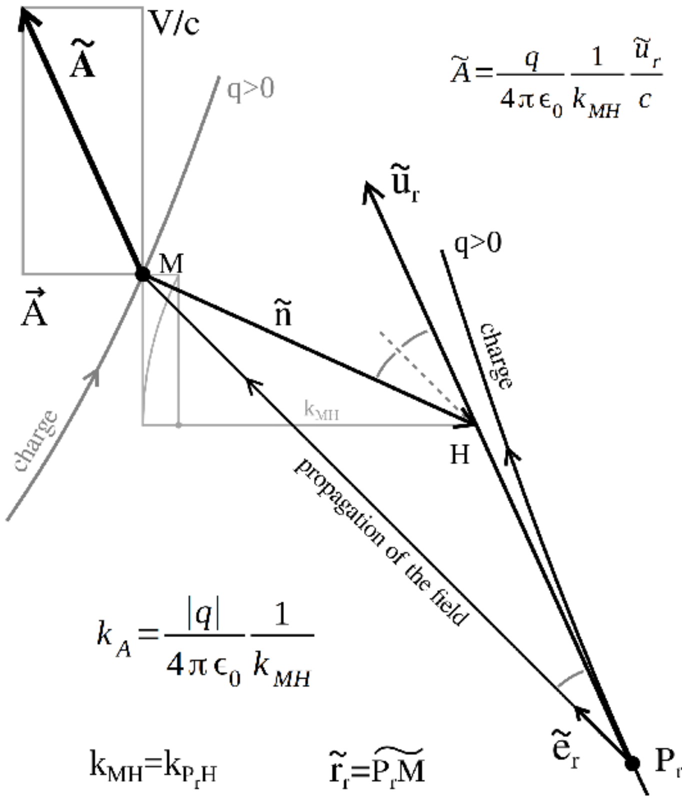
$$\Rightarrow\ \widetilde{A}=\frac{q}{4\pi\epsilon_0}\left[\frac{\widetilde{u}}{\widetilde{r}\cdot\widetilde{u}}\right]_{ret} \quad \text{with} \quad \widetilde{r}=r(1,\vec{e})=(r,\vec{r}) \quad \text{and} \quad \widetilde{r}\cdot\widetilde{u}>0$$

The four-potential is expressed using covariant quantities, so it is a four-vector. The four-potential $\widetilde{A}$ at M is collinear to the retarded four-velocity $\widetilde{u}_{ret}$ at $\mathrm{P_r}$. $\widetilde{A}$ is thus timelike, it points to the future for a particle with a positive electric charge, and to the past for a negative charge.

For a distribution of charges in motion at the retarded positions $\mathrm{P_i}$, the total four-potential at M will be the sum of the individual four-potentials. The four-potential created by a charge distribution is therefore also timelike.

47 Landau,§ *Radiation damping in the relativistic case*.

◦ **Geometric construction of the 4-potential**



Let's consider the world lines of two charged particles. We are looking for the field created on a particle at M by a second at P. We have drawn the Minkowski diagram in an inertial frame of reference. As the field propagates at the finite speed *c*, the event P is anterior on the past cone of M. We obtain H by orthogonal projection of M on the retarded four-velocity. The intensity of $\widetilde{A}$ at M is equal to the inverse of the intensity of $\widetilde{HM}$ multiplied by $|q|/4\pi\varepsilon_0$.

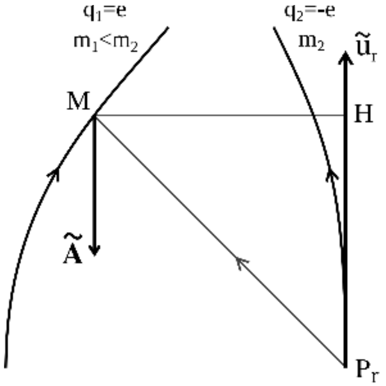
This property can be read directly on the graph:

$$\widetilde{r}\cdot\widetilde{u}=\left(\widetilde{n}+\widetilde{HP}\right)\cdot\widetilde{u}=k_{HP}\,k_u=k_{MH}\,c$$

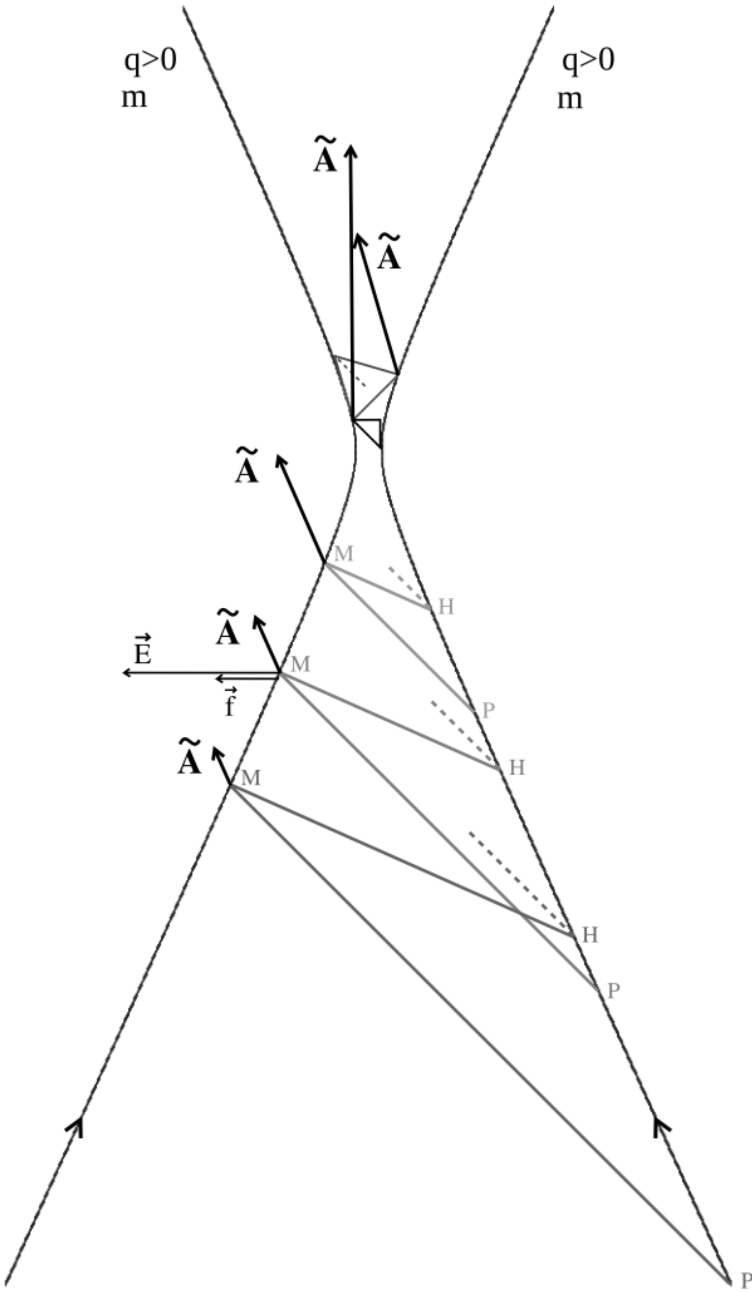
The intensities of $\widetilde{HP}$ and $\widetilde{MH}$ are equal, because these two vectors point to hyperbolas with the same parameters (90° rotation to switch from one hyperbola to the other).

The projections of $\widetilde{A}$ along the axes of the study frame of reference provide the potential *V* and the vector potential $\vec{A}$. The spatial and temporal variations of these two quantities then determine the $\vec{E}$ and $\vec{B}$ fields. It is therefore clear that by projecting along the axes of another inertial frame, we would have other values for the electric and magnetic fields. The intensities of the fields change, while the fourpotential remains the same.

Attractive case.
Charges of opposite signs,
first at rest, then move closer together.



Minkowski diagram of a collision :



Exercises

1. ▲△△ Units

Sometimes we need to switch from one unit system to another. We have in the books equations where c=1, or old systems of units with $1/4\pi\varepsilon_0$ that have disappeared.
We want the expressions in the new international system approved in 1946 (SI MKSA : m-kg-s-A).

We find in a book the expression of the radiated power *P* by a charge *e* of acceleration *a*:

$$P=\frac{2e^2}{3c^3}a^2$$

Restore if necessary the SI units.

Answers page 439

2. ▲▲△ Relativistic equation of motion

We consider motion in a one-dimensional Cartesian inertial frame of reference.

In classical mechanics we have: $a_x=\frac{dv_x}{dt}=\frac{F_x}{m}$.

What relationship do we have in special relativity between a_x and f_x ?

Notations: $\vec{a}=\frac{d\vec{v}}{dt}$ *and* $\vec{f}=\frac{d\vec{p}}{dt}$ *with* $\vec{p}$ *the momentum.*

Answers page 439

3. ▲▲▲ Radiation damping 4-force

Damping four-force $\widetilde{g}$ properties:

1- Show that $\widetilde{g}\cdot\widetilde{u}=0$.

2- For a rectilinear motion along *x*, determine the expression of *g* as a function of *v, a* and *da/dt*.

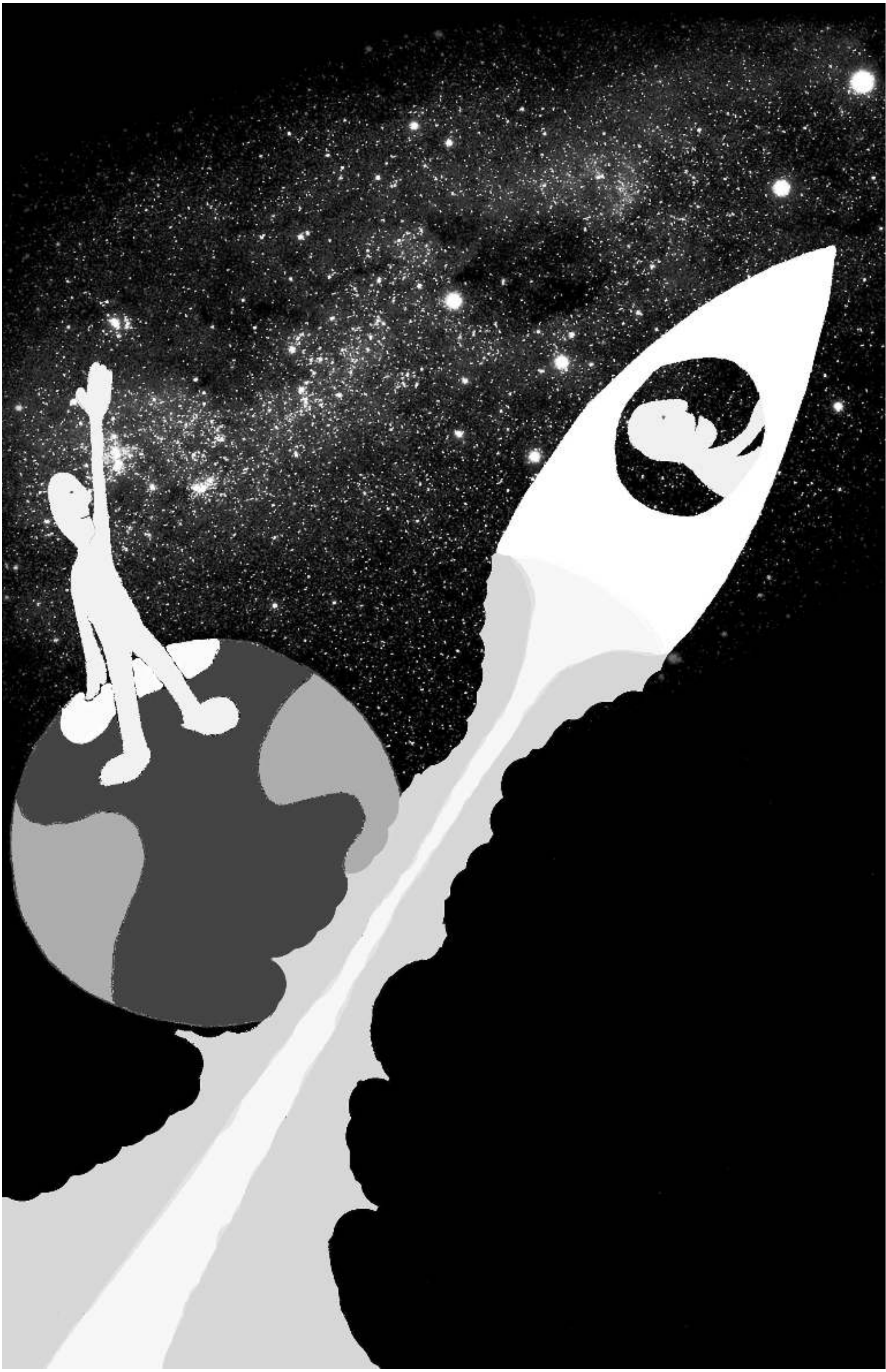
Notations: $g=g^x=g^1$ $a=a^x=dv^x/dt$ $v=v^x=dx/dt$

Answers page 439

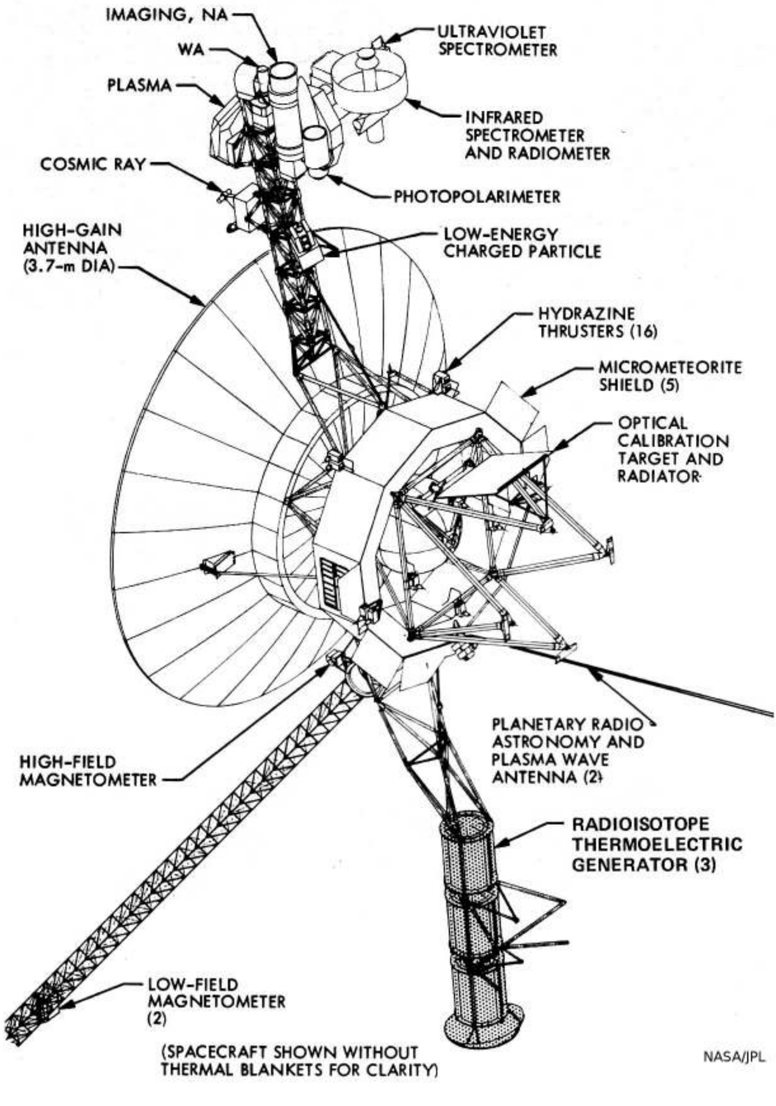
4. ▲△△ Four-potential magnitude

Express the intensity of $\widetilde{A}$ in terms of *r* and β.

Answers page 440



Voyager 1 and 2 probes



9

Interstellar travel and antimatter

Introduction

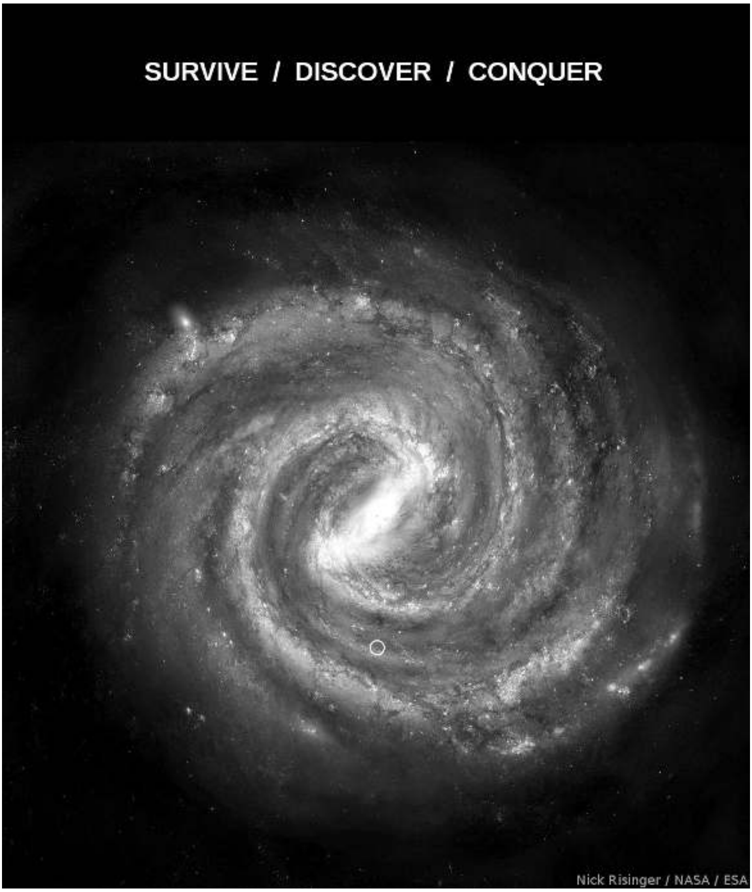
Who says travel, says to leave his place of life for several reasons:

- by necessity, for reasons of survival
- in the spirit of adventure and discovery
- to conquer and colonize

For all these reasons, we have for centuries:

- explored our planet Earth
- we are right now exploring our solar system
- and, one day, surely, we will leave our system to explore other stars

Our planet is fragile, and even if we managed to live on it in harmony, it may seem risky to stay in only one place.



A representation of a picture of our galaxy, the Milky Way. At night on a beautiful starry night without clouds and without Moon, we clearly see a milky band arching the celestial vault, the cross section of our galaxy. Our Sun is at the center of the small circle, and most of the stars we see at night are our neighbors and are contained in this zone.
Of course, this is not a real picture, we have never placed a camera in a place outside our own galaxy. This is a computer-generated reconstruction from real photos.

For example, it is very likely that a meteorite, like the one responsible for the disappearance of the dinosaurs, will hit the Earth again one day, in a few years, or, millions of years, we don't know. Hence the idea of a multi-planetary humanity, with as a starting point the establishment of autonomous colonies and extraterrestrial bases.

Some, such as Elon Musk are targeting the planet Mars with a manned mission planned for the near future, and subsequently the establishment of a Martian base and the terraforming of the planet. This project is exciting, but before a group of humans can live on Mars without being dependent on freight arrivals from Earth, it may take several centuries.

The planet Mars is perhaps the best candidate among the eight planets that orbit our Sun. But probably not among the thousands of exoplanets already discovered that orbit other stars!

The idea is to join an exoplanet that has a greater similarity to Earth than Mars, a twin planet of Earth, so, despite a longer journey, the colony could establish itself much faster.

Some will tell you that the other stars are far too far away and that interstellar travel is unrealistic, when in fact we are already making interstellar travel with *Voyager* probes.

They were built with the technologies of the 70s. They have already crossed the heliopause, the limit of our solar system, and are now traveling through the interstellar medium. These probes were designed to explore only the solar system, but, simply, with current technologies, they could be adapted to reach other stars. For example, the radioisotope thermoelectric generator will stop in 2025 and the transmission with. They can easily be replaced by batteries with an isotope with a much longer lifetime. The *Voyager* probes travel at about 61,000 km/h and would reach the closest star to our Sun, Proxima Centauri located 4 light-years away, in 70,000 years[48].

This is a lot compared to the life span of an individual, but very little compared to the age of mankind. As we will see, the spaceship can be large and reach this speed on the same principle. We can then design, still with currently accessible technologies, a *seedship*.

A manned journey over such a length of time is difficult to conceive, people would be born and die in the vessel over several generations, this type of vessel is called a *generation ship*.

On the other hand, the seedship contains only

48 In fact, over such periods of time we can no longer consider the stars motionless from one another. Nevertheless, in order not to complicate the presentation unnecessarily and to get to the point, we will consider the star Proxima Centauri fixed at 4 light-years.

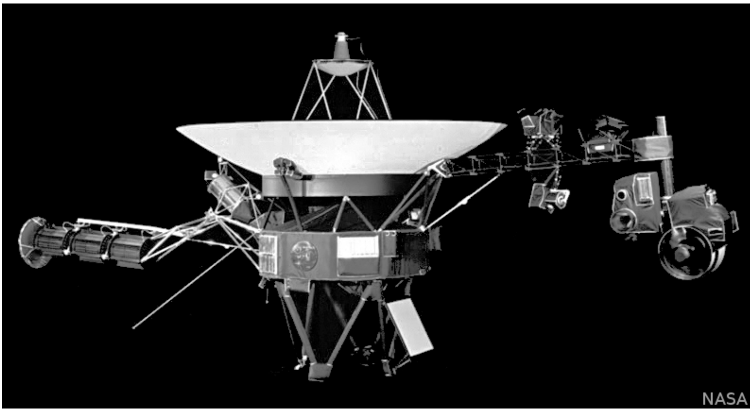
frozen ovocytes and spermatozoa (no risk of them hitting each other!). Once close to Earth's twin planet, an automated process starts the incubators and the first generation of children will be raised by robots with artificial intelligence.

At this rate, an extraterrestrial human civilization can establish itself and re-launch a new interstellar seeding ark in 100,000 years. Thus, step by step, in small leaps of 10 light-years, humanity can colonize the entire galaxy in less than a billion years. Reasonable duration, compared to the age of our Sun, 4.5 billion years, and the appearance of the first cells 3.8 billion years ago.

We will first talk about the *Voyager* probes and then detail other technologies that would allow us to reach the other stars much faster.

ꝏ *VOYAGER* PROBES

The two *Voyager 1* and *Voyager 2* probes were built identically and were launched in 1977. They each have a mass of 820 kg including 90 kg of propellants.

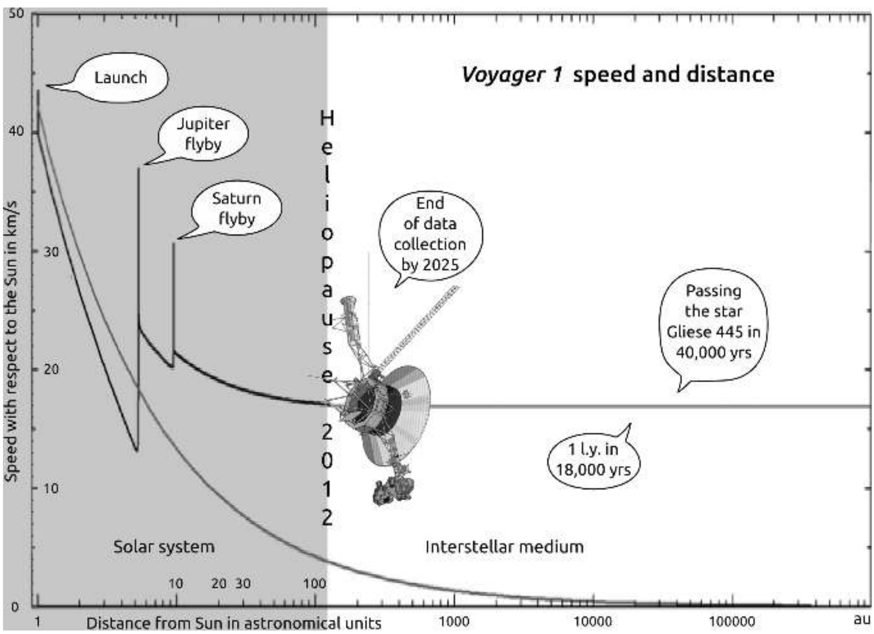


In astronautics, the term propellant, refers to the chemical substance that allows the propulsion of the rocket. For your car to work you must regularly take your vehicle to the pump to fill the fuel tank. But your car would not be able to run on the Moon, because for the combustion of the fuel it also needs the oxygen naturally present in the Earth's atmosphere. A rocket operates in vacuum and therefore has to carry both the fuel (the reductant), and the oxidant, the combination of the two is called propellant.

From the ground the probes left the terrestrial attraction on board Titan rockets containing tons of propellants. In addition to the speed thus gained, is added the speed of the Earth in its orbit around the Sun. But even so the speed of the probes was insufficient to break away from the solar attraction. And it is not the few kg of propellants carried by the probe that would allow it, they are used for trajectory corrections. The *Voyager* probes cleverly used the gravity assist of the planets to escape from the Sun's gravitational well.

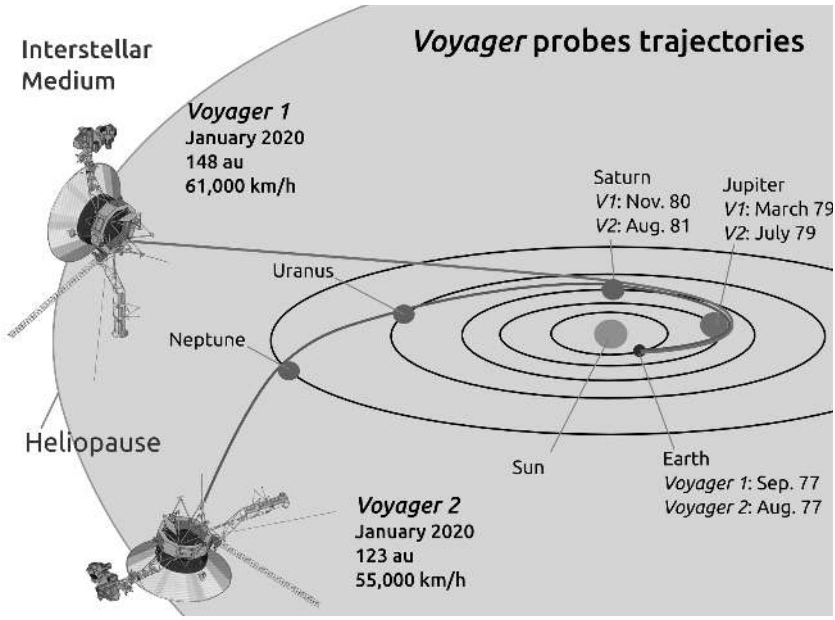
ꝏ Sling effect

We use the speed of revolution of the planets around the Sun. For example, Jupiter orbits at 13 km/s around the Sun and the *Voyager 1* spacecraft after its deflection by the planet has gained more than 12 km/s.

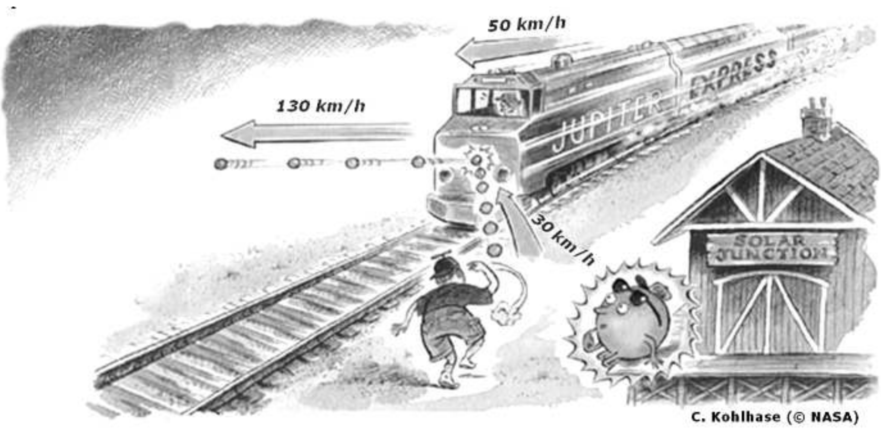


The black line represents the speed of the probe as a function of the distance to the Sun (multiplicative scale). By flying over Jupiter, the probe escapes its orbit around the Sun. The shaded line crossed corresponds to the speed necessary to escape from our stellar system. The astronomical unit corresponds to the distance Earth-Sun, one light-year is about 60,000 au.

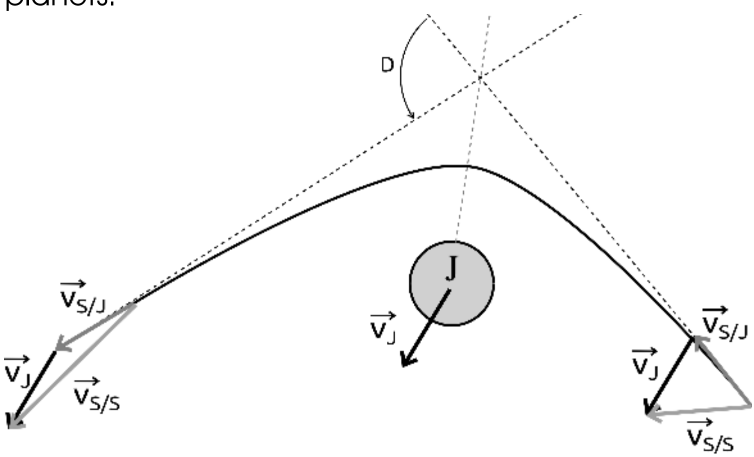
The *Voyager 2* probe even took advantage of the slingshot effect of four planets: Jupiter, Saturn, Uranus and Neptune.



We have a small drawing, that follows, which allows us to understand simply the sling effect. A train moves towards you at 50 km/h and you throw a ball at 30 km/h to make it bounce on the front of the locomotive. Let's now put ourselves in the position of the train driver, he sees by additivity of velocities the ball arriving faster, at 80 km/h, the sum of the velocities, with respect to the ground, of the train and the ball. If the collision is perfectly elastic, the ball starts again, with respect to the train, with the same speed and in the opposite direction. So the ball thrower sees the ball bounce back with a speed of 130 km/h with respect to the ground. By throwing the ball frontally, the speed of the ball increases by twice the speed of the train.



If now you throw the ball at a certain angle, the effect will be weaker but the principle remains the same. The same happens with the probe and the planets.



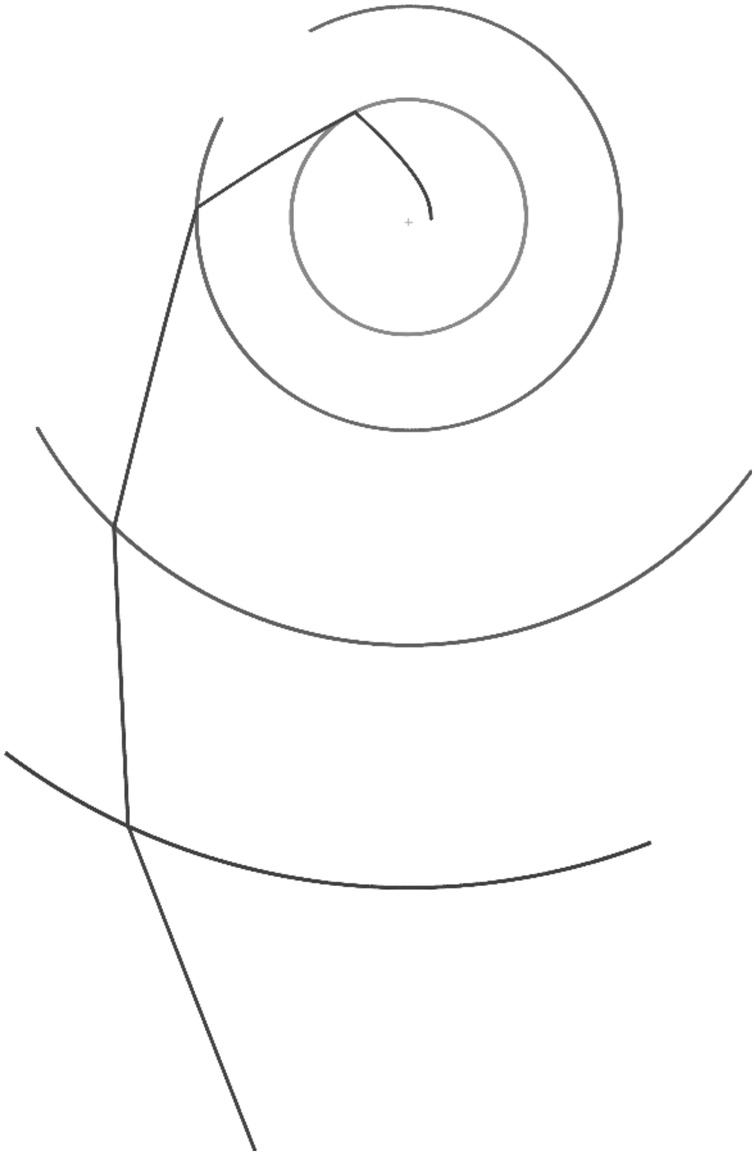
Jupiter in the center and the hyperbolic trajectory of the probe in the frame of reference which has for origin Jupiter. The velocity of the spacecraft $\vec{v}_{S/J}$ *with respect* *to Jupiter changes in direction but not in magnitude. The velocity of Jupiter with respect to the Sun must be added $\vec{v}_J$ to obtain the velocity of the probe $\vec{v}_{S/S}$ with respect to the Sun. We see in our figure that this speed increases, this is the slingshot effect. In the example of the train, there was a half-turn of the ball and the deviation D was 180°. For the passage of the Voyager 1 probe in March 1979, the deviation was 80° and the heliocentric speed of the probe increased by 12.5 km/s[49]. The object which benefits from the gravitational assistance can have an important mass without modifying the effect (its mass must remain small in front of the mass of Jupiter...).*

ꝏ *Voyager 3* PROJECT

The *Voyager* probes were not designed for interstellar travel, but to explore the solar system. For the *Voyager 3* project, we are optimizing the slingshots to gain speed and reach nearby stars. For example, we could take advantage of an opportunity: in 25,000 years, Proxima will be as close as possible to the Sun, 3 light-years away instead of 4.
This is a great project for mankind that also allows humanity to project itself into the future.

Next page, a numerical simulation of the trajectory of the spacecraft with the successive deviations at the flyby of Jupiter, Saturn, Uranus and Neptune.

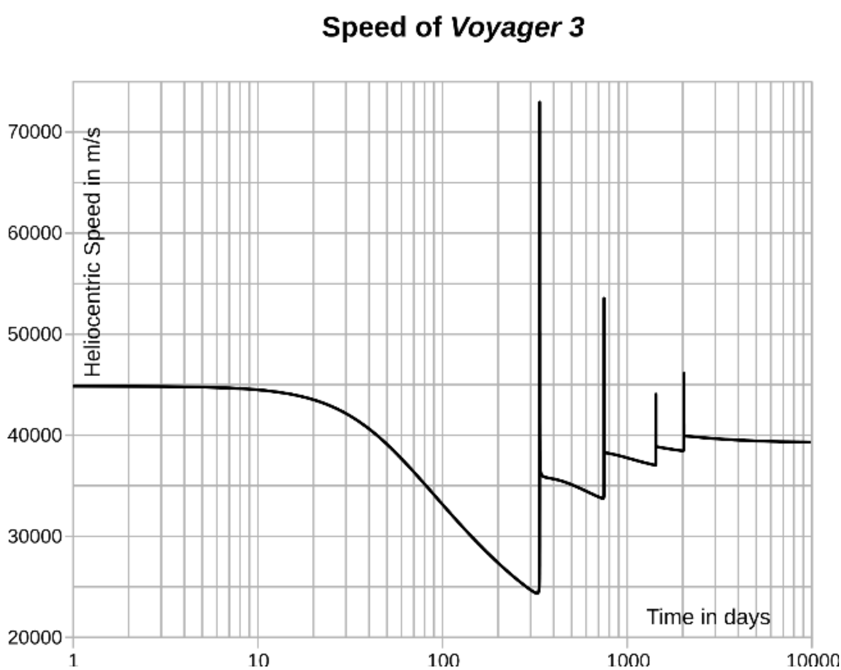
49 Document: *La fronde gravitationnelle*, Pierre Magnien, 2019. Real time position of the *Voyager* probes: voyager.jpl.nasa.gov.



Voyager 3: the probe is propelled at the level of the Earth's orbit and it then chains four slings around the gas giants. The final speed is 140,000 km/h. Two differences compared to the historical Voyager probes: additional propellant is used and the effect of the slings is optimized.

The mass of the whole, the probe and the propellant, is very reasonable: only about ten tons, which can be sent into space with the current rockets.

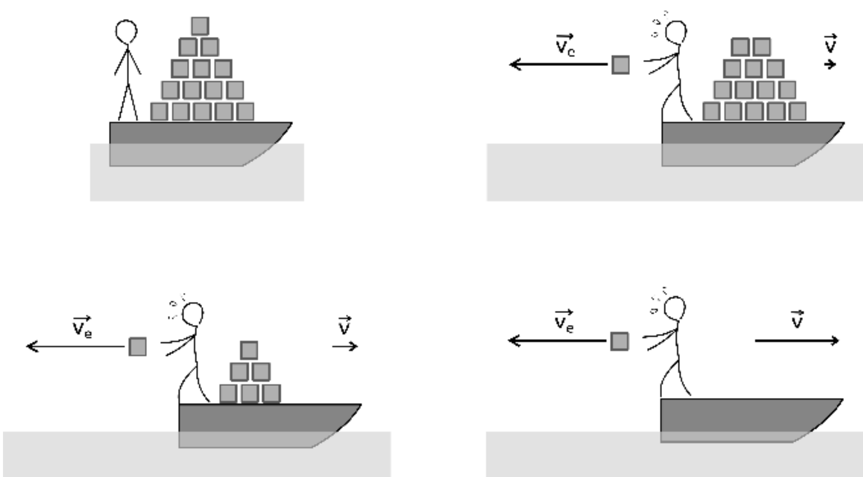
Below is the speed profile of the probe. We see an initial velocity surplus of 5 km/s given by the propellants. Each slingshot borders the upper atmospheres of the gaseous planets for maximum speed gain.



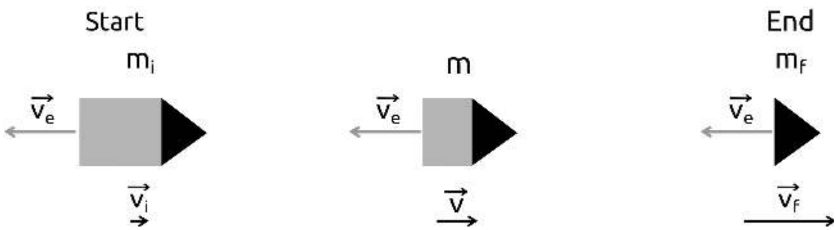
ROCKET EQUATION

We would like to go even faster towards the stars by thrusting the probe with propellants. The propellants burn and the resulting gases are ejected backwards and allow the rocket to gain speed by reaction. The law of astronautics gives the speed increase Δv of the rocket as a function of the initial mass m_i of the rocket, of its final mass m_f and of the speed of ejection v_e of the gases.

We can begin by illustrating this law with the example of a small boat on which a person throws stones backwards as far as possible with all his strength:

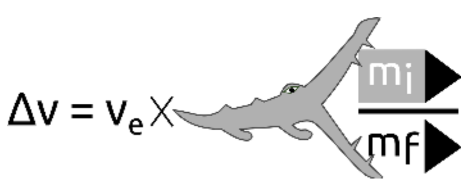


The boat is at first immobile with all its reserve of stones. The person on the boat throws a first stone backwards. The boat then starts to move slightly. This is the conservation of momentum. The friction with the water is neglected: the acquired speed is preserved. The person throws the stones until the stock is consumed and the speed of the boat increases with each throw. The last stone increases the speed much more than the first one because at the end the boat is much lighter. The first stones are not very effective because the boat is initially very heavy and they are used above all to move the stock of stones in waiting.



The initial mass of the rocket is that of the probe and the propellants, the final mass corresponds to the probe alone. The speed variation Δv is the difference between the final speed and the initial speed. The mass of propellant required increases very quickly, much faster than the speed reached.

Rocket equation:



The crocodile illustrates that in spite of a mass ratio made important by the increase in the quantity of propellants, this ratio is massively crushed by the need to also increase the speed of these same propellants before their combustion.

For a conventional chemical propellant we have an ejection speed of approximately 4 km/s. Let's imagine that we want to go twice as fast to reach Proxima with a Voyager-type probe. How much propellant would we have to take on board?
We then have Δv=60,000 km/h, or 16 km/s. The mass of fuel to be embarked increases exponentially and it would take 40 tons of propellants to get to Proxima in 35,000 years... To get there in 50 years, we would far exceed the mass of the Universe!

Duration of a trip to 4 light-years (current Sun-Proxima distance) with a Voyager type probe using traditional propellants (chemical energy / probe with a mass of 800 kg):

Duration of the trip	Mass of propellants required	$\frac{m_i}{m_f}$	$\ln\left(\frac{m_i}{m_f}\right)$
70 000 yrs	0 ton	1	0
35 000 yrs	40 tons	50	4
1 000 yrs	Mass greater than that of the observable Universe	∞	140
50 yrs		∞	2800

Once the star system is reached we can slow down the probe by sling effect. For the journey twice as fast, if we don't want to simply fly over the distant star system, the gravity assistance will not be sufficient to put ourselves in orbit around the star and we must also bring fuel to slow down the probe. As we have a factor of 50, we need 2000 tons of propellants at the departure from Earth to be able to be in orbit at the level of the exoplanet at the arrival!

To get around this monstrous increase in mass, the ejection speed would have to be increased instead. We would then have to use other technologies. We can use nuclear energy or mass energy.
For one kilogram of propellant, which substance allows the maximum release of energy?

Let's compare energy efficiencies. It is the energy released compared to mass energy. For example, one gram of antimatter releases more energy than a thousand tons of chemical propellants:

Propellant	**Efficiency**		**Details**
Chemical	1 / 6 billions	0.00000002 %	Oxygen-Hydrogen
Fission	1 / 1000	0.1 %	Uranium 235
Fusion	1 / 250	0.4 %	Deuterium-Tritium
Antimatter	1	100 %	$E=mc^2$

In the current state of scientific knowledge, antimatter appears to be the ideal fuel. The entire mass is then converted into energy and motion of the rocket.

Duration of a one-way trip for Proxima Centauri for a Voyager-type probe using an antimatter reactor (10% efficiency):

Travel time to Proxima	Antimatter mass required
70 000 yrs	0
35 000 yrs	230 grams[50]
10 000 yrs	1.4 kg
1 000 yrs	16 kg
50 yrs	333 kg

Calculations for a distance of 4 light-years. In fact, Proxima Centauri will be closest to the Sun at 3 ly in 25,000 years. For an equivalent quantity of propellants, we gain 10,000 years.

We see that the problem of the mass of propellants to carry has disappeared. We will therefore focus on antimatter: its nature, its collection and its storage.

∞ ANTIMATTER

Paul Dirac in 1928 constructed a theory to unify special relativity and quantum physics. It was then that antimatter imposed itself in the equations, it was later discovered experimentally as early as 1932 with the positron. Theoretical prediction appears as symmetry in the Dirac equation. In nature, to each elementary particle corresponds a "twin" particle, a particle with exactly the same mass but with an opposite electric charge.

50 One gram of antimatter releases as much energy as an atomic bomb.

For example, to the electron corresponds the anti-electron commonly called positron, or positon. In 1955, the antiproton was discovered by creating it with a particle accelerator. In 1995, the first atom of antimatter was created, the atom of anti-hydrogen. When a matter particle meets its antimatter counterpart, the two disappear and annihilate each other in pure energy. Hence perhaps the name antimatter, but, to avoid any confusion related to this name, let us specify that antimatter is matter.

We can produce antimatter artificially with a particle accelerator, but it also exists - although in much smaller quantities than matter - in nature.
The production of antimatter in the laboratory requires a lot of time and energy. For example, to create antiprotons, protons are accelerated and when they collide at high energy, they create proton/antiproton pairs:

$$p+p \rightarrow p+p + p+\overline{p}$$

You create a proton for nothing and the productivity is low. It is very interesting and precious to understand the secrets of matter on a small scale, but, to produce the propellant for a rocket, it is perhaps not the most judicious[51].

51 In 2020, world energy production corresponds to the energy released by the annihilation of 3.5 tons of antimatter, however, with the existing current means, even to produce just one gram of antimatter would be prohibitively expensive.

It would be simpler to collect it in the nature. Positrons are released by beta-positive radioactivity, by cosmic rays or even storms. Antiprotons are a fuel of choice because they have a mass energy much higher than positrons. However, unlike positrons, antiprotons are not directly produced in our solar system. The Sun, the most powerful source of energy in our star system, only rises in energy to the level of fusion and the solar wind does not contain antiprotons.

We must, therefore, look for a source of antimatter outside our system. This source exists, it was discovered in 1912, it is the cosmic rays. It is made up of particles of very high energy capable of creating antiprotons. The precise sources of this radiation are not yet known, but it is now believed that they are mainly located in our galaxy. This galactic radiation is constantly passing through the solar system, and it is estimated that 200,000 tons of antimatter crosses the heliosphere every year[52].

The density of antiprotons is higher in the planetary magnetospheres. For example, around the Earth, there is an antimatter belt with a zone a thousand times denser than the surrounding cosmic rays[53]. Cosmic antiprotons are trapped, and moreover,

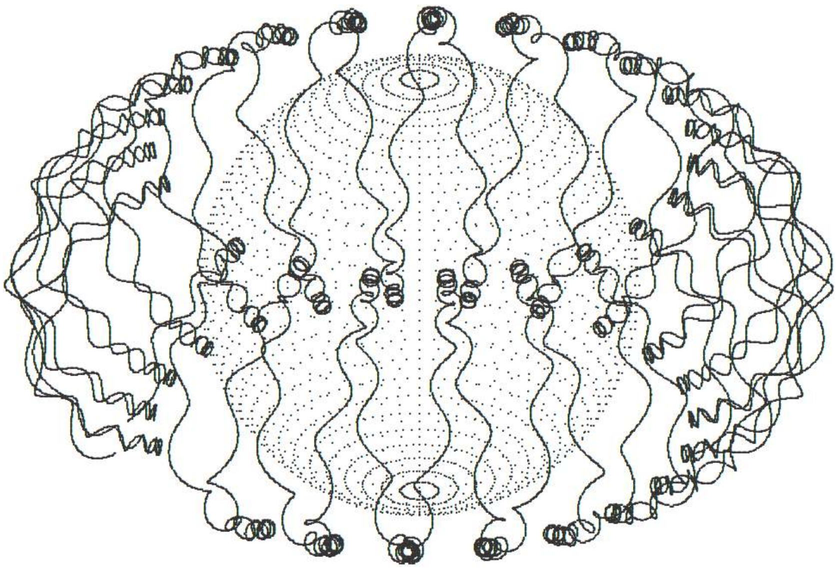
52 A lot of data is taken from a very comprehensive article from the Draper Laboratory: *Extraction of antiparticles concentrated in planetary magnetic fields*, 77 pages, 2006.

53 Analysis of results from the PAMELA detector installed on a satellite in Earth orbit: *The discovery of geomagnetically trapped cosmic ray antiprotons*, 2011.

others are directly created by the interaction of cosmic rays with the upper layer of the Earth's atmosphere. The Earth's antiproton belt is located several hundred kilometers above sea level in the Van Allen radiation belt.

ꝏ JUPITER: THE SOLAR SYSTEM GAS PUMP

The Earth generates a magnetic field that traps charged particles at altitude, such as electrons contained in the solar wind. Sometimes during a destabilization of the magnetosphere, for example following a solar flare, electric particles are released at the poles and create beautiful polar auroras. The magnetosphere acts as a giant magnetic bottle that stores all kinds of charged particles. The Earth's magnetosphere is subjected to a flux of about 4 grams of antiprotons per year. But it is mainly the large gas giant planets, and, without a doubt, the gigantic magnetosphere of Jupiter that could contain the largest amount of antimatter with a flux estimated at 9 kg per year.

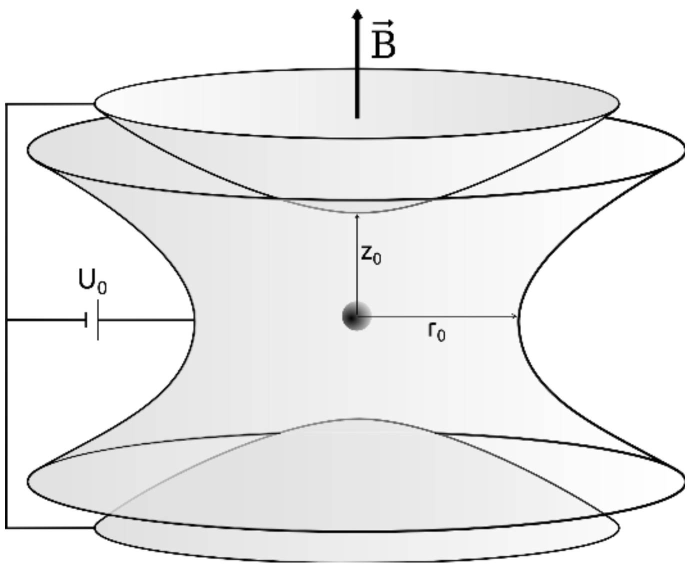


A picture of the antiproton belt around the Earth. Here, an antiproton moving at 70% of the speed of light. The Earth's magnetic field curves its trajectory and traps it using three types of combined motions: the fastest, a cyclotron rotation that makes it make small circles, then, an oscillation between the poles, and finally, a slower drift that makes it go around the Earth.

Satellites could collect and store this antimatter. The ships would then refuel at Jupiter before leaving for the stars.

ꝏ ANTIMATTER STORAGE

We currently know how to store antiprotons for more than a year. The temperature is maintained below one Kelvin and the measurements of the characteristics of the antiproton are extremely accurate[54]. Nevertheless, the quantities are very small and the mass of the trap is very large compared to the mass of antimatter stored.



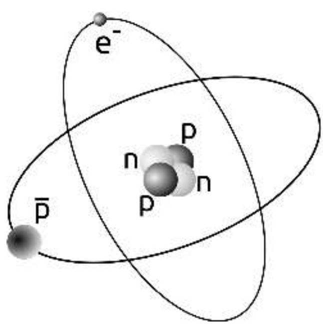
Penning trap. By combining a magnetic field and an electric field, charged particles can be trapped in the laboratory.

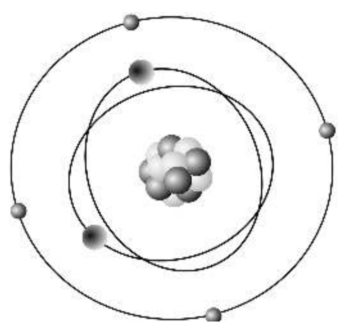
The ideal would be to store antimatter on a microscopic scale. The antimatter thus trapped and

54 BASE experiment: *A parts-per-billion measurement of the antiproton magnetic moment*, review Nature, 2017.

confined at the atomic or molecular scale could then be stored like matter. We would have a flexible and versatile use of this new fuel, both for space travel and in our daily lives. For example, a car could travel around the Earth on a single tank of a few milligrams of antimatter.
Let's call *Proximium* this hypothetical fuel of the future. A luminal fuel that would allow us to reach the stars and bring us into a new energy era. Could this dream come true? Only experimentation will allow us to make progress on this question. Let's start by letting our imagination consider different options.

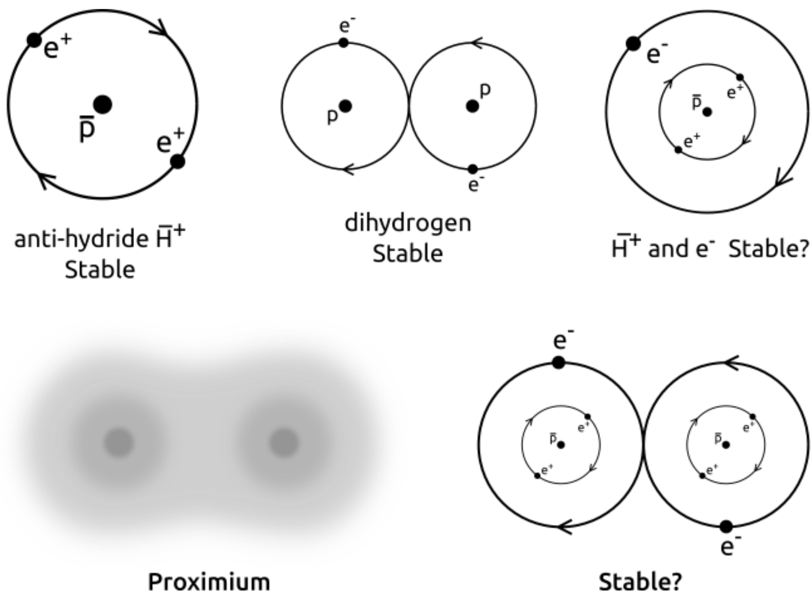
1 - Exotic atoms where an electron would be replaced by an antiproton:





Examples of helium and carbon atoms where one or more e^- have been substituted by a $\bar{p}$. Antimatter density of the structures: 20% and 14%. The first compound, sometimes called antiprotonic helium and noted $\bar{p}He^+$, was discovered by serendipity at the Japanese CEC laboratory in 1991, and then studied at the CERN antiproton decelerator. Normally an antiproton is stopped *by matter and annihilates on a nucleus in a time of the order of a picosecond. In this experiment, where a beam of slow antiprotons encounters a liquid helium target, we naturally obtain the metastable $\bar{p}He^+$ state in which the trapped antiproton can be stored for several microseconds*[55].

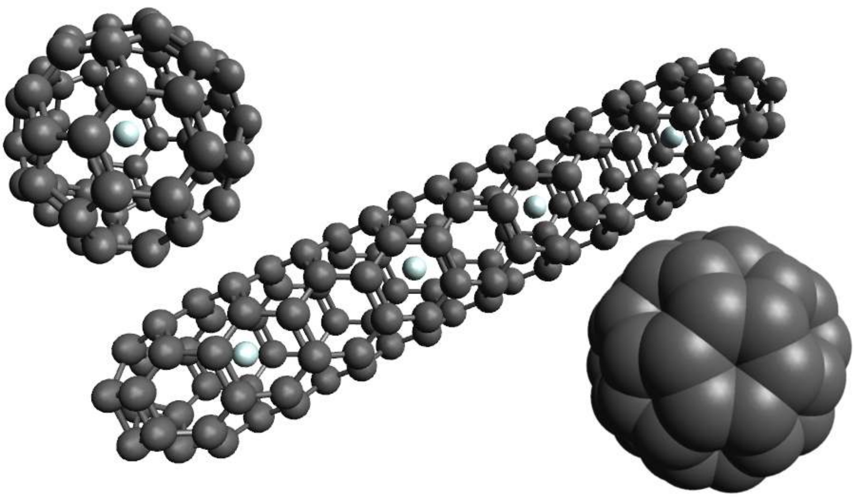
2 - An antihydrogen atom ionized with an additional positron $\bar{H}^+$, could replace the nucleus of a hydrogen atom. Two such exotic atoms would constitute a Proximium molecule:



The storage density in this case would be almost 100%. Experimental research can first focus on the synthesis of an anti-proximium molecule. Experiment easier to implement for a molecule that has the same stability.

55 Article of Hayano *Spectroscopy of antiprotonic helium atoms and its contribution to the fundamental physical constants*, Japon, 2010.

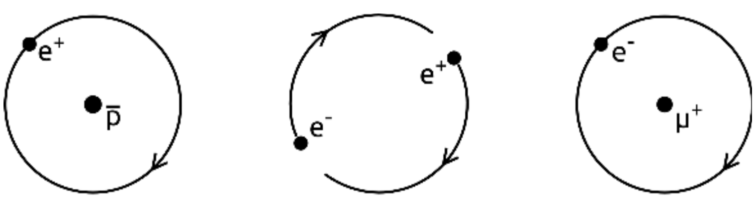
3 - A cage molecule. There are many cage molecules in chemistry that allow the encapsulation of molecules. We can imagine such a molecule that contains an antiproton as in a microscopic Penning trap. We have, for example, fullerene-type molecules and nanotubes:



Different carbon-based structures. In the top left corner, we represented the C_{60} fullerene. Different types of atoms have already been trapped in these structures. Fullerene can easily be negatively ionized and could thus be a good antiproton trap. Bottom right, the same structure using a model showing the electrostatic spheres of influence of electronic clouds. Diagonal, a nanotube with 4 confined antiprotons.

And so on... We can start by measuring the life span of such structures, and maybe one day we will have the pleasant surprise of finding a stable one. Scientific research makes it possible to test multiple combinations. It's worth the effort because even if we don't find what we're looking for, we'll have learned a lot about matter.

Scientists have already studied different exotic atoms. We have created and studied anti-hydrogen atoms that have proven to be stable. Another hydrogen derivative, positronium, which consists of an electron and a positron that revolve around each other, has a stability of 100 nanoseconds. The muonium, on the other hand, replaces the nucleus of a hydrogen atom by a muon, the stability is 2 microseconds.



Anti-Hydrogen Stable | Positronium 100 ns | Muonium 2 μs

Stability can also depend on the context. For example, a neutron in the nucleus of an atom is stable, whereas in its free, isolated state, the neutron has a lifetime of only 10 minutes.

Conclusion

By learning to master antimatter we could reach the first stars in 50 years and explore the entire galaxy in a few million years. This type of vessel could be manned and would quickly overtake the previously sent seed ships. Both scenarios deserve to be developed in parallel over the next decades.

Elon Musk projects a colony on Mars of one million humans by 2050 and a progressive empowerment. Also planned are microprobes for Proxima propelled by giant lasers placed on Earth.

Often for interstellar travel, nuclear fission or fusion are proposed as a source of energy and antimatter is little considered. The aim of this conference is to show the important potential of antimatter as a key element for the future.

Exercises

1. ▲△△ Figures

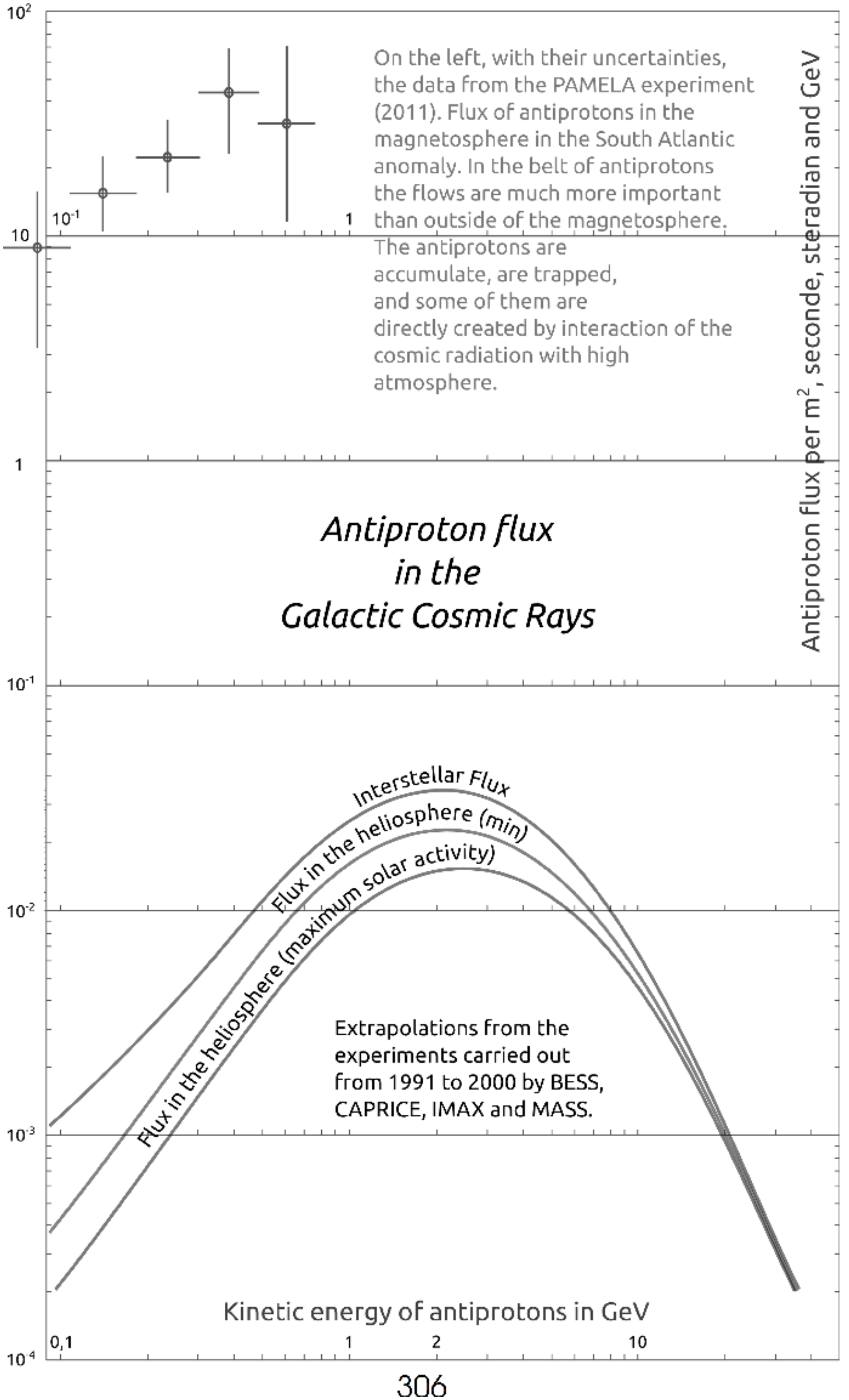
Find the numerical values of the conference:

- A probe goes at 61,000 km/h to a 4 ly star.
  Do you find 70,000 years of travel?

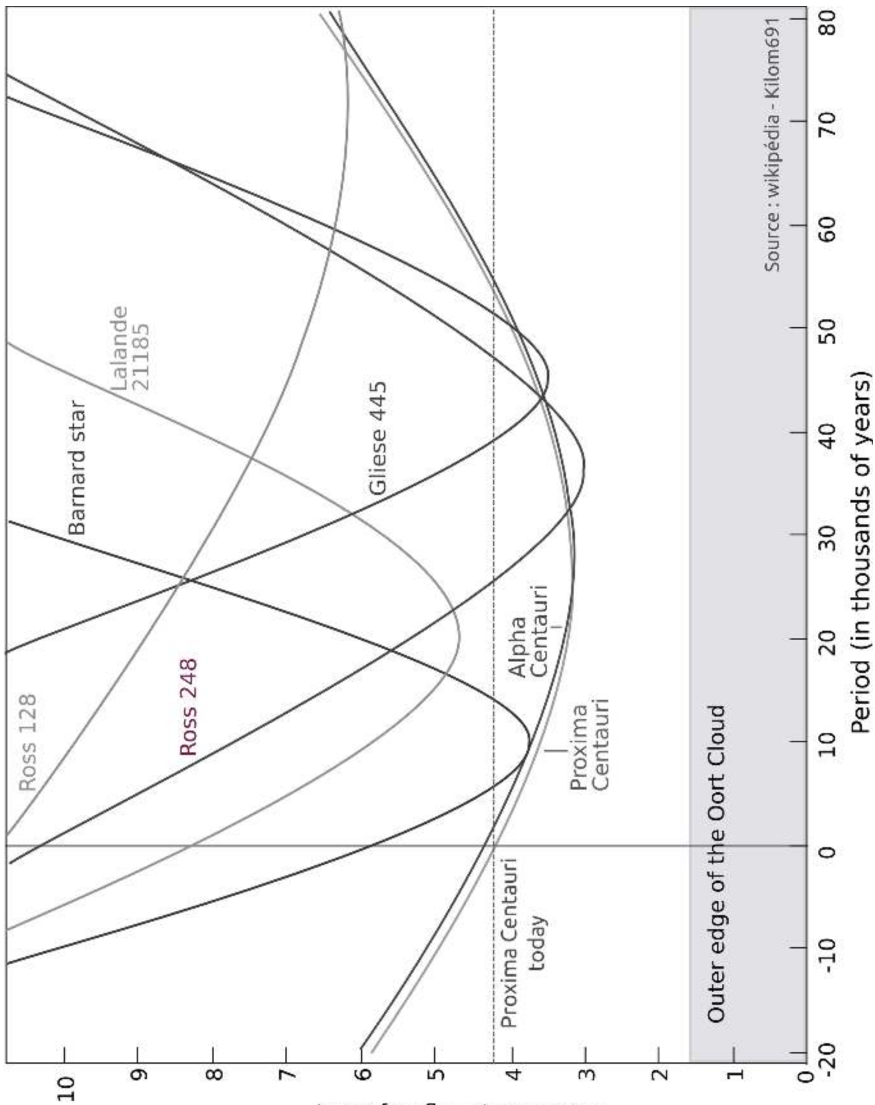
- World energy consumption is estimated at 15,000 Mtoe in 2020. The toe (ton of oil equivalent) is worth 42 GJ. Show that this energy is equivalent to the energy released by the annihilation of 3.5 tons of antimatter.

- Using the data in the table on page 34 of the article *Extraction of antiparticles concentrated in planetary magnetic fields*, find the 200,000 tonnes of antimatter that crosses the heliosphere each year. For example, for Jupiter the flux is 9.1 kg of antiprotons for a cross section of 45 R_J radius (zone of influence of the Jovian magnetosphere with R_J the radius of Jupiter). The effective radius of the Sun is taken at heliopause, limit of the influence zone of the solar magnetic field. If we now take the interstellar flux of cosmic radiation, external to the heliosphere, evaluate how much the antimatter flux is by using the following curve.

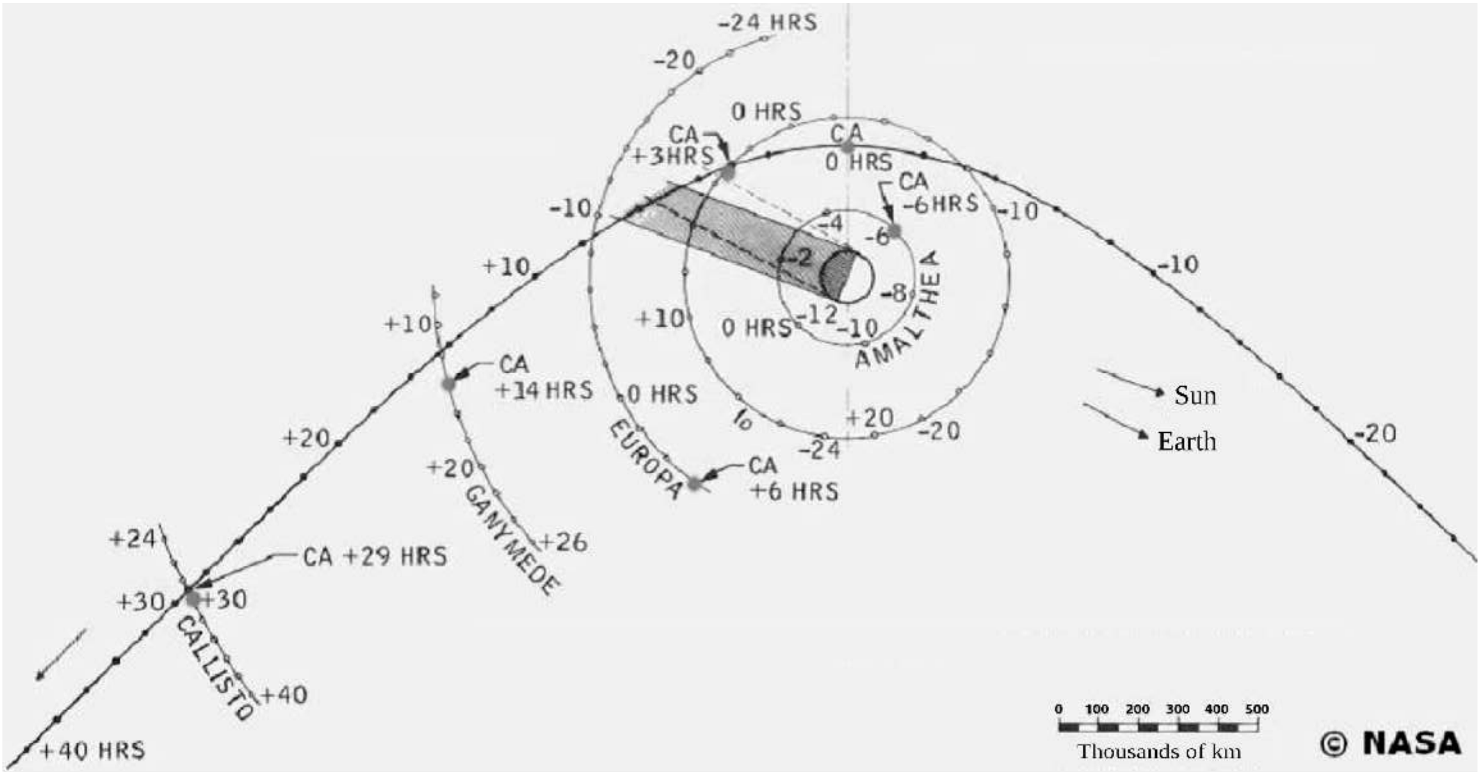
Answers on page 441.



On the left, with their uncertainties, the data from the PAMELA experiment (2011). Flux of antiprotons in the magnetosphere in the South Atlantic anomaly. In the belt of antiprotons the flows are much more important than outside of the magnetosphere. The antiprotons are accumulate, are trapped, and some of them are directly created by interaction of the cosmic radiation with high atmosphere.
Antiproton flux in the Galactic Cosmic Rays
Antiproton flux per m², seconde, steradian and GeV
Interstellar Flux
Flux in the heliosphere (min)
Flux in the heliosphere (maximum solar activity)
Extrapolations from the experiments carried out from 1991 to 2000 by BESS, CAPRICE, IMAX and MASS.
Kinetic energy of antiprotons in GeV
10²
10
1
10⁻¹
10⁻²
10⁻³
10⁻⁴
0,1
1
2
10



Over significant periods of time, several thousand years, the stars can no longer be considered fixed to one another. The three stars of the Alpha Centauri system will be closest to the Sun in 25,000 years at three light years.



At the center Jupiter. The hyperbolic trajectory of Voyager 1 in the inertial reference frame centered on Jupiter. Five Jovian satellites with the minimum approach positions (CA).

2.✎ ▲△△ The distances of stars over time

In the conference the distance Sun-Proxima is set to 4 light-years. For fast journeys the stars can be considered fixed, but for slow journeys of more than 10,000 years the variations in distance are no longer negligible. We have placed the curve in the previous pages. Show that the *Voyager 1* and *2* probes could not reach Proxima Centauri. What should be the minimum speed of the probes? How fast does a probe have to go to reach the Alpha Centauri system when it is closest?

Answers p442.

3.✎√ ▲▲▲ Sling effect

We consider the flyby of *Voyager 1* at the level of Jupiter.

a - With an initial probe speed of 12.6 km/s and a Jovian speed of 12.8 km/s, find the speed variation of *Voyager 1* (heliocentric velocities). The motions are assumed to be coplanar and the trajectory of Jupiter in the heliocentric reference frame circular. You will estimate the required angles using the curve on the previous page.

Help: it is not easy to visualize the asymptotes, trajectories at a great distance from the probe, the view is too close. Two indications: the inner angle between the two

asymptotes of the hyperbola is 82° and the impact parameter b is 13 R_J (b: distance between the barycenter of Jupiter and the asymptotes — R_J: radius of Jupiter).
Definition of angles : $\alpha_i = \widehat{(\vec{v}_J, -\vec{v}_i)}$ *and* $\alpha_f = \widehat{(\vec{v}_f, \vec{v}_J)}$.

b - Evaluate on the NASA graph the maximum speed of the probe at the *periastron*. Does the result correspond to the peak on the graph page 283 ? Estimate the speed of the probe 38 hours after its passage at the periastron. Deduce, by calculation, the speed of the probe to infinity. Evaluate the minimum approach distance of *Voyager*, and deduce by calculation the *impact parameter b* of the probe.

Help: For an isolated system, in a Galilean frame of reference, there is conservation of mechanical energy and angular momentum.

c - Conic parameters.
Find the *semi-latus rectum p*, the *eccentricity e* and the deviation *D*.

Aids: The general solution of the Kepler problem provides the polar equation of a conic (hyperbola, parabola and ellipse):

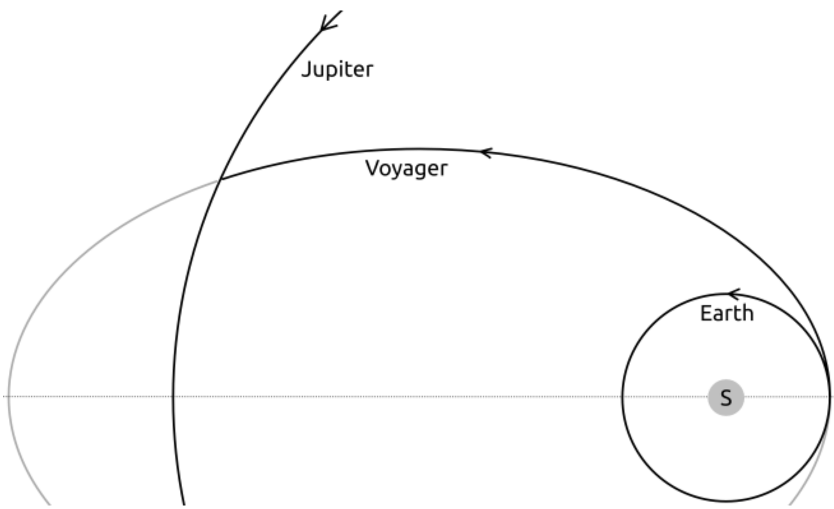
$$r = \frac{p}{1 + e\cos\theta} \qquad p = \frac{L^2}{\alpha m} \qquad \alpha = GmM$$

Origin of the reference system: center of mass of Jupiter.
Angles origin: main axis of the hyperbola.
p : semi-latus rectum of the conic.
e : eccentricity. L : angular momentum of the probe.
$M = M_J = 1.90\times 10^{27}$ kg. m : mass of the probe.
Distance Sun-Jupiter: 800×10^6 km. $M_S = 2\times 10^{30}$ kg.

d - We want to increase the sling effect.

• All else being equal, for what value of α_f do we get a maximum v_f? Determine the corresponding Δv. If the probe then left the solar system directly, what would be its interstellar speed?

• The trajectory of the probe from the Earth is considered to correspond to an orbit of the Hohmann transfer elliptic orbit type.
What is the semi-major axis a of this ellipse?



We can also find again the angle of approach. How could we increase the interstellar speed of the probe? We must not get too close to Jupiter. The equatorial radius of Jupiter is 71,492 km and an altitude of 1,000 km places the probe as close as possible, in an atmosphere sufficiently tight that its influence can be neglected.

Aids: Mechanical energy for a conic:

$$E_m = \frac{\alpha}{2p}\left(e^2 - 1\right). \quad \textit{Ellipse:} \quad E_m = -\frac{\alpha}{2a} \quad \text{and} \quad p = \frac{b^2}{a}$$

• Explain why Mars does not allow to have a consequent slingshot effect despite its high orbital speed.

• Retrieve the characteristics of the speed profile of *Voyager 1* by considering the two slings one after the other (Jupiter then Saturn). A spreadsheet can be used for a systematic calculation for *n* slings. Conservation of the angular momentum and mechanical energy between the slings, properties of the hyperbola during a sling.

• Model the succession of the four fronds from Jupiter to Neptune. Show that it is possible to obtain, by optimizing the successive effects, an interstellar speed of 100,000 km/h (on the principle of *Voyager* probes and using only gravitational assistance). Show that by giving, at the level of the Earth's orbit, a speed surplus of 4.8 km/s using propellants, the probe reaches an interstellar speed of more than 137,000 km/h.

• Globally, all the planets revolve around the Sun in the same plane, called the ecliptic. In our model for the succession of the fronds, the probe leaves the solar system in this plane. However, most of the stars are out of the ecliptic. For example, at the closest, in 25,000 years, the star Proxima will be located 39°

below the ecliptic plane[56]. The velocity given by the gravity assist has a value but also a specific direction. The direction of the velocity is just as important as its magnitude: what's the point of going fast if it's not to the right place? Do you have a proposal to have a correctly directed velocity without using huge quantities of propellants?

• The probe at the end of its 25,000-year journey flies over the Alpha Centauri star system. How should we proceed to slow down the probe in order to trap it in the star system? Should additional propellant reserves be provided for this purpose?

Answers p442.

4.$^{01}_{10}$ √ ▲▲▲▲ Numerical simulations of the slings

The simulations make it possible to recover the results established in the previous exercise, which used Kepler's formulas. Also, simulations give a great deal of freedom and help to envisage a number of situations. The counterpart is the necessary computing power. We will use that of a personal computer. This will be sufficient for a first approach and to explain the basic principles.

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56 Calculations in the exercise *Motion of the stars* on page 322. Current ecliptic coordinates of the stars: heasarc.gsfc.nasa.gov/cgi-bin/Tools/convcoord/convcoord.pl. Often only equatorial coordinates are given, all conversions on this site.

We will study the problem of N bodies in gravitational interaction. The modeling is very ambitious and the computation time can be very long: the number of interactions evolves in N factorial and from N=3 we can have chaotic regimes. Each body has 6 degrees of freedom, three for the position and three for the velocity components. We will, therefore, simplify with a set of reasonable hypotheses.

For the *Voyager* probes the motions will be considered in the same plane: indeed, it is a reality, basically all the planets orbit in the plane of the ecliptic, moreover, it is shown that the two-body motion is done in one plane.

We will assume that the Sun is motionless. This way we have one less body to consider. The heliocentric reference frame is then Galilean. No need to consider the center of mass of the solar system and the Copernican frame of reference, because the mass of the Sun is very large in front of those of the other bodies.

We will not consider the forces between the planets. Always to simplify the equations, reduce the number of relations, and the computation time. Only the Sun exerts its force on a planet. Only the probe remains connected to all the bodies.

Newton's equations of motion give a system of coupled differential equations:

$$\frac{d\overrightarrow{OM_i}}{dt}=\vec{v}_i \quad \text{and} \quad \frac{d\vec{v}_i}{dt}=\sum_{j\neq i} Gm_j \frac{\overrightarrow{M_i M_j}}{r_{ij}^{\;3}}$$

For each body, we have two vectorial differential equations of order one. For a 2D motion, we have four variables per body: x_i, y_i, v_{xi} and v_{yi}. Finally, for the probe and the four gas giants we have 20 equations. It is already a lot.

The principle of digital resolution is simple, it is a step-by-step method. We have the initial conditions at *t=0*, positions and velocities of all bodies. After a small interval of time *Δt*, we evaluate the new velocities and positions using differential equations. We thus pass, step by step, from t_n to t_{n+1}:

$$x_{i,n+1}=x_{i,n}+v_{x,i,n}\Delta t \text{ , ... ,}$$

$$v_{x,i,n+1}=v_{x,i,n}+F_{x,i}\left(x_{j,n}, y_{j,n}\right)\Delta t \text{ , ...}$$

This is the *Euler method*. We will then study the much more precise *Runge-Kutta method*.

Mechanically, as in a line of dominoes that fall one after the other, we move causally from one stage to the next. At each step, we make a small local error that accumulates to the one of the previous step. We will take a step small enough to be able to properly linearize each segment and minimize the global error. Since we are not mathematicians, in this initiation exercise we will be content to control, as good physicists, the conservation of mechanical energy and angular momentum.

We will use a spreadsheet program. No need to download any special programming software, a worksheet will be enough.
Let's start by practicing on simple models for which the analytical solutions are known.

1 - *Revolution of the Earth around the Sun:*
Let us take as initial conditions the Earth at its perihelion: r_{min}=147,098,074 km and v_{max}=30,287 m/s.
Sun mass: $M_S=1.9891\times10^{30}$ kg.
Gravitational constant: $G=6.6743\times10^{-11}$ N.m^2/kg^2.

a- *Kepler law's:* The previous data comes from Wikipedia. Determine, from them, the semi-latus rectum p of the conic, the eccentricity e, r_{max}, v_{min}, the semi-major axis a and the period T.

b- First simulation with a step h=1 day.
Do you get a satisfactory simulation on a revolution? What is the percentage of error on the radius after one revolution? How does this percentage change for h/2, h/4 and h/8?
Do you find the right values for the period of revolution and the values at aphelion ?
Even already on the first step from t=0 to t=h, do you notice an anomaly ?
How to explain it?
We have calculated the values at t_{n+1} from those at t_n. For example, the velocity $v_{x,\,n+1}$ is calculated with $v_{x,\,n}$, x_n, and y_n . On the same principle, the position x_{n+1} is calculated with $v_{x,\,n}$ and x_n. But it would also be quite possible to determine the positions x_{n+1} and y_{n+1}

with the velocities at rank n+1. Indeed, it is no more false to take the velocity at the end of the interval than at the beginning. Run the simulation again for h=1 day with this modification for the calculation of the positions. Do you now find better estimates for the period and the aphelion? What is then the global error for the radius after one revolution? What is the value of the variation of mechanical energy over 365 days? Conclusion.

2 - *Runge-Kutta method of order 4 (RK4)*:
The global error with the Euler method was of the order of h, with the midpoint method (for example, the modified Euler method seen previously) according to h^2, and with RK4 in h^4. Although the calculation for one step will be a little longer, the total calculation time for the same global error will be immensely shorter. Rather than using only one slope, the one at the beginning of the interval, as for the Euler method, we will use four slopes judiciously distributed and weighted over the interval.
We give the general Runge-Kutta scheme for two degrees of freedom, and let you generalize. The degrees of freedom are named X and Y. For example, in physics, for a one-body motion in one direction, we would have X=x and $Y=v_x$.
X(t) and Y(t) obey the following differential equations:

$$\frac{dX}{dt}=A(X,Y) \quad \text{and} \quad \frac{dY}{dt}=B(X,Y)$$

With the initial conditions X(0) and Y(0) known.

We determine the values X_{n+1} and Y_{n+1} from those of the previous rank X_n and Y_n over the interval [nh, (n+1)h] with the following iterative method. For each degree of freedom we have four slopes to calculate. For example, for X, A_1 corresponds to the slope at the beginning of the interval, A_2 and A_3 are estimates of the slope in the middle of the interval, and A_4 is an estimate at the end of the interval:

$$A_1 = A(X_n, Y_n) \qquad B_1 = B(X_n, Y_n)$$

$$A_2 = A\left(X_n + \frac{h}{2} A_{1,} Y_n + \frac{h}{2} B_1\right)$$

$$B_2 = B\left(X_n + \frac{h}{2} A_{1,} Y_n + \frac{h}{2} B_1\right)$$

$$A_3 = A\left(X_n + \frac{h}{2} A_{2,} Y_n + \frac{h}{2} B_2\right)$$

$$B_3 = B\left(X_n + \frac{h}{2} A_{2,} Y_n + \frac{h}{2} B_2\right)$$

$$A_4 = A\left(X_n + h A_{3,} Y_n + h B_3\right)$$

$$B_4 = B\left(X_n + h A_{3,} Y_n + h B_3\right)$$

$$X_{n+1} = X_n + \frac{h}{6}\left(A_1 + 2A_2 + 2A_3 + A_4\right)$$

$$Y_{n+1} = Y_n + \frac{h}{6}\left(B_1 + 2B_2 + 2B_3 + B_4\right)$$

We take again the case of the revolution of the Earth around the Sun with this method.

a- Establish the RK4 scheme to solve this problem: define the variables, write the differential equations of order 1 while naming the functions and the slopes.

b- Start the numerical calculation for a step of one day and compare the precision of the method with the previous simulations.
The RK4 method will now be the preferred method.

3 - *Voyager 1*: Establish the Runge-Kutta scheme (here we have 48 slopes to calculate for each iteration). Find the characteristics of the speed profile, the approach distances and check the values and the conservation of mechanical energy and angular momentum between two slings.
It will be necessary to adapt the step at the moment of the slings because the curvature is then important. The motion is plane and on each step you can calculate the angular variation on the osculating circle to check a good tracking of the trajectory.

4 - *Voyager 3 Project*: retrieve the speed profile. Adjusting the initial conditions to perfectly chain the four slings can be tedious. It can be judicious to proceed as in reality, with, for example, the use of a bit of propellant for a trajectory correction at the Uranus periastron (minimum energy consumption: powered flyby and Oberth effect).

Answers p458.

5.√ ▲▲△ Calculation of propellant masses

The aim is to retrieve all the values given during the conference.

1 - You are out for some repairs outside your space station. But a small loss of attention and you are detached from your rope drifting freely in space with your adjustable wrench in your hand. You slowly move away from the station. How could you get back?
By throwing the one kilo wrench with all your strength, it can reach a speed of 36 km/h. Your mass, including your suit, is 100 kg. What will be your speed after the throw? What quantity is conserved before and after? Is energy a quantity that is conserved? Is the kinetic energy acquired by the key the same as yours?

2 - Resume the calculation for a rocket. In this case the mass varies over time and must be integrated. The gas ejection speed is considered constant. Show how the formula fits for antimatter.

3 - In the relativistic case of *Voyage to Proxima*, calculate, for an ideal photon rocket, the antimatter masses for a round trip.
Duration of the outward journey: 3 years of proper time. Constant acceleration: 1 *g*.

4 - Calculate the mass of propellants required for the *Voyager 3 Project*.

Answers p473.

6. ▲△△ Planetary alignments

For the slings, the planets must have particular relative positions. We can use the alignments as markers. For example, for a slingshot around Jupiter after a departure from Earth, we start by looking for the Sun-Earth-Jupiter alignment dates. The alignments searched are approximate. Perfect alignments are very rare or do not exist. For example, the global alignment of the Earth with the Moon and the Sun happens twice a lunar month. On the other hand, exact alignments occur only at eclipse times.

We consider circular and coplanar trajectories. Periods of revolution of gas giants:

$$T_{Jupiter} \simeq 11.86\, yrs \qquad T_{Saturn} \simeq 29.44\, yrs$$

$$T_{Uranus} \simeq 84.05\, yrs \qquad T_{Neptune} \simeq 164.86\, yrs$$

1 - Show that two planets A and B are aligned according to the period:

$$T_{AB} = \frac{T_A T_B}{T_B - T_A}$$

where B is further from the Sun than A. T_{AB} is the synodic period.

2 - Determine the Jupiter-Earth synodic period and the next alignment date with the help of ephemerides[57].

57 Institut de Mécanique Céleste et de Calcul des Éphémérides de l'Obs. de Paris / CNRS : vo.imcce.fr/webservices/miriade/?forms
Form. : p:Earth, p:Jupiter / heliocenter / Ecliptic.

3 - Set a date for the Earth-Jupiter-Saturn alignment.

4 - How often does the alignment of the four gas giants with the Earth take place?

Answers p477.

7.√ ▲△△ Motion of the stars

For a quick trip to the nearby stars we can consider them fixed. In the case of slow travel over 25,000 years, we must anticipate the motion of the star to launch the probe in the direction it will be at the time of arrival. The velocity of a star is divided into its transverse and radial parts. The transverse components are known with good resolution thanks to the Hipparcos satellite, and now with the even more precise Gaia satellite, which took over in 2013. The Gaia spectrometer allows, by Doppler method, to improve the accuracy on the radial part.

1 - *Determination of the velocity of a star:*

The databases give the current distance d_0 of the star, the *proper motion* μ, and, the *radial velocity* v_r. The proper motion indicates the angular displacement per unit time. This angular change is itself split into two orthogonal components, along longitude and latitude in equatorial coordinates: μ_α and μ_δ.

α : right ascension / δ : declination

units: milliarcseconds per year

Proxima Centauri :

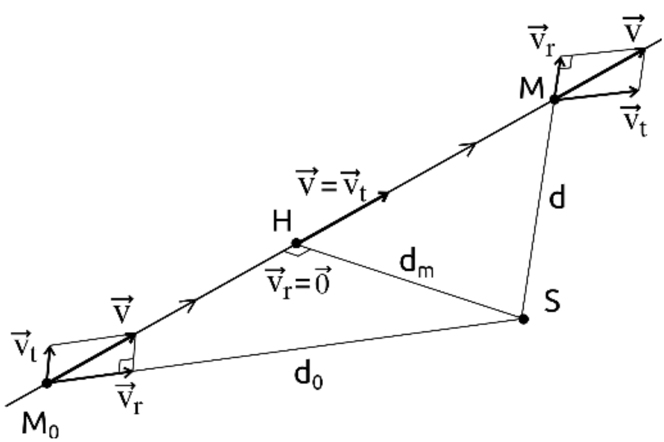
d_0 (ly)	$v_{r\,0}$ (km/s)	$\mu_{\alpha\,0}$ (mas/yr)	$\mu_{\delta\,0}$ (mas/yr)	α_0	δ_0
4.244	-22.2	-3781.3	769.8	14h29^{m}43^s	-62°40'46"

Determine μ, the tangential velocities $v_{t\alpha}$, $v_{t\delta}$, v_t, and the velocity v of the star Proxima Centauri.

What will be the equatorial coordinates of Proxima in a century?

2 - ▲▲ *Linear motion approximation*:

We neglect the Sun gravity and the galactic gravitational potential[58]. At first order, the velocity vector of the star can be considered as constant. The motion of the star is then rectilinear and uniform:



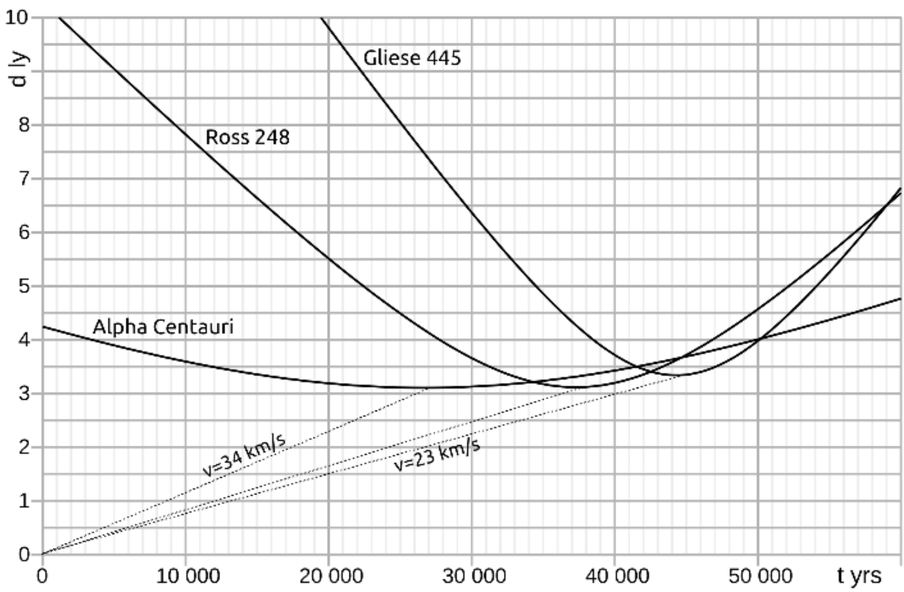
58 *The Close Approach of Stars in the Solar Neighbourhood*, Matthews, 1993. *Close encounters of the stellar kind*, Bailer-Jones, 2014.

a- Determine the distance d of the star from the Sun as a function of time.

b- Determine the minimum approach distance d_m and the corresponding date t_m.

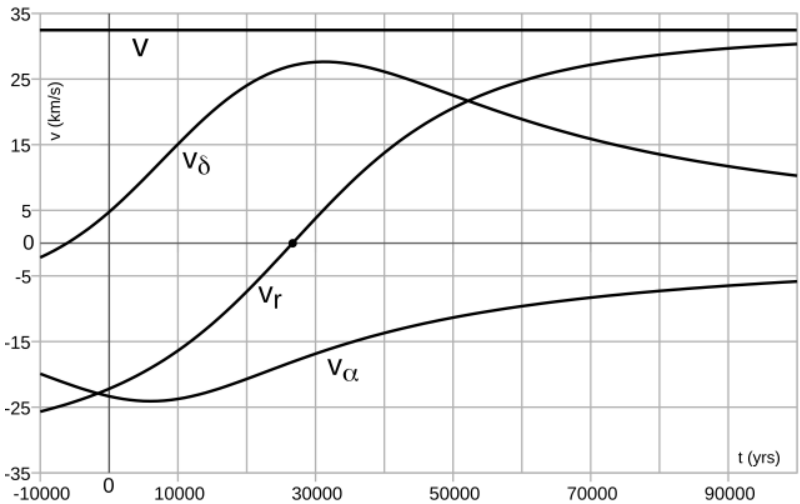
c- What are the coordinates of the star at the closest approach distance?

Distance of stars over time:



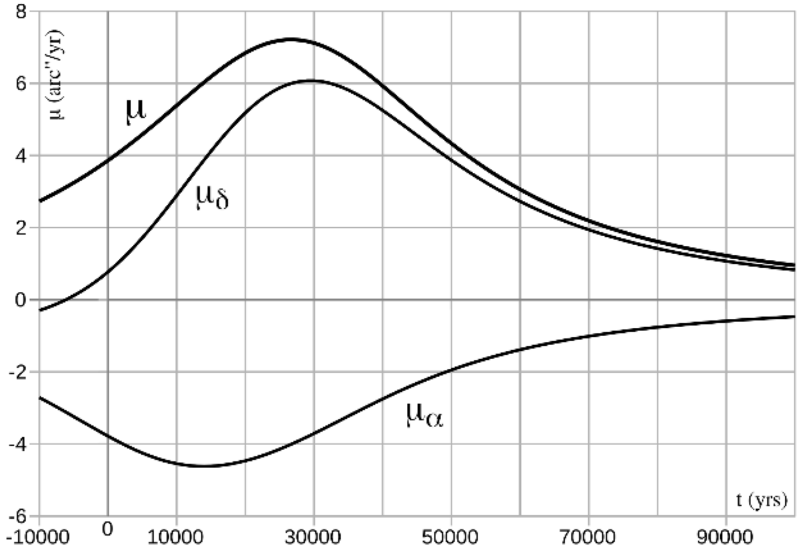
Three stars that can be reached in less than 50,000 years by a probe that uses gravitational assistance.

Radial and tangentials velocities
of Proxima Centauri

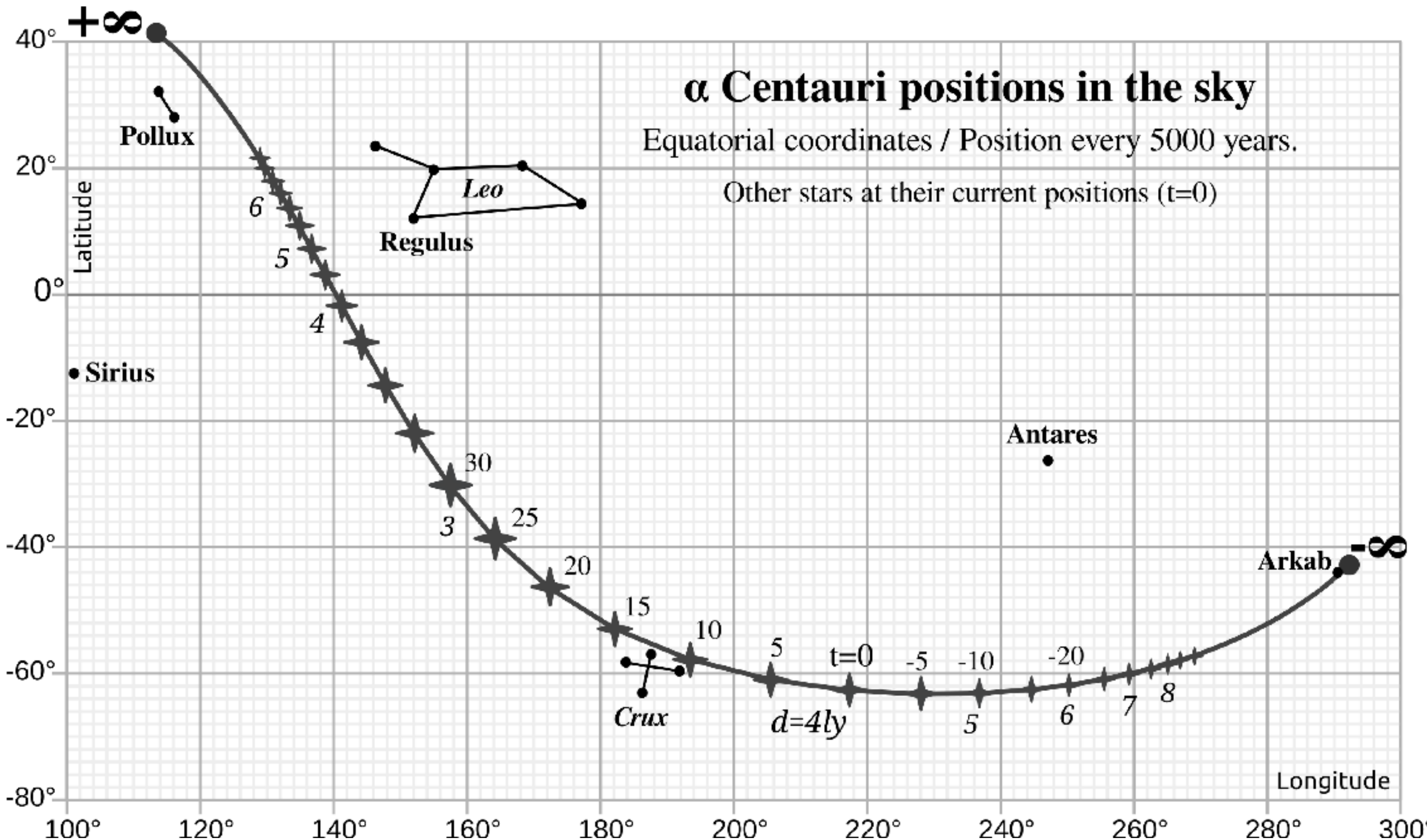


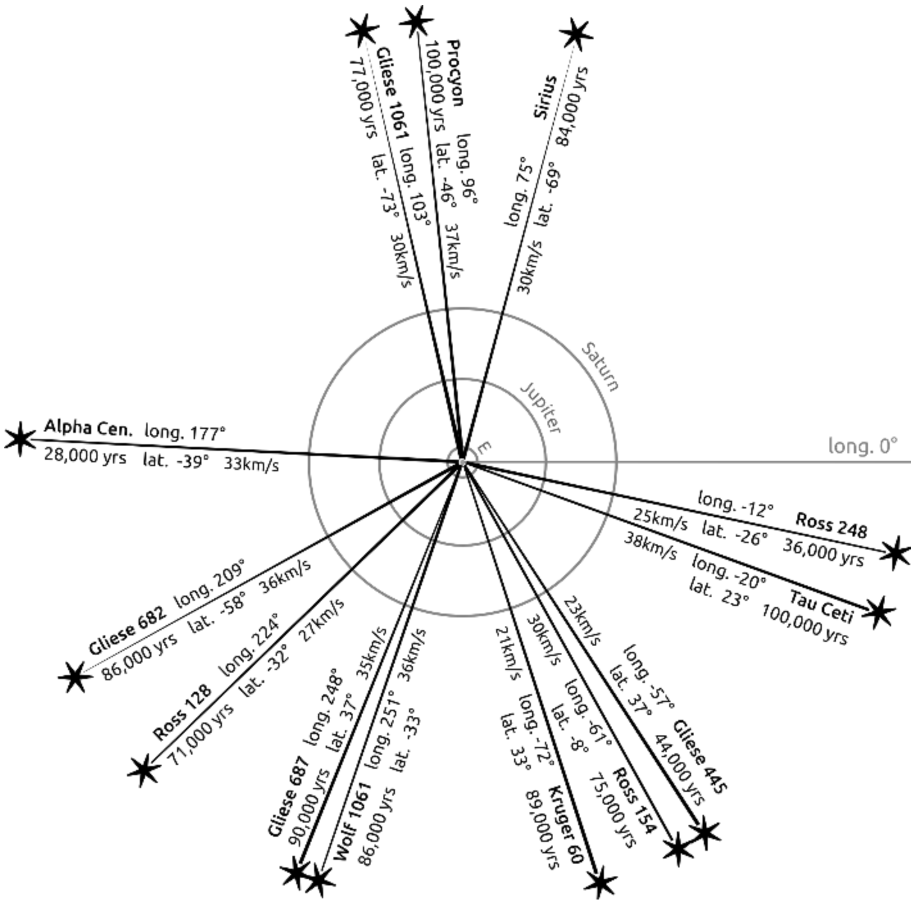
The motion of a star is rectilinear and its speed v is constant. However, its three components, normal to each other, vary with time. At the perihelion time, the radial velocity is zero and the tangential velocity is maximum: $v_t = \sqrt{v_\alpha^2 + v_\delta^2}$. *At infinite times, the velocity becomes purely radial and the tangential components tend towards zero.*

Proper motions of Proxima Centauri



Proper motion of a star for a terrestrial observer. We have the annual angular variations on the celestial sphere in equatorial coordinates of the position of a star. These proper motions are not constant and vary over the millennia. The distant stars can be considered fixed and the closer they are to our Sun, the more apparent their motion becomes.
One second of arc = one 3600th of a degree.


α Centauri positions in the sky
Equatorial coordinates / Position every 5000 years.
Other stars at their current positions (t=0)
+∞
-∞
Pollux
Leo
Regulus
Sirius
Antares
Arkab
Crux
t=0
d=4ly
30
25
20
15
10
5
-5
-10
-20
3
4
5
6
5
6
7
8
Latitude
Longitude
40°
20°
0°
-20°
-40°
-60°
-80°
100°
120°
140°
160°
180°
200°
220°
240°
260°
280°
300°



The position of the stars in ecliptic coordinates at the time when the spacecraft will have joined the distant star system. 13 stars at less than 100 000 years and 40 km/s.

Answers p478.

8.√ ▲▲△ Can a pair of primordial black holes be used as a stargate?

Researchers explain in a 2019 paper[59] how the existence of primordial black holes beyond Neptune's orbit would explain, both, the anomalous orbits observed for transneptunian objects, and, an excess in gravitational microlensing events observed by the OGLE experiment[60]. The primordial black holes (PBH) would have been created in the first moments of the Big Bang. They could explain the origin of gamma-ray bursts and part of the dark matter. These small black holes have not yet been observed, they would be the size of a fist and a few earth masses.

In this exercise we assume the existence of such black holes beyond Neptune, and we imagine that they sometimes form pairs in rapid rotation around their barycenter.

Characteristic data for PBHs: Radius R=4.5 cm.
Mass $M=5\ M_T$. Distance from Sun D=300 au.

1 - Show how such a pair of primordial black holes could help to reach dizzying speeds by gravity assist. Could we, from there, reach Proxima in less than 50 years?

2 - As we get closer to primordial black holes, the tidal forces increase. Would a manned mission be viable? *Answers p485.*

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59 *What if Planet 9 is a Primordial Black Hole?* J. Scholtz, J. Unwin.

60 *Optical Gravitational Lensing Experiment* is a Polish astronomy project based at the University of Warsaw.

9.√ ▲▲△ Antiproton-proton collision

1 - In a particle accelerator, what must be the minimum speed of protons incident on a hydrogenated target to create a pair p $\bar{p}$?
Mass of a proton: 938 MeV/c^2.

2 - The same thing can happen when an antiproton collides with a proton. Do the antiprotons of cosmic rays have sufficient kinetic energies to create pairs? The quantity of cosmic protons is much greater than that of antiprotons. Could we obtain a consequent flux of $\bar{p}$ using energetic p?

Data on page 7 of "The discovery of geomagnetically..." and on page 13 of "Extraction of particles...": there are about 10,000 times more protons than antiprotons in this energy range.

Answers p488.

10.√ ▲▲△ Helical motion

This kinematic and geometric study will help us to interpret the dynamics of the antiproton in the Earth's magnetic field.

Parametric equations of the trajectory in Cartesian coordinates for uniform helical motion:

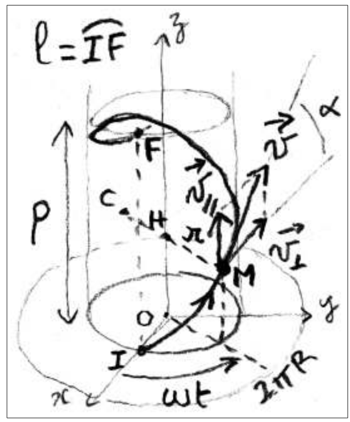
$$\begin{cases} x(t)=r\cos\omega t \\ y(t)=r\sin\omega t \\ z(t)=v_z t \end{cases} \qquad r=cst>0 \qquad \omega=cst \qquad v_z=cst$$

1 - Write the equations in cylindrical coordinates.

2 - Determine the components of the velocity $\vec{v}$ and the acceleration $\vec{a}$.

3 - Calculation of v, a, dv/dt and the radius of curvature R.

4 - Relation between R, the radius $r=HM$ of the helix and the pitch p ($|\Delta z|$ for one complete helix turn).



5 - Calculation of the arc length l traveled by the particle on one turn as a function of: r and p, then of, v and $v_\perp$, and eventually, of R and α (angle between $\vec{v}$ and the horizontal).

Answers p489.

11.√ 01 10 ▲▲▲ The magnetosphere

The field lines of the Earth's magnetosphere are similar to that of a giant bar magnet with its south magnetic pole close to the geographic north pole.

1 - Show that in a magnetic field the speed of a particle is constant.

Help: In relativistic mechanics $\vec{f}=\frac{d\vec{p}}{dt}=\frac{d\,m\gamma\vec{v}}{dt}$ *and we have, here, for the Lorentz force* $\vec{f}=q(\vec{E}+\vec{v}\wedge\vec{B})$.

For the energy aspect $\vec{f}\cdot\vec{v}=\frac{dE}{dt}$ *with* $E=T+mc^2$.

2 - Give the trajectory of a charged particle in a uniform magnetic field.

3 - Give the shape of the field lines of a magnetic dipole. Characteristics and components of the magnetic field of a dipole in spherical coordinates.

4 - Show the mirror effect on the example of a narrowing field tube.

5 - Show the drift phenomenon in the simple case of two areas with uniform magnetic fields of different intensities.

6 - <u>Trapped antiproton</u>: We will carry out a numerical simulation with the Runge-Kutta method of order 4 (method described page 313).

a- Establish the expression of the components of the magnetic field of a dipole in Cartesian coordinates.

b- Give the equations of motion of a charged particle in a magnetic field.

c- Write the RK4 scheme.

d- Carry out the numerical simulation. On a spreadsheet it can be too computationally intensive. In this case we preferred to program in php and to make the calculations on server.

Answers p490.

12.√ ▲▲▲ Penning trap

This charged particle trap, designed in 1936, uses a quadrupole electric field and a uniform magnetic field. Penning traps are commonly used at CERN to store antiprotons. The electric field is created by a set of electrodes that follow the hyperboloidal equipotentials of the quadrupole. The globally uniform magnetic field in the storage area is the one created inside a solenoid.

1 - Expression of the electric field:

$$\vec{E}=\frac{U_0}{r_0^2}(-x\,\vec{i}-y\,\vec{j}+2\,z\,\vec{k})$$

Show that $\vec{E}$ derives from a potential that we will determine.

2 - Show that the origin O is an equilibrium position. Discuss the stability along the (Oz) axis and then in the plane (Oxy). Calculate the pulsation ω_z of the oscillations along Oz.

3 - To stabilize the trajectory of the antiproton we add a uniform magnetic field:

$$\vec{B}=B_0\,\vec{k}$$

a - Is the motion along (Oz) modified?

b - According to (xOy): show that the antiproton is trapped if B_0 is greater than a critical value B_c to be determined (to do this, establish the differential equation verified by $\rho=x+j\,y$, $j^2=-1$, with $\omega_c=e\,B_0/m$).

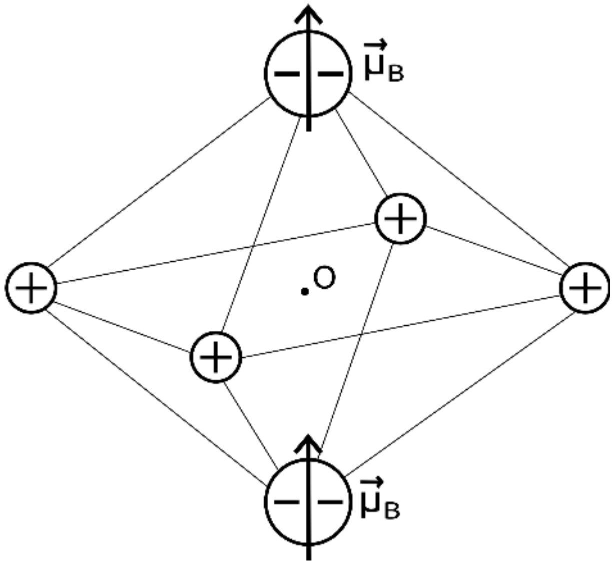
c - Solve and highlight two angular frequencies ω_c' and ω_m (magnetron frequency).

Numerical Applications: U_0=9.3 V, r_0=29.1 cm, B_0=0.55 T, $e=1.6\times10^{-19}$C, $m_p=1.67\times10^{-27}$kg.

d - Plot the trajectory.

4 - <u>Microscopic cage</u>: Could we create a Penning trap at the microscopic scale? We are going to propose a model to try to give elements of an answer. For the quadrupole electric field we can use cations and anions. For the magnetic field we have paramagnetic atoms which have a permanent magnetic moment (iron is an example among many others). Let's take six atoms arranged in a bipyramid with a square base. The two atoms at the vertices have a charge 2⊖ and an elementary magnetic moment μ_B. The four atoms at the base are cations of elemental charge ⊕.

Data (usual order of magnitude):
Edges of the regular octahedron equal to: a=100 pm.
Elementary charge: $e=1.6\times10^{-19}$ C. $\varepsilon_0=8.85\times10^{-12}$ $C^2.m^{-2}.N^{-1}$.
Elementary magnetic moment: that created by a classical electron orbiting in a hydrogen atom, called Bohr magneton: $\mu_B=9.27\times10^{-24}A.m^2$. All atomic magnetic moments are equivalent to a few elementary magnetons (orbital and spin moments combined).



Representation of a hypothetical microscopic Penning trap within a crystal lattice or molecular structure. The paramagnetic atoms placed at the top and bottom create a globally uniform magnetic field around the center O. These atoms at the apexes of the bipyramid correspond to the upper and lower caps of a macroscopic Penning trap, and the cations at the square base, to the ring electrode.

a - Show that this atomic structure is not a monopole, nor an electric dipole.

b - Evaluate the magnetic field B_0 created at the center of the bipyramid. You can use the expressions on page 491.

c - Estimate the factor U_0/r_0^2.
You can consider the Oz axis to identify the expressions.

d - Is the magnetic field sufficient to trap an antiproton? Conclusions.

Answers p507.

Answers

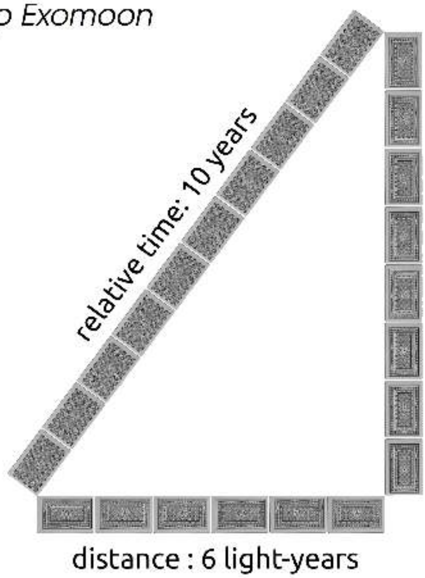
1

.1. *The Crystals of the Pop Exomoon (Barnard system)*

Exercise p25.

Distance and relative time measured in the galactic frame *R'*.
Arrival at the galactic year **2120** (=2010 + 10).
Rocket speed in *R'* :
$v = 6/10\ c$
$= \mathbf{60\%\ of\ c}$.

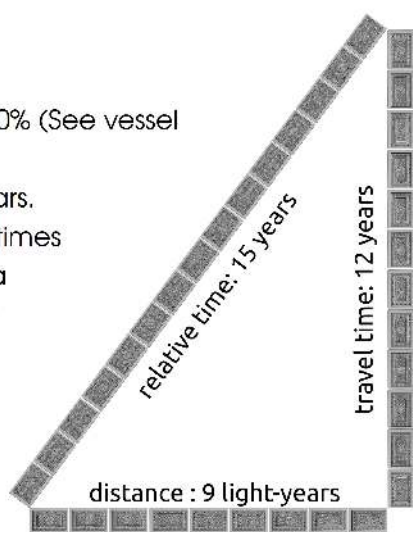
It is a double triangle of the 3-4-5.



2. One-way ticket for *Sirius with an old β6*

(Exercise p26)

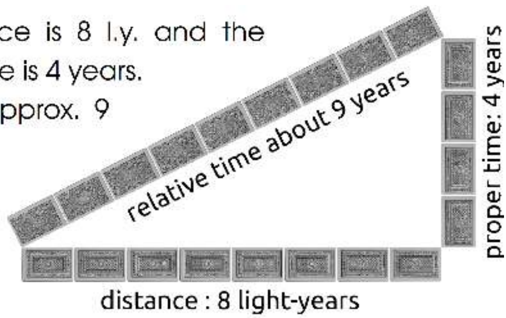
speed = (distance / time)$_{R'}$
then: relative time = 9 l.y. / 60% (See vessel characteristics), so:
$\Delta t' = 9 / 0.6 = 9\text{x}10/6 = 15$ years.
We try to build a triangle of times with a base of 9 cards and a hypotenuse of 15 cards. The proper time is 12 years.
Arrival at **42 years** (=30+12).
Arrival date: 2169 (2154 + 15). You arrive one year after the first



festival, you have to wait for the one of **2178**.
It's a triple triangle of 3-4-5.

1. 3. *Parcel delivery* *(Exercise p26)*

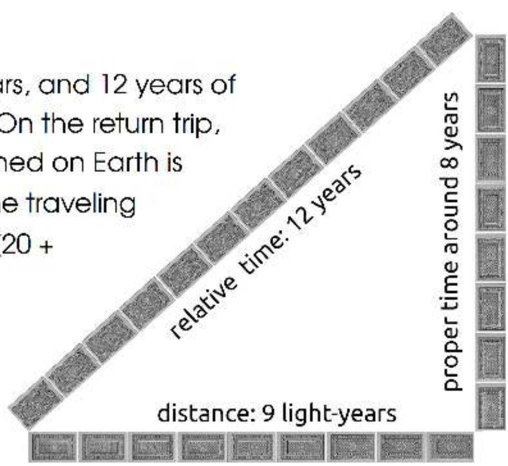
The round trip distance is 8 l.y. and the cumulative proper time is 4 years.
The relative time is approx. 9 years.
So the customers are delivered in **4 in a half years**, the speed of the ship in the galactic frame is 8/9 ≃ **89% of *c***, and the time dilation is 9/4 ≃ **2.2**.



1. 4. *Twin on his way to Sirius*

Exercise p26.

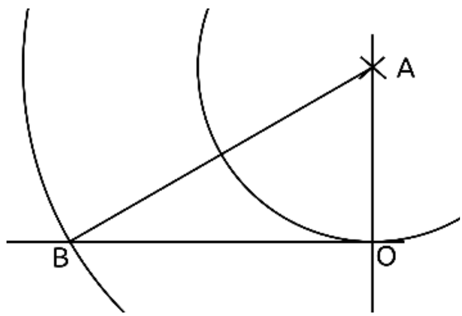
2156 - 2132 = 24 years, and 12 years of relative time to go. On the return trip, the twin who remained on Earth is **44 years** old, and the traveling twin is **36 years** old (20 + 8x2).
Rocket speed:
9/12 or **75% of *c***.



1. 5. *Cruel dilemma?* *(Exercise p27)*

If Denys stays at home, in the galactic center, he will die in 3053, therefore impossible to attend the festival. Moreover, if he does not defuse the bomb, it will explode in 3052 and due to the propagation of gamma rays the center of the galaxy will be destroyed 26 years later in 3078, so no party either...

On the other hand, if Denys travels with a dilation of two, for 32 years of life in the ship, it takes 64 years of galactic time.



Let us consider that Denys goes to disarm the bomb, his relative time AB is twice the proper time AO, and BO is 26 ly.

Construction: we draw a vertical line, we fix a point A, we draw a circle of arbitrary radius, for example 5 cm. One obtains a point O, one draws a perpendicular straight line on it. A second circle with a double radius of 10 cm is drawn. Measure the distance BO ≃ 8.7 cm.

5 cm	? years
10 cm	? years
8.7 cm	26 l.y.

The proper time is therefore:
5 / 8.7 x 26 ≃ 15 years.
And 30 years for the relative time.
Denys arrives in time to defuse the bomb, in 3051. Then he is back at the galactic center 60 years later in 3081, one year before the festival. And he is not dead yet because he has only aged 30 years, he will die in 3083 and will therefore be able to attend the party in 3082!

1. 6. M*uons* *(Exercise p28)*

A muon at rest disintegrates in 1.5 µs, so the half-life is a proper time. We can name R the proper frame related to the muon.
The terrestrial reference frame named *R'* is in rectilinear and uniform translation with respect to *R*, at the speed v = 0.999 c. Over the short duration of the experiment *R'* is a very good inertial reference frame, and *R*, therefore, also. The relative decay time of the muon is:

$$\Delta t' = \gamma \Delta t \quad \text{and} \quad \gamma = 1/\sqrt{1-\beta^2}$$

$$\text{then, here,} \quad \gamma = 1/\sqrt{1-0.999^2} \simeq 22.4$$

The half-life time in the terrestrial frame is 22 times longer than in its proper frame.
The distance traveled by the muon in the terrestrial frame during its relative half-life time is:

$$d = v\,\Delta t' = \beta\, c\, \gamma\, t_{1/2} \simeq 10\,km$$

About **half of these muons reach ground level**, the other half will have disintegrated, before, in altitude.

In the context of classical mechanics, or if we forget to take into account time dilation, the muons would have 1.5 µs instead of 33.5 µs to reach the ground. Thus, after 450 meters, half of them would have already disintegrated. After 900 m only a quarter would still be there, and on the ground, after 10 km, one out of 2^{22} would have survived . Finally, such a muon would have only one chance out of four million to reach the ground, which is very different from one chance out of two!
Muons were discovered in 1936. The measurement of the muon flux as a function of altitude made it possible to verify the validity of special relativity.

1. 7. High-speed train journey *(Exercise p29)*

The average speed of the train is:

$$v = d/t = 2300/8 = 287.5\,km/h \simeq 79.86\,m/s$$

This speed is tiny compared to the speed of light, gamma is very close to one, and a calculation of γ with a standard calculator will give you 1, as if there was no time dilation. However, as we will see, this dilation is very easy to measure with atomic clocks.
For low speeds, compared to the maximum speed, it is more convenient and meaningful to use series expansions:

$$\Delta t' = \gamma\,\Delta t = \left(1-\beta^2\right)^{-1/2}\Delta t \simeq \left(1+\beta^2/2\right)\Delta t$$

The difference of time between the clock that stayed in Beijing and the one that traveled 4600 km is:

$$\Delta t' - \Delta t \simeq \left(1+\beta^2/2\right)\Delta t - \Delta t = \beta^2/2\,\Delta t = \frac{v^2}{2c^2}\,\Delta t$$

Hence the difference for the 16 hours round trip:

$$\Delta t' - \Delta t \simeq \frac{79.86^2}{2\left(3.10^8\right)^2}\,16\times 60\times 60 \simeq 2.04\,ns$$

The clock that stayed in the station is **two nanoseconds** ahead of the one that traveled.

Let's check if our clocks are accurate enough: A clock drift of 10^{-14} seconds per second, gives, for a trip of 57600 seconds, a global drift of 0.6 ns. The uncertainty is small compared to the measured difference: the time dilation is confirmed.

1. 8. *Satellite* (Exercise p29)

We add proper times on a revolution. We calculate the duration of a tour:

$$\Delta t=\frac{2\pi R}{v}=\frac{2\pi\times 6900.10^{3}\times 3.6}{27000}\simeq 5781\,s\simeq 1\,hour\,36\,minutes$$

This leads to the following time difference between the two clocks (same formulas as in the previous exercise):

$$\Delta t'-\Delta t\simeq\frac{v^{2}}{2c^{2}}\Delta t\simeq\frac{(27000/3.6)^{2}}{2\times 9.10^{16}}5781\simeq 1.8\,\mu s$$

The clock in the satellite is **1.8 µs** younger than the one that remained stationary in the geocentric frame of reference.
We can achieve a Taylor series expansion, because the satellite speed is very small in comparison with the speed limit (one forty thousandth of *c*).

1. 9. *Hafele-Keating experiment* (Exercise p30)

Two sources of time dilatation are present here: speed and gravitation. In an airplane, speed increases and gravitation decreases. The two effects act in opposite directions. The *clock hypothesis* is generalized and the time Δt' spent in the plane, reference frame *R'*, compared to that Δt for a stationary clock in the geocentric reference frame *R*, is written as follows:

$$\Delta t'\simeq\left(1+\frac{gh}{c^{2}}-\frac{v^{2}}{2c^{2}}\right)\Delta t \quad \text{and} \quad \Delta t'-\Delta t\simeq\left(gh-\frac{v^{2}}{2}\right)\frac{\Delta t}{c^{2}}$$

Towards the east, the speeds are added and we have the following time difference:

$$\Delta t' - \Delta t \simeq \left(9.81 \times 10000 - \frac{(2674/3.6)^2}{2}\right)\frac{40 \times 3600}{9.10^{16}} \simeq -284\,ns$$

To the west, the speeds subtract and we have the time difference:

$$\Delta t' - \Delta t \simeq \left(9.81 \times 10000 - \frac{(674/3.6)^2}{2}\right)\frac{40 \times 3600}{9.10^{16}} \simeq 129\,ns$$

Furthermore, the difference between the clock on the ground, terrestrial reference frame *R''*, and the stationary one in the geocentric reference frame *R*, is:

$$\Delta t - \Delta t'' \simeq \frac{(1674/3.6)^2}{2}\frac{40 \times 3600}{9.10^{16}} \simeq 173\,ns$$

We add the two equations:

To the east: $\Delta t' - \Delta t'' \simeq 173 - 284 \simeq -111\,ns$.

The clock on the ground advances **111 ns**.

To the west: $\Delta t' - \Delta t'' \simeq 173 + 129 \simeq 302\,ns$.

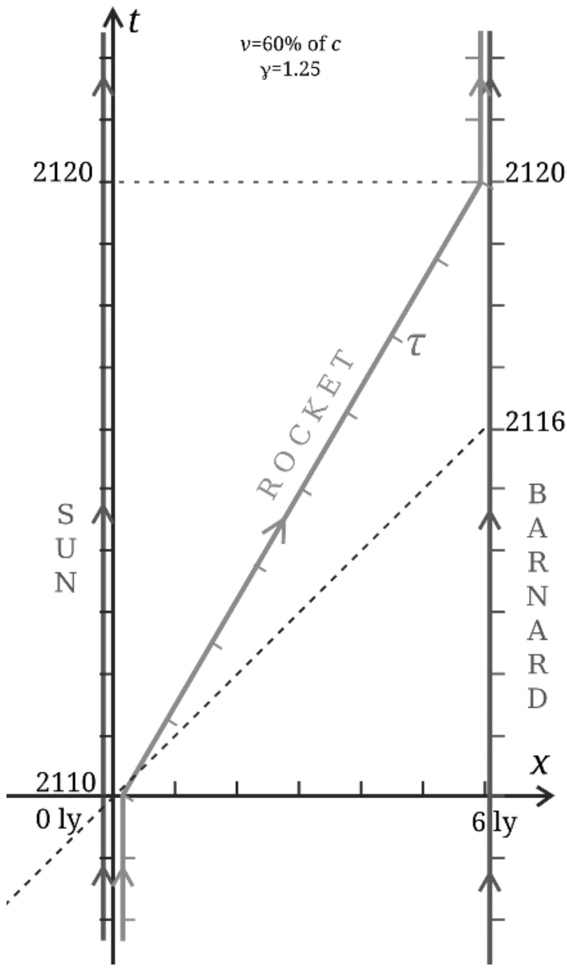
The clock on the ground retards by **302 ns**.

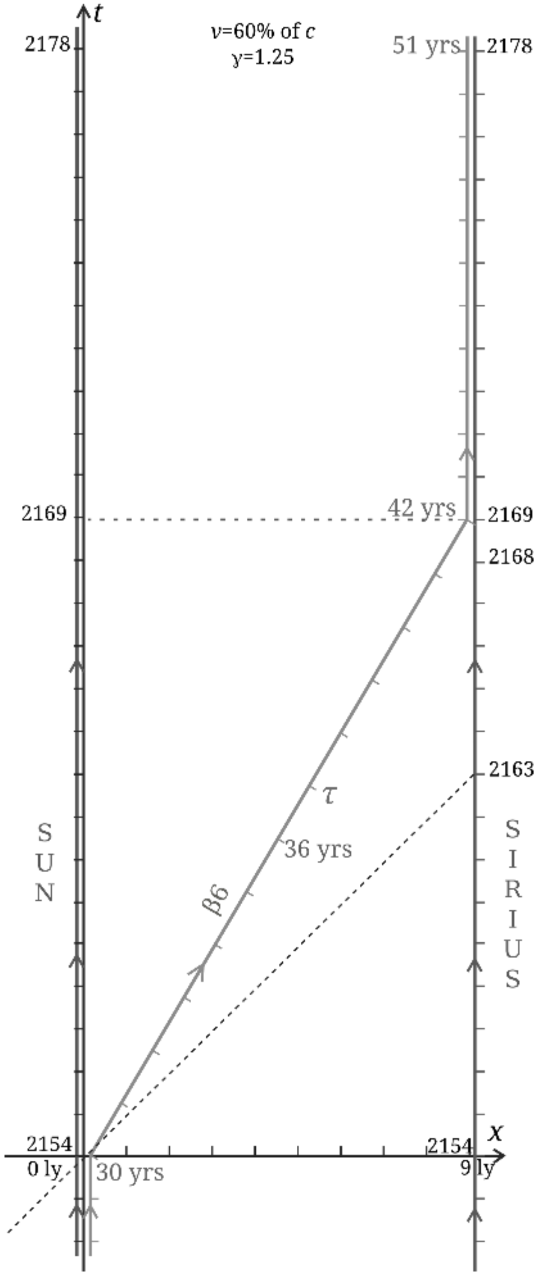
Our results obtained with our simple model are consistent with those of the 1971 experiment.

It is normal that the results differ, the actual flights, with multiple stopovers, were only globally equatorial, the speeds and altitudes had different average values than those chosen in the exercise.

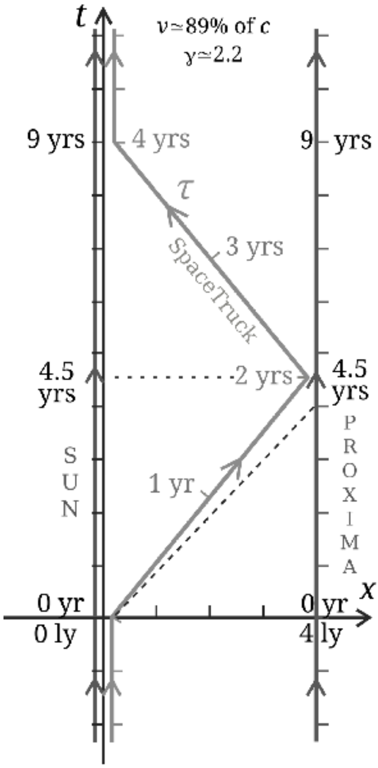
2.1. Exercise *p45.*

" The Crystals of the Pop exomoon "

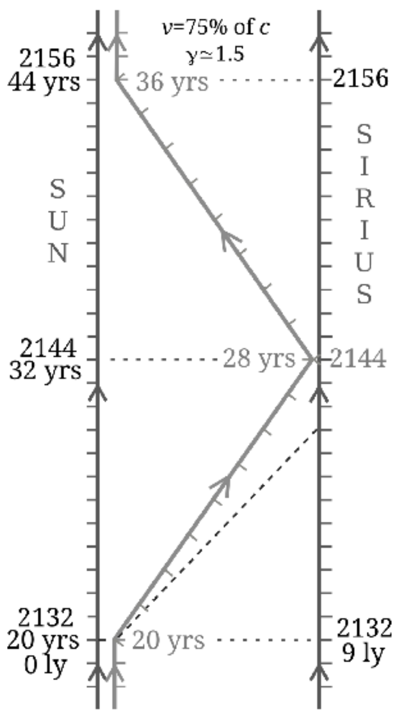




" A spacious and comfortable ship to go to the two suns festival "

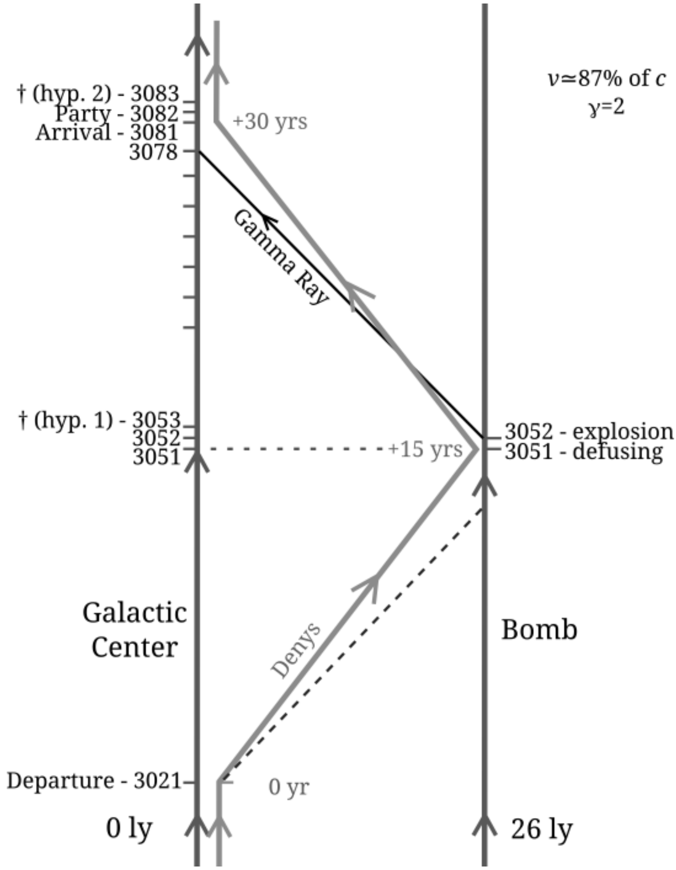


" Parcel delivery with a SpaceTruck "



" The twins "

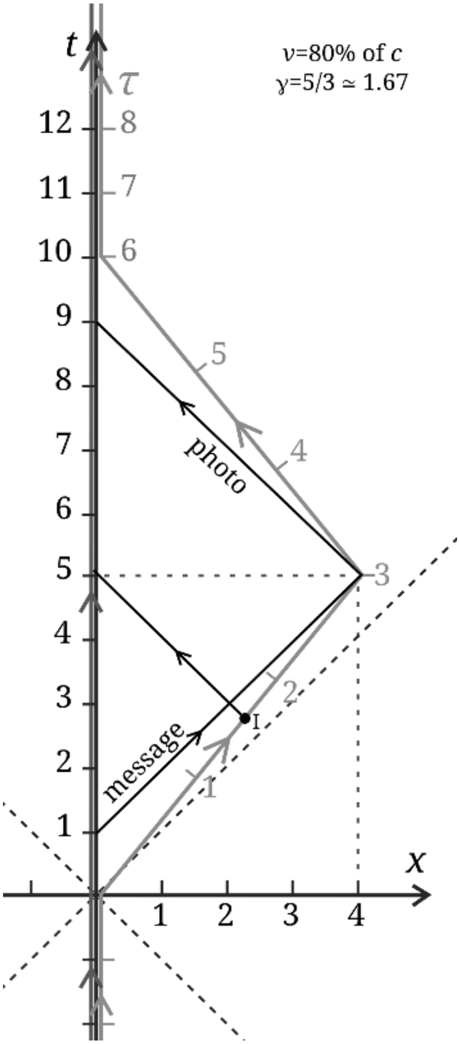
" Denys will die in exactly 32 years, and there is no cure... "



2.2. Interstellar communications

Exercise p45.

Let us reason from the galactic frame of reference, at the moment when the traveling twin lands on the exoplanet, 5 years have elapsed since the departure for the twin on Earth, the light ray then takes 4 more years to reach the Earth - the speed of light in vacuum is independent of the inertial frame of reference - and the photo is received 9 years after the departure and only 1 year before the return!



Although the landing and five years indicated on the Earth's clock are simultaneous events in the galactic frame of reference, the information is propagating at finite speed and it is necessary to wait for the images to arrive. And here also 4 years of propagation, in his telescope, so the twin will see his brother landing and receive the picture at the same time, 9 years after the departure.

Now, if the twin on Earth looks through his telescope at the date t=5, he will see photons emitted earlier, he will not see his brother landing on Proxima b at all (even if that's what he actually does at that time), he will see him in his ship on his way to Proxima, 1 year and 8 months before his arrival in *terra incognita*.

This result is given by the calculation of the point of intersection of the two straight lines:

worldline of the ship: $t=\frac{1}{\beta}\frac{x}{c}$

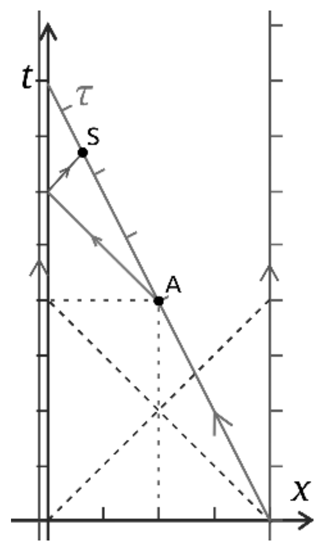
worldline of the photon : $t=-\frac{x}{c}+t_P$ with $t_P=5$

Then for the intersection point I : $t_I = \frac{t_P}{1+\beta}$ and $\tau_I = \frac{t_I}{\gamma} = \frac{5}{3}$

In order for a message to be received from Earth at the time of landing, it must be sent in advance, only one year after departure.

2.3. Call for help

Exercise p46.



Notations :

Cruise ship speed relative to *c*: $\beta_1 = 0.5$

Emergency shuttle speed: $\beta_2 = 0.9$

Worldlines :

Vessel: $ct = -\frac{1}{\beta_1}(x - D)$

Messenger photons:

$$ct = -x + \frac{D}{2\beta_1} + \frac{D}{2}$$

Shuttle: $ct = \frac{1}{\beta_2}x + \frac{D}{2\beta_1} + \frac{D}{2}$

Coordinates of point S :

$$-\frac{1}{\beta_1}x_S + \frac{D}{\beta_1} = \frac{1}{\beta_2}x_S + \frac{D}{2\beta_1} + \frac{D}{2} \quad \text{then} \quad x_S = \frac{\beta_2(1-\beta_1)}{\beta_1+\beta_2}\frac{D}{2}$$

$$\text{and} \quad ct_S - ct_A = \left[-\frac{1}{\beta_1}\frac{\beta_2(1-\beta_1)}{\beta_1+\beta_2}\frac{D}{2} + \frac{D}{\beta_1}\right] - \left[\frac{D}{2\beta_1}\right]$$

$$\text{so} \quad c\Delta t_{AS} = \frac{D}{2}\frac{1+\beta_2}{\beta_1+\beta_2} \quad \text{and} \quad \tau_{AS} = \frac{\Delta t_{AS}}{\gamma_1} = \frac{D}{2c}\frac{1+\beta_2}{\beta_1+\beta_2}\sqrt{1-\beta_1^2}$$

Numerical application: $\tau_{AS} \simeq 2.35\,yrs$

Passengers have to wait two years and four months before help arrives!

2.4. Tim, Tam, Tom

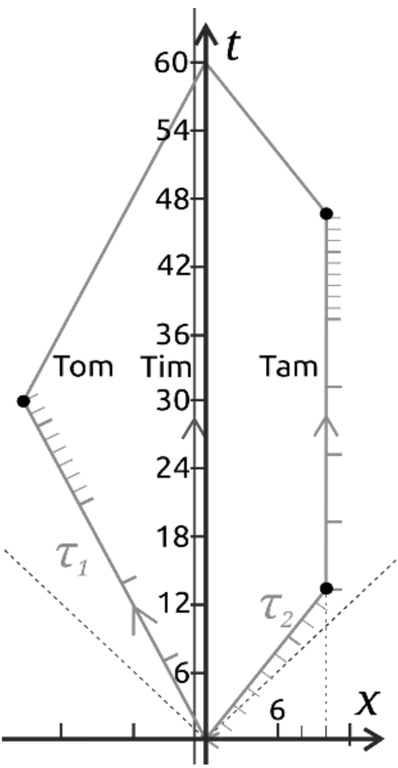
Exercise p46.

Notations :

Speed relative to *c* of Tim : $\beta_1 = \frac{v_1}{c} = \frac{10}{20} = 0.5$

Speed of Tam : $\beta_2 = \frac{v_2}{c} = \frac{15}{20} = 0.75$

For Tom, we apply the time dilation factor $\gamma_1 \simeq 1.1547$ and $\tau_1 = \frac{\Delta t}{\gamma_1} \simeq 51.96\, min$ then an arrival at 10:52 to his watch.



For Tam by bike $\gamma_2 \simeq 1.5119$.

To go, $\Delta t = \Delta x / \beta_2 \simeq 13.3\, min$ and $\tau_2 = \frac{\Delta t}{\gamma_2} \simeq 8.82\, min$ then an arrival at 10:51 to his watch.

3.1. Composition of velocities *Exercise p71.*

a - 80% of *c*. b - 50% of *c*.

2. Two vessels *Exercise p71.*

a - In practice we do not know the position of an object continuously but at regular intervals. We estimate the velocities and accelerations of a point by using average values with the help of neighbouring points. For example, on an air-cushion table, with a video, or a radar, we have such measurements.

<u>Average velocity</u> between two points M_n and M_{n+1} of a trajectory:

$$\vec{v}_{n\ n+1} \simeq \frac{\overrightarrow{M_n M_{n+1}}}{t_{n+1}-t_n} \simeq \frac{\overrightarrow{OM_{n+1}}-\overrightarrow{OM_n}}{\Delta t_{n\ n+1}} .$$

Estimated velocity at the intermediate date:

$$t_{n\ n+1} = \frac{t_{n+1}+t_n}{2} .$$

- Between t_1 and t_2 :

$$t_{12} = (t_2+t_1)/2 = 2 \quad \text{and} \quad \Delta t_{12} = t_2 - t_1 = 4$$

$$\vec{v}_{A12} \simeq (x_{A2}-x_{A1}, y_{A2}-y_{A1}, z_{A2}-z_{A1})/\Delta t_{12}$$

then: $\vec{v}_{A12} \simeq (1/2, 0, 0)$ and $v_{A12} \simeq 1/2$

so $\beta_{A12} \simeq 1/2$ and $\gamma_{A12} \simeq 2/\sqrt{3}$.

The velocities are not expressed here in the international system in m/s, but in their natural units in ly/yr, i.e. as a percentage of *c*.

$$\vec{v}_{B12} \simeq (4-2,2,2)/4 \quad \text{then} \quad \vec{v}_{B12} \simeq (1/2,1/2,1/2)\ .$$

$$v = \|\vec{v}\| = \sqrt{v_x^2 + v_y^2 + v_z^2} \quad \text{and} \quad v_{B12} \simeq \sqrt{3}/2$$

$$\text{so} \quad \beta_{B12} \simeq \sqrt{3}/2 \simeq 0.866 \quad \text{and} \quad \gamma_{B12} \simeq 2\ .$$

- Between t_2 and t_3 :

$$t_{23} = 6\ , \quad \Delta t_{23} = 4 \quad \text{and} \quad \vec{v}_{A23} = \vec{v}_{A12}$$

$$\vec{v}_{B23} \simeq (1/4, 1/4, 1/4) \quad \text{then} \quad v_{B23} \simeq \sqrt{3}/4$$

$$\text{so} \quad \beta_{B23} \simeq \sqrt{3}/4 \simeq 43\% \quad \text{and} \quad \gamma_{B23} \simeq \sqrt{16/13} \simeq 1.11\ .$$

b -

<u>Average acceleration</u>

between the instants $t_{n\,n+1}$ and $t_{n+1\,n+2}$:

$$\vec{a} \simeq \frac{\Delta \vec{v}}{\Delta t} \quad \text{and} \quad \vec{a}_{n\ n+2} \simeq \frac{\vec{v}_{n+1\ n+2} - \vec{v}_{n\ n+1}}{t_{n+1\ n+2} - t_{n\ n+1}}\ .$$

Estimated acceleration at the intermediate date:

$$t_{\text{int}} = \frac{t_{n+1\ n+2} + t_{n\ n+1}}{2}\ , \ \text{moreover} \quad \Delta t = t_{n+1\ n+2} - t_{n\ n+1}\ .$$

- Between t_1 and t_3 for *A* : $\vec{a}_A \simeq \vec{0}$.

- Between t_1 and t_3 for *B* : $t_{\text{int}} = 4$ and $\Delta t = 4$,

$$\vec{a}_B \simeq (1/4 - 1/2, -1/4, -1/4)/4$$

$$\text{so} \quad \vec{a} = \vec{a}_B \simeq (-1/16, -1/16, -1/16) \ \text{(deceleration)}$$

$$a = a_B \simeq \sqrt{3}/16 \simeq 0.108\ ly/yr^2$$

c - Any reference frame in translation, rectilinear and uniform with respect to a reference frame of inertia is also of inertia. The reference solid of the vessel has a translational, rectilinear and uniform motion with respect to the frame of reference *R* (the velocity vector of point *A* is considered constant in *R*). As *R* is inertial *R'* is also inertial.

Given the three events at our disposal, the trajectory of the vessel *B* in *R* can be rectilinear (the three positions provided are aligned).

Let us now determine the coordinates of the events of *B* in *R'* using the Lorentz transformation:

$$\begin{cases} x'_B/c=\gamma\,(x_B/c-\beta t) \\ y'_B=y_B \\ z'_B=z_B \\ t'=\gamma(t-\beta x_B/c) \end{cases} \quad \text{For } t_1=0 : \quad \begin{cases} x'_B=\frac{2}{\sqrt{3}}(2-0) \\ y'_B=2 \\ z'_B=2 \\ t'_1=\frac{2}{\sqrt{3}}(0-\frac{1}{2}2) \end{cases}$$

Then: $E_{R',B,1}(x'_B=\frac{4}{\sqrt{3}}, y'_B=2, z'_B=2, t'_1=-\frac{2}{\sqrt{3}})$

$$\text{For } t_2=4 : \quad \begin{cases} x'_B=\frac{2}{\sqrt{3}}(4-\frac{1}{2}4) \\ y'_B=4 \\ z'_B=4 \\ t'_2=\frac{2}{\sqrt{3}}(4-\frac{1}{2}4) \end{cases}$$

Then: $E_{R',B,2}(x'_B=\frac{4}{\sqrt{3}}, y'_B=4, z'_B=4, t'_2=\frac{4}{\sqrt{3}})$

For $t_3=8$: $\begin{cases} x'_B=\frac{2}{\sqrt{3}}(5-\frac{1}{2}8) \\ y'_B=5 \\ z'_B=5 \\ t'_3=\frac{2}{\sqrt{3}}(8-\frac{1}{2}5) \end{cases}$

Then: $E_{R',B,3}(x'_B=\frac{2}{\sqrt{3}}, y'_B=5, z'_B=5, t'_3=\frac{11}{\sqrt{3}})$

Two remarks concerning the worldline of vessel *B* interpreted from *R'*: on the one hand the three events are no longer separated by equal time intervals as they are from *R*. and on the other hand, the spatial part projected in *R'* corresponds to non-aligned points, contrary to *R*.

d - $\vec{v'}_{B12} \simeq (x'_{B2}-x'_{B1}, y'_{B2}-y'_{B1}, z'_{B2}-z_{B1})/\Delta t'_{12}$

$\vec{v'}_{B12} \simeq (0,2,2)\sqrt{3}/6$ and $\vec{v'}_{B12} \simeq (0, \sqrt{3}/3, \sqrt{3}/3)$

at $t'_{12}=(t'_1+t'_2)/2=1/\sqrt{3}$ with $\beta'_{B12} \simeq \sqrt{2/3} \simeq 0.816$

$\vec{v'}_{B23} \simeq (-2/\sqrt{3}, 1, 1)\sqrt{3}/7$ so $\vec{v'}_{B23} \simeq (-\frac{2}{7}, \frac{\sqrt{3}}{7}, \frac{\sqrt{3}}{7})$

at $t'_{23}=\frac{15}{2\sqrt{3}}$ with $\beta'_{B23} \simeq \frac{\sqrt{10}}{7} \simeq 0.452$

e - $\vec{a'}=\vec{a'}_B \simeq (-2/7, -4\sqrt{3}/21, -4\sqrt{3}/21)2\sqrt{3}/13$

then $\vec{a'} \simeq (-4\sqrt{3}/91, -8/91, -8/91)$

and $a'=a'_B \simeq 4\sqrt{11}/91 \simeq 0.1458\ ly/yr^2$

$$\text{at} \quad t'_{123} = \frac{t'_{12} + t'_{23}}{2} = \frac{17}{4\sqrt{3}}$$

f - We can reason in *R* or *R'*. Let us observe the proper frame R_B of vessel *B* from the reference frame *R,* and let us consider the inertial reference frame *R''* which coincide at t=4 yrs with R_B. We show on page 113 that the acceleration of *B* measured in *R''* is equal to the one felt by the passengers in the reference frame of the vessel *B*. And the relation between the accelerations of a point measured from two inertial frames of reference *R''* and *R* is given by the law of transformation of accelerations. We choose a new abscissa axis for *R* according to the rectilinear trajectory of *B*. The velocity of *R''* with respect to *R* is approximately:

$$\vec{v}_{B12} \simeq (4-2,2,2)/4 \quad \text{and} \quad u = v_{B2} \simeq 3\sqrt{3}/8$$

$$\text{so} \quad \beta = \beta_{B2} \simeq 3\sqrt{3}/8 \ , \quad \gamma = \gamma_{B2} \simeq 1.32$$

$$\text{and} \quad a_{proper} = a'' = \gamma^3 a \simeq 2.275\, a_B \simeq 0.246 \ ly/yr^2 \ .$$

g - 1 yr = 365.25x24x3600 = 31,557,600 s
1 ly = 3×10^8x 31,557,600 = 9.46728×10^{15} m

$$\text{Then:} \quad 1\frac{ly}{yr^2} \simeq 9.51\frac{m}{s^2}$$

$$\text{and} \quad a_{proper} \simeq 2.34\, m/s^2 \simeq 24\% \ of \ g$$

Comparisons:

17% on the Moon's surface and 38% for Mars.

3. Low speeds limit *Exercise p72.*

The law of composition gives on a standard calculator 180 km/h because the difference is extremely small. We will therefore perform a Taylor series expansion:

$$v_{relativist}=\frac{v_{classical}}{1+\frac{v_1 v_2}{c^2}}=v_{clas}\left(1+\frac{v^2}{c^2}\right)^{-1}\simeq v_{clas}\left(1-\frac{v^2}{c^2}\right)$$

$$\Delta v=v_{clas}-v_{rel}\simeq 2\frac{v^3}{c^2} \quad \text{and} \quad \frac{\Delta v}{v_c}\simeq\frac{v^2}{c^2}=\beta^2$$

Numerical applications:

$$\Delta v\simeq 347.10^{-15}\,m/s\simeq 0.35\,pm/s \quad \text{and} \quad \frac{\Delta v}{v_c}\simeq 6.9\times 10^{-15}$$

4.1. The suicidal physicist *Exercise p95.*

$$\lambda_{\text{Red}} = \sqrt{\frac{1+\beta}{1-\beta}}\lambda_{\text{Green}} \quad \text{then} \quad \beta = \frac{\lambda_R{}^2 - \lambda_G{}^2}{\lambda_R{}^2 + \lambda_G{}^2}$$

32% of *c* also 350 millions of km/h.

2. Laser sail *Exercise p95.*

a - The force is expressed as the variation of the momentum per time interval:

$$F = \Delta p/\Delta t.$$

Energy of a photon: e=pc.
Variation of momentum per reflected photon: $\Delta p=2e/c$ (e/c at incidence and e/c at reflexion).
Number of photons received during Δt : $\Delta N=\Phi\Delta t/e$.
Φ(J/s) : flux / power / luminous energy received per second on the sail.
Force: $F=2\Phi/c$.
Radiation pressure: $P=F/S=2\Phi/Sc$.

b - Due to the Doppler effect, in the reference frame of the sail moving away from the laser sources, the photons are less energetic and less numerous, the apparent power is reduced by a Doppler factor squared: $\Phi_a = \frac{1-\beta}{1+\beta}\Phi$.
Same factor for force and pressure.

c - Here $\beta=0.2$ and $\frac{P_a}{P} = \frac{2}{3}$.
The force is reduced by one third.

3. Optical molasses

Exercise p96.

a - When the atom is stationary, the radiation pressures produced by the two lasers, and thus the F forces exerted on the cross section on the two opposite sides of the atom, balance each other.

When an atom moves towards a laser, the radiation pressure increases by the Doppler factor squared and decreases by the same factor in the opposite direction:

$$F_{resultant}=\left(\frac{1+\beta}{1-\beta}-\frac{1-\beta}{1+\beta}\right)F=\frac{4\beta}{1-\beta^2}F$$

The resulting force is in the opposite direction of the velocity, so it is indeed a force that slows the atom.

b - For v«c so $\beta \ll 1$: $1-\beta^2 \simeq 1$ and $F_r=\frac{4v}{c}F$

Or in vectorial form: $\vec{F}_r=-\frac{4F}{c}\vec{v}$

c - At rest, the atom does not interact with the laser because its absorption line is above that of the laser. The atom therefore remains confined.

When the atom possesses kinetic energy and moves towards a laser, it sees in its own referential the laser frequency increased by Doppler. When this frequency corresponds to its resonance frequency, the atom absorbs a photon. The momentum is conserved and therefore the atom slows. A photon is re-emitted after a duration of the order of the lifetime of the energy level of the atom in a random direction. The emitted photon has a higher energy than the absorbed one, hence the decrease of the atom's kinetic energy.

$$A + \gamma \rightarrow A^* \rightarrow A + \gamma$$

d - $e=\frac{3}{2}k_B T \simeq \Delta E \geqslant \frac{\hbar}{2\tau}$ and $T_{min} \simeq \frac{\hbar}{3k_B \tau}$

N.A.: τ=27ns, $\hbar = h/2\pi$ and $T_{min} \simeq 0.1 mK$.

We find the right experimental order of magnitude.

e - $e=\frac{3}{2}k_B T = \frac{1}{2}m v^2$ where v is the root mean square velocity. Thus $v=\sqrt{\frac{3 k_B T}{m}}=\sqrt{\frac{\hbar}{\tau m}}$.

N.A. : $M_{Rb}=87\times 10^{-3}$kg/mol, $m=M/N_A$, $v\simeq 16$ cm/s.

4. Detection of exoplanets

Exercise p98.

a - $T^2=\frac{4\pi^2 a^3}{G(m_1+m_2)}$

with $m_2=m$, $m_1=km$, $k=10\geq 1$ and $a=R$.

$$d_{GP}=d=R_{Planet}=\frac{m_1}{m_1+m_2}a \text{ and } R_{Star}=\frac{m_2}{m_1+m_2}a=\frac{m_2}{m_1}d$$

$$v_{Star}=\frac{2\pi R_S}{T}=\frac{2\pi m_2 d}{2\pi m_1}\sqrt{\frac{G(m_1+m_2)m_1^3}{d^3(m_1+m_2)^3}}=\frac{1}{1+k}\sqrt{\frac{G m_1}{d}}$$

N.A.: $v_{Star}\simeq 1.3\, km/s \ll c$

Precision already accessible by Doppler in the 50s.

b - When the star gets closer: $\lambda'=\sqrt{\frac{1-\beta}{1+\beta}}\lambda$

For β small: $\lambda' = (1-\beta)^{1/2}(1+\beta)^{(-1/2)}\lambda \simeq (1-\beta)\lambda$
And when the star moves away: $\lambda' \simeq (1+\beta)\lambda$

c - $\lambda_{max} = (1+\beta)\lambda$ and, after T/2, $\lambda_{min} = (1-\beta)\lambda$, then

$\Delta\lambda = \lambda_{max} - \lambda_{min} = 2\dfrac{v}{c}\lambda$ and $\dfrac{\Delta\lambda}{\lambda} = 2\dfrac{v}{c}$

N.A.: $\dfrac{\Delta\lambda}{\lambda} \simeq 8.7 \times 10^{-6}$

5. Calculations for the moving ruler *Exercise p100.*

a -

$$\begin{cases} E_1:\ ct = \dfrac{1}{\beta}\left(x + \dfrac{L}{2\gamma}\right) \\ E_2:\ ct = \dfrac{1}{\beta}\left(x - \dfrac{L}{2\gamma}\right) \\ \\ y = 0 \end{cases}$$

b - $\overrightarrow{MC} = (x, y-D, c\,t_C - c\,t)$
To ensure collinearity we make a cross product:

$$\overrightarrow{MC} \wedge \vec{u} = \vec{0} \quad \text{gives} \quad \begin{cases} y - D - bc(t_C - t) = 0 & (1) \\ ac(t_C - t) - x = 0 & (2) \\ bx - ay + aD = 0 & (3) \end{cases}$$

We now introduce the constraints of the end worldlines.

For E_1 :

(3) : $bx_1 = -a\,D$, $b = \dfrac{\pm 1}{\sqrt{1+\left(\dfrac{x_1}{D}\right)^2}}$ and $a = \dfrac{\pm 1}{\sqrt{1+\left(\dfrac{D}{x_1}\right)^2}}$

(2) : $-x_1 + a\dfrac{1}{\beta}\left(x_1 + \dfrac{L}{2\gamma}\right) - a\,c\,t = 0$

then by substituting a :

$$(1-\beta^2)\,{x_1}^2 - 2\beta(ct - L/2\gamma\beta)\,x_1 + \beta^2[(c\,t - L/2\gamma\beta)^2 - D^2] = 0$$

Quadratic equation for x_1, same approach for x_2, then :

$$L_a = \gamma\,L + \gamma\,\beta\left[\sqrt{\left(\gamma\,\beta\,c\,t - \frac{L}{2}\right)^2 + D^2} - \sqrt{\left(\gamma\,\beta\,c\,t + \frac{L}{2}\right)^2 + D^2}\right]$$

We retrieve the long-distance limits:

$$\lim_{t \to \pm\infty} L_a = \gamma\,L(1 \mp \beta)$$

6. Velocity transformation and the aberration of light

Exercise p101.

a - According the Lorentz transformation:

$x' = \gamma(x - \beta c\,t)$, $y' = y$, $z' = z$ and $c\,t' = \gamma(c\,t - \beta\,x)$

hence for infinitesimal variations:

$$dx' = \gamma(dx - \beta\,c\,dt),\quad d\,y' = d\,y\ ,\quad d\,z' = d\,z$$

$$\text{and}\quad c\,dt' = \gamma(c\,dt - \beta\,dx)$$

$$\text{with}\quad \beta = \frac{u}{c}\quad\text{and}\quad \gamma = \frac{1}{\sqrt{1-\left(\dfrac{u}{c}\right)^2}}$$

And by dividing the two equations:

$$\frac{dx'}{c\,dt'}=\frac{dx-\beta c\,dt}{c\,dt-\beta\,dx}, \quad \frac{v_x'}{c}=\frac{\frac{v_x}{c}-\beta}{1-\beta\frac{v_x}{c}} \quad \text{and} \quad v_x'=\frac{v_x-u}{1-\frac{u v_x}{c^2}}$$

$$\frac{dy'}{c\,dt'}=\frac{dy}{\gamma(c\,dt-\beta\,dx)} \quad \text{from where} \quad v_y'=\frac{v_y}{\gamma\left(1-\frac{u v_x}{c^2}\right)}$$

and likewise: $$v_z'=\frac{v_z}{\gamma\left(1-\frac{u v_x}{c^2}\right)}$$

u is the velocity of R' with respect to R.

b - $\vec{v}=(-c\cos\theta,-c\sin\theta,0)$

c - $$\vec{v}'=\left(-\frac{c\cos\theta+u}{1+\frac{u\cos\theta}{c}}\ ,\ -\frac{c\sin\theta}{\gamma(1+\frac{u\cos\theta}{c})}\ ,\ 0\right)$$

$$\vec{v}'\cdot\vec{v}'=\frac{(c\cos\theta+u)^2+c^2\sin^2\theta(1-\beta^2)}{(1+\frac{u\cos\theta}{c})^2}=...=c^2.$$

d - $$\tan\theta_a=\frac{v_y'}{v_x'}=\frac{c\sin\theta}{\gamma(c\cos\theta+u)}=\frac{\sin\theta}{\gamma(\cos\theta+\beta)}\ .$$

7. Composition of velocities and accelerations. 3D generalization *Ex. p101.*

a - $\vec{v}=(0,v_{2,}0)$ and $\vec{u}=\vec{\beta}c=(v_{1,}0,0)$.

now $\quad \vec{v}\,'=\left(\dfrac{v_x-u}{1-\dfrac{u\,v_x}{c^2}},\dfrac{v_y}{\gamma\left(1-\dfrac{u\,v_x}{c^2}\right)},\dfrac{v_z}{\gamma\left(1-\dfrac{u\,v_x}{c^2}\right)}\right)$

then $\quad \vec{v}\,'=\left(-v_1,\dfrac{v_2}{\gamma},0\right)$ and $\beta\,'=\sqrt{\beta_1^2+\beta_2^2-\beta_1^2\beta_2^2}$

NA : $v\,'\simeq 66\,\%\,of\;c$

b - $\quad \vec{v}=(v_2\cos\theta,v_2\sin\theta,0)$

gives $\quad \vec{v}\,'=\left(\dfrac{v_2\cos\theta-v_1}{1-\beta_1\beta_2\cos\theta},\dfrac{v_2\sin\theta\sqrt{1-\beta_1^2}}{1-\beta_1\beta_2\cos\theta},0\right)$

and $\quad \beta\,'=\dfrac{\sqrt{(\beta_2\cos\theta-\beta_1)^2+\beta_2^2\sin^2\theta\,(1-\beta_1^2)}}{1-\beta_1\beta_2\cos\theta}$

eventually $\boxed{\beta\,'=\dfrac{\sqrt{\beta_1^2+\beta_2^2-2\,\beta_1\beta_2\cos\theta-\beta_1^2\beta_2^2\sin^2\theta}}{1-\beta_1\beta_2\cos\theta}}$

NA : $\beta_1=\beta_2=\sqrt{3}/2$ and $\theta=30°$ then $v\,'\simeq 70\,\%\;of\;c$.

c -1 -

<u>First method</u>:

We apply the formula of the previous question. We are going to determine the angle $\theta=\widehat{(\vec{v}_1,\vec{v}_2)}$. $\vec{v}_1$ is along the edge of a cube and $\vec{v}_2$ is along the space diagonal, so we reason in a right triangle with sides 1, √2 and √3, then $\cos\theta=1/\sqrt{3}$, $\sin\theta=\sqrt{2/3}$ and $\theta\simeq 54.7°$. Withc $\beta_1=1/2$ and $\beta_2=\sqrt{3}/2$, we find the same result:

at t_{12} : $\beta\,'=\dfrac{\sqrt{1/4+3/4-2\sqrt{3}/4\times 1/\sqrt{3}-3/16\times 2/3}}{1-\sqrt{3}/4\times 1/\sqrt{3}}=\sqrt{\dfrac{2}{3}}$

at t_{23} : same calculation with $\beta_2=\sqrt{3}/4$

$$\beta'=\frac{\sqrt{1/4+3/16-2\sqrt{3}/8\times1/\sqrt{3}-3/64\times2/3}}{1-\sqrt{3}/8\times1/\sqrt{3}}=\frac{\sqrt{10}}{7}$$

<u>Second method</u>:

We calculate the three components of the velocity with the general transformation of the velocities from *R* in *R'* with $\beta=1/2$ and $\gamma=2/\sqrt{3}$:

For t_{12} : $\vec{v}=(v_x,v_y,v_z)=(1/2,1/2,1/2)$

$$\vec{v}'=\left(\frac{1/2-1/2}{1-1/4},\frac{\sqrt{3}\times1/2}{2(3/4)},\frac{\sqrt{3}\times1/2}{2(3/4)}\right)=\left(0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$$

and $v'=\sqrt{\frac{2}{3}}$

For t_{23} : $\vec{v}=(1/4,1/4,1/4)$

$$\vec{v}'=\left(\frac{1/4-1/2}{1-1/8},\frac{\sqrt{3}\times1/4}{2(7/8)},\frac{\sqrt{3}\times1/4}{2(7/8)}\right)=\left(-\frac{2}{7},\frac{\sqrt{3}}{7,},\frac{\sqrt{3}}{7}\right)$$

and $v'=\frac{\sqrt{10}}{7}$

c -2 -

<u>Transformation of accelerations</u>:

$$a_y=\frac{dv_y}{dt} \quad \text{and} \quad a_{y'}'=\frac{dv_{y'}'}{dt'}$$

$$a_{y'}'=\frac{dv_{y'}'}{dt}\frac{dt}{dt'}=\frac{a_y\left(1-\frac{uv_x}{c^2}\right)+v_y\frac{u}{c^2}a_x}{\left(1-\frac{uv_x}{c^2}\right)^2}\frac{1}{\gamma\left(1-\frac{uv_x}{c^2}\right)}$$

eventually:

$$\vec{a}'=\begin{cases} \dfrac{a_x}{\gamma^3\left(1-\dfrac{uv_x}{c^2}\right)^3} \\ \dfrac{1}{\gamma^2\left(1-\dfrac{uv_x}{c^2}\right)^2}\left(a_y+\dfrac{\dfrac{uv_y}{c^2}a_x}{\left(1-\dfrac{uv_x}{c^2}\right)}\right) \\ \dfrac{1}{\gamma^2\left(1-\dfrac{uv_x}{c^2}\right)^2}\left(a_z+\dfrac{\dfrac{uv_z}{c^2}a_x}{\left(1-\dfrac{uv_x}{c^2}\right)}\right) \end{cases}$$

The transformation law is very different from that of classical mechanics where, between two inertial frames of reference, $\vec{a}=\vec{a}_a=\vec{a}_r=\vec{a}'$. $\vec{a}'$ depends here of $\vec{a}$ and $\vec{v}$ (which only occurs for non-Galilean frames in classical mechanics).

We consider $\vec{a}$ and $\vec{v}$ at t=4 :

$$\vec{a}=(-1/16,-1/16,-1/16) \quad \text{and} \quad \vec{v}=(3/8,3/8,3/8)$$

$$\beta=1/2 \quad \text{and} \quad \gamma=2/\sqrt{3}$$

$$\vec{a}'=\left(-\frac{3\times 8^2}{13^3}\frac{\sqrt{3}}{2},-\frac{3\times 8^2}{13^3},-\frac{3\times 8^2}{13^3}\right)$$

$$a'=\frac{3\times 8^2}{13^3}\frac{\sqrt{11}}{2}\simeq 0.145\ ly/yr^2$$

Results, both for velocities and accelerations, in agreement with those found previously.

8. Starry sky at the halfway point *Exercise p102.*

a - $M = -2.5\log\left(\frac{5.10^{-5}\times 25^2}{4^2\times 37}\right)\simeq 11 > 6$

We divide the emitted powers by the distance squared.

Proxima Centauri is not visible to the naked eye.

b - $M = -2.5\log\left(\frac{5.10^{-5}\times 25^2}{2^2\times 37}\right)\simeq 9 > 6$

Proxima would still not be visible to the naked eye.

c - $\theta_a = 0$ and $I_a = \frac{1+\beta}{1-\beta} I = 39 I$ with $\beta = 0.95$.

$$M = -2.5\log\left(\frac{39\times 5.10^{-5}\times 25^2}{2^2\times 37}\right)\simeq 5 < 6$$

Proxima is now visible to the naked eye!

d - $M = -2{,}5\log\left(\frac{1\times 25^2}{2^2\times 37}\right)\simeq -1{,}5 < 6$

The Sun would be well visible to the naked eye (equivalent to the brightness of Sirius seen from Earth).

e - $\theta_a = \pi$ and $I_a = \frac{1-\beta}{1+\beta} I = \frac{I}{39}$

$$M = -2.5\log\left(\frac{1\times 25^2}{39\times 2^2\times 37}\right)\simeq 2.4 < 6$$

The Sun is still visible to the naked eye with a brightness comparable to that of the star Polaris.

f - The Sun in the night sky of the exoplanet Proxima b would be visible to the naked eye with a magnitude of zero, a brightness comparable to Vega from Earth.

9. Numerical simulation of the sky   *Exercise p104.*

a - Uniform spherical probability law:

- We have a uniform distribution with respect to φ:

$$\Psi = U(0,360) = 360\,U \quad \text{with} \quad U = U(0,1)$$

Indeed, the infinitesimal area elements between φ and $\varphi + d\varphi$ are all of the same size:

$$dS = \int_{\theta=0}^{\theta=\pi} d\theta \sin\theta \, d\varphi = 2\,d\varphi.$$

Corresponds to the area bounded by two meridians on the surface of a sphere.

- We do not have a uniform distribution with respect to θ. The surface elements between and θ and $\theta + d\theta$ are not all the same size:

$$dS = \int_{\varphi=0}^{\varphi=2\pi} d\theta \sin\theta \, d\varphi = 2\pi \sin\theta \, d\theta.$$

It is analogous to the area delimited by two latitudes on the surface of a sphere. The surface at the equator is larger than at the poles.
We have a probability density function $f(x)$ proportional to $sin(x)$:

$$f(x) = k\sin(x) \quad \text{and} \quad \int_0^{\pi} f(x)\,dx = 1 \quad \text{then} \quad k = \frac{1}{2}.$$

Cumulative function: $F(x) = \int_{-\infty}^{x} f(x)\,dx = \dfrac{1-\cos(x)}{2}.$

Inverse transformation method: $\Theta = F^{-1}(U)$.

Then: $\Theta = arcos(1 - 2U)$.

b - We enter on a first column of a spreadsheet the values of φ:

=ALEA.ENTRE.BORNES(0;360) (for LibreOffice)

We enter on a second column of a spreadsheet the values of θ: =ACOS(1-2*ALEA())/PI()*180

Before calculating the apparent angles, the energy and the number of photons received, we check, at the beginning of the spreadsheet, the consistency of the results with the theory:

$$S=\int_{\theta_1}^{\theta_2} 2\pi\sin\theta\, d\theta=2\pi(\cos\theta_1-\cos\theta_2)$$

Surface S obtained for a unit sphere, then divide by 4π for the percentage. For example, at the North celestial pole between 0° and 20° of colatitude, we have 3% of the surface and therefore of the stars, whereas between 80° and 100°, therefore for the same amplitude of angle, we have 17%. This corresponds well to a uniform spherical distribution. On the graph the uniformity is not obvious; it is the same problem when we want to represent a sphere on a plane. For example, on a world map in Mercator projection, Greenland and Africa seem to have equivalent surfaces, whereas in fact Africa is 14 times larger.

File: www.voyagepourproxima.fr/docs/CielRelativiste.ods

10. A bit of math... *Exercise p105.*

a - cos(a+b)=cos(a)cos(b)-sin(a)sin(b)

sin(a+b)=sin(a)cos(b)+cos(a)sin(b)

Mnemonics:

COCO MINUS SISI, SICO PLUS COSI

Quickly re-demonstrates itself by going to $\mathbb{C}$:

$$e^{i(a+b)}=e^{ia}\,e^{ib}=\cos(a+b)+i\sin(a+b)=\ldots$$

$$\text{So}\quad \tan(a+b)=\frac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}.$$

b - $a=b=\theta/2$ gives $\tan(\theta)=\dfrac{2\tan(\theta/2)}{1-\tan^2(\theta/2)}$.

c - $\forall\ (\theta,\theta_a)\ \in\ ]0,\theta_0[\cup]\theta_{0,}\pi[$:

$$\tan(\theta_a)=\frac{2\tan(\theta_a/2)}{1-\tan^2(\theta_a/2)}=\frac{\sin\theta}{\gamma(\beta+\cos\theta)}=\frac{1}{k}$$

Then: $x^2+2kx-1=0$ with $x=\tan(\theta_a/2)$

$$x=-k\pm\sqrt{k^2+1}=\gamma\frac{-(\beta+\cos\theta)\pm\sqrt{(\beta+\cos\theta)^2+(1-\beta^2)\sin^2\theta}}{\sin\theta}$$

$$x=\gamma\frac{-\beta-\cos\theta\pm\sqrt{\beta^2+2\beta\cos\theta+\cos^2\theta+\sin^2\theta-\beta^2\sin^2\theta}}{\sin\theta}$$

$$x=\gamma\frac{-\beta-\cos\theta\pm|1+\beta\cos\theta|}{\sin\theta}=\gamma\frac{-\beta-\cos\theta+1+\beta\cos\theta}{\sin\theta}$$

$$x=\gamma(1-\beta)\frac{1-\cos\theta}{\sin\theta}=\sqrt{\frac{1-\beta}{1+\beta}}\tan(\theta/2)\qquad \text{QED}$$

11. Energy distribution

Exercise p106.

a - We have: $d\tan(\theta_a/2)=\dfrac{d\theta_a}{\cos^2\theta_a}=\sqrt{\dfrac{1-\beta}{1+\beta}}\dfrac{d\theta}{\cos^2\theta}$

$$\text{so}\quad d\theta=\sqrt{\frac{1+\beta}{1-\beta}}\frac{\cos^2\theta}{\cos^2\theta_a}d\theta_a$$

$$\text{Besides:}\quad \sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\frac{\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}}$$

because $\cos^2\theta = \frac{1}{1+\tan^2\theta}$ and $\sin^2\theta = \frac{\tan^2\theta}{1+\tan^2\theta}$

$$d\Omega = 2\pi\sin\theta\, d\theta = 4\pi(1-\beta^2)\frac{\tan\frac{\theta_a}{2}}{\left(1-\beta+(1+\beta)\tan^2\frac{\theta_a}{2}\right)^2}\frac{d\theta_a}{\cos^2\theta_a}$$

$$d\Omega = 2\pi(1-\beta^2)\frac{\sin\theta_a}{(1-\beta\cos\theta_a)^2}d\theta_a = \frac{1-\beta^2}{(1-\beta\cos\theta_a)^2}d\Omega_a$$

$$\frac{n_\beta(\theta_a)}{n_{\beta=0}} = \frac{1-\beta^2}{(1-\beta\cos\theta_a)^2}$$

$$\frac{n_\beta(0)}{n_{\beta=0}} = \frac{1+\beta}{1-\beta} \quad \text{and} \quad \frac{n_\beta(\pi)}{n_{\beta=0}} = \frac{1-\beta}{1+\beta}$$

$$n_{\beta=0,5}(0) = 3\, n_{\beta=0} \quad \text{and} \quad n_{\beta=0,5}(\pi) = n_{\beta=0}/3$$

b -
$$N_{\beta=0} = \int_{\theta=0}^{\pi} n_{\beta=0}\, d\Omega = 4\pi\, n_{\beta=0}$$

$$N_\beta = \int_{\theta_a=0}^{\pi} n_\beta(\theta_a)\, d\Omega_a = \frac{N_{\beta=0}}{2}\int_{\theta_a=0}^{\pi}\frac{1-\beta^2}{(1-\beta\cos\theta_a)^2}\sin\theta_a\, d\theta_a$$

The integral calculation gives 2, we have a constant number of stars.: $N_\beta = N_{\beta=0} = N$.

c -
$$E = \int_{\theta=0}^{\pi} n_{\beta=0}\, I\, d\Omega = 4\pi\, n_{\beta=0}\, I$$

$$E_a = \int_{\theta_a=0}^{\pi} n_\beta(\theta_a) I_a(\theta_a)\, d\Omega_a = \frac{E}{2}\int_{\theta_a=0}^{\pi}\frac{(1-\beta^2)^2}{(1-\beta\cos\theta_a)^4}\sin\theta_a\, d\theta_a$$

$$E_a = \int_{\theta_a=0}^{\pi} n_\beta(\theta_a) I_a(\theta_a)\, d\Omega_a = \frac{\gamma^2}{3}(3+\beta^2)\,E$$

We have well the expression of the course.

d -

$$E_a(0\leq\theta_a\leq\frac{\pi}{2})=\int_{\theta_a=0}^{\frac{\pi}{2}} n_\beta I_a d\Omega_a=\frac{(1+\beta)^2}{(1-\beta)}\frac{(\beta^2-3\beta+3)}{6}E$$

$$E_a(\frac{\pi}{2}<\theta_a\leq\pi)=\int_{\theta_a=\frac{\pi}{2}}^{\pi} n_\beta I_a d\Omega_a=\frac{(1-\beta)^2}{(1+\beta)}\frac{(\beta^2+3\beta+3)}{6}E$$

For β=0.5, forward $E_a(\theta<\pi/2)\simeq 1.3$ E and backward $E_a(\theta>\pi/2)\simeq 0.13$ E. In total $E_a\simeq 1.44$ E, and forward $E_a(\theta<\pi/2)/E_a\simeq 90.8\%$.

12. Number of photons *Exercise p107.*

a - Infinitesimal energy received from the infinitesimal surface dS: $dE=n\,I\,d\Omega=n(\theta)\,I(\theta)2\pi\sin\theta\,d\theta$.

The energy of a photon is given by the following relation: $e=h\,f=h\,c/\lambda$. Hence the expression of the number of photons received from this surface element: $dN_{photons}=dE/e$.

In the vessel's reference frame:

$$\frac{I_a}{I}=\frac{1-\beta^2}{(1-\beta\cos\theta_a)^2},\quad \frac{\nu_a}{\nu}=\frac{\sqrt{1-\beta^2}}{1-\beta\cos\theta_a},\quad \frac{n_\beta(\theta_a)}{n_{\beta=0}}=\frac{1-\beta^2}{(1-\beta\cos\theta_a)^2}$$

So: $dN_{photons,a}=dE_a/e_a=n_a I_a d\Omega_a/e_a$

$$dN_{p,a}=\frac{(1-\beta^2)^{\frac{3}{2}}}{(1-\beta\cos\theta_a)^3}\sin\theta_a d\theta_a\times\frac{2\pi n_{\beta=0} I\lambda}{hc}$$

In total:

$$N_{p,a}=Constant\times\int_{\theta_a=0}^{\theta_a=\pi}\frac{(1-\beta^2)^{\frac{3}{2}}}{(1-\beta\cos\theta_a)^3}\sin\theta_a d\theta_a=\gamma N_p$$

The number of photons received does increase by a gamma factor.

b - By integrating from 0 to $\pi/2$:

$$N_{p,a}(0<\theta_a<\frac{\pi}{2})=\frac{1}{2}(1+\beta)^2(1-\frac{\beta}{2})$$

For example, at 50% of *c*, there are, in total, 15% more photons received. But above all they are differently distributed: 84% of the photons come from the front hemisphere, for 91% of the total energy.

13. Power emitted by a star *Exercise p108.*

a - $P=\iiint i(\lambda)d\lambda d\Omega dS=\int i(\lambda)d\lambda\int d\Omega\int dS$

and $P=I\times 2\pi\times 4\pi R^2$

with $I=\int i(\lambda)d\lambda=\int_0^{+\infty}\frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda k_B T}}-1}d\lambda$

The integral I is numerically estimated at www.integral-calculator.com. h, Planck's constant is 6.63×10^{-34} J.s. k_B is Boltzman's constant, $k_B=R/N_A$, R=8.31 J/mol/K (ideal gas constant) and $N_A=6.02\times 10^{23}$ mol^{-1} (Avogadro constant).
One finds $P\simeq 4.34\times 10^{26}$ W, which corresponds to the expected value.

b - Values proportional to the areas under the luminance curve:

$$P_{visible} = \int_{\lambda=400\,nm}^{\lambda=800\,nm} i(\lambda)\,d\lambda\,d\Omega\,dS \simeq 1.80\times10^{26}\,W$$

$$P_{IR} = \int_{\lambda=800\,nm}^{+\infty} i(\lambda)\,d\lambda\,d\Omega\,dS \simeq 2.26\times10^{26}\,W$$

$$P_{UV} = \int_{\lambda=0}^{\lambda=400nm} i(\lambda)\,d\lambda\,d\Omega\,dS \simeq 0.29\times10^{26}\,W$$

and $\frac{P_{vis}}{P}\simeq 41\,\%$, $\frac{P_{IR}}{P}\simeq 52\,\%$ and $\frac{P_{UV}}{P}\simeq 7\,\%$.

c - For Proxima Centauri, we find $P_P \simeq 1.1\times10^{24}$ W, hence $P_P/P_S \simeq 0.25\%$, which corresponds basically to the value found on the wiki.

5.1. Half-time

Exercise p129

Using the results and notations of the lecture, we have for the galactic time at half-time $t_{1/8}=T/8 \simeq 1.397$ years. Hence the distance covered at Earth's half-time: $\mathbf{x(t_{1/8}) \simeq 0.74}$ **ly**. Seen from the Earth, the halfway distance is 2 ly and the value of the distance at the half-time of the halfway is less than one light-year, because the motion is not uniform but accelerated.

In the reference frame of the vessel we will be at a different position, because the time of the vessel is each time more dilated and passes less quickly with the increase of the speed.

For the proper half-time: $\tau_{1/8}=\tau/8 \simeq 0.855$ yrs.

But $\frac{g t}{c}=sh\left(\frac{g \tau}{c}\right)$ so we find $\frac{g t_{\tau/8}}{c}$ => $\mathbf{x(\tau_{1/8}) \simeq 0.41}$ **ly**

In classical mechanics the distances would be, of course, the same, because the times are identical:

$$x=\frac{1}{2} g t^2 , \quad \frac{D}{2}=\frac{1}{2} g\left(2 t_{1/8}\right)^2 \text{ and } x\left(t_{1/8}\right)=\frac{D}{8}=0.5\, ly .$$

The classical calculations are, of course, false here, because the speed is not small in front of c, the speed of the rocket would even exceed the speed of light at the halfway point.

5.2. Reality show

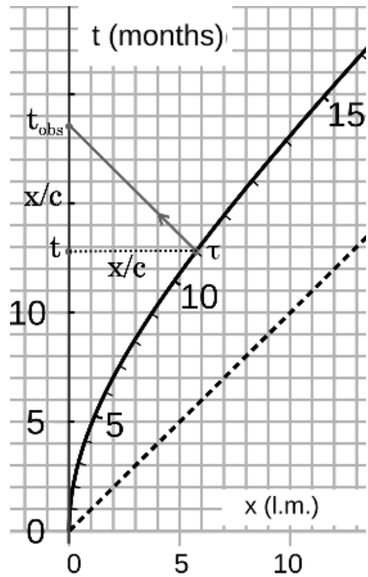
Exercise on page 129

a - We shown that $\gamma=\sqrt{1+\frac{g^2 t^2}{c^2}}$,

then $x=\frac{c^2}{g}(\gamma-1)$.

moreover with $t=\frac{c}{g}sh\left(\frac{g\tau}{c}\right)$

we obtain $\gamma=ch\left(\frac{g\tau}{c}\right)$.



b - 1- 2- We have:

$$t_{obs}=t+\frac{x(\tau)}{c}$$

but $x=\frac{c^2}{g}\left[ch\left(\frac{g\tau}{c}\right)-1\right]$,

then

$$t_{obs}=\frac{c}{g}\left[sh\left(\frac{g\tau}{c}\right)+ch\left(\frac{g\tau}{c}\right)-1\right]$$

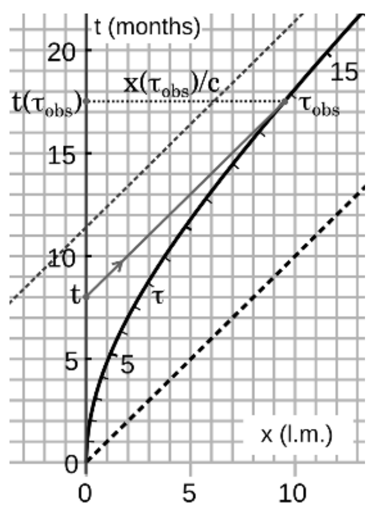
so $t_{obs}=\frac{c}{g}\left(e^{\frac{g\tau}{c}}-1\right)$

and $\tau=\frac{c}{g}\ln\left(\frac{g t_{obs}}{c}+1\right)$.

3- $\tau=6\,months$ gives

$t_{obs}\simeq 7.9\,months$.

4- We can calculate τ's for t_{obs}'s for 12 months and 12 months and one day.



We subtract the two τ and we have the answer. Another more elegant method is to calculate the derivative of t_{obs} with respect to τ. Indeed, the variations are small, one day, over the duration of the trip, one year. The curve can be linearized around the point studied.

$$\frac{\Delta\tau}{\Delta t_{obs}} \simeq \frac{d\tau}{d t_{obs}}, \quad \frac{d\tau}{d t_{obs}} = \frac{1}{\frac{g t_{obs}}{c}+1} \quad \text{and} \quad \frac{d t_{obs}}{d\tau} = 1 + \frac{g t_{obs}}{c}$$

For t_{obs}=1 year and Δt_{obs}=1 day, we find $\Delta\tau \simeq$11h and 41 minutes. The daily reality show on Earth will have to be satisfied, one year after departure, with describing only half a day of life on the ship.

For t_{obs}=10 years, we find $\Delta\tau \simeq$2h and 5 minutes. The reality show will narrate 2 hours aboard the ship each day. If it's while the spacemen are sleeping, there won't be much to say!

c - 1- 2- we have:

$$t(\tau_{obs}) = t + \frac{x(\tau_{obs})}{c} \quad \text{then} \quad t = \frac{c}{g}\left(1 - e^{-\frac{g\tau_{obs}}{c}}\right)$$

$$\text{and} \quad \tau_{obs} = \frac{c}{g}\ln\left(\frac{1}{1-\frac{g t}{c}}\right) \quad \text{with} \quad t < t_{\lim} = \frac{c}{g}$$

N.A.: $t_{\lim} \simeq 11.4\, months$, instant precisely reached on December 14, 2100 at 17h20m00s.

3- $t = 6\, months$ gives $\tau_{obs} \simeq 8.5\, months$.

4- $$\frac{d\tau_{obs}}{dt}=e^{\frac{g\tau_{obs}}{c}}$$

For τ_{obs}=1 year and $\Delta\tau_{obs}$=1 day, we find $\Delta t \simeq$8h and 22 minutes. The daily reality show aboard the ship will be largely satisfied with 8 hours of life on Earth. There is always plenty to talk about, at any given moment on the globe people are getting up, living and going to bed.

For τ_{obs}=10 years, we find $\Delta t \simeq$2.3 seconds! The reality show in the rocket will be limited by the event horizon located on December 14, 2100 at 17:20m00s (Earth calendar). In 2110, the ship's calendar, 10 years after their departure, they will have images of the Earth on December 14, 2100, from 17h06m30s until 17h06m32s! And the day after, they will have two seconds more...

d - <u>Doppler effect for an accelerated frame</u>:

1- From the inertial frame: $$\frac{dt_{obs}}{d\tau}=\frac{T_{Received}}{T_{Emitted}}$$

then: $$f_R=\frac{f_E}{1+\frac{gt_{obs}}{c}}$$ and $$\lambda_R=\lambda_E\left(1+\frac{gt_{obs}}{c}\right)$$

From the accelerated frame: $$\frac{d\tau_{obs}}{dt}=e^{\frac{g\tau_{obs}}{c}}$$

then: $$f_R=f_E e^{-\frac{g\tau_{obs}}{c}}$$ and $$\lambda_R=\lambda_E e^{\frac{g\tau_{obs}}{c}}$$

2- $\frac{\lambda_R}{\lambda_E}=2=e^{\frac{g\tau_{obs}}{c}}$ then $\tau_{obs}=\frac{c}{g}\ln 2=t_{\lim}\ln 2.$

The blue light emitted from the Earth will be perceived red on board the spacecraft after 7.9 months lived by the astronauts in the ship.

3- $t_{obs}=\frac{c}{g}\ln 2$ then $\lambda_R=(1+\ln 2)\lambda\simeq 677\, nm.$

Orange-red light received on Earth.

4- Contrary to the case of inertial reference frames, the Doppler effect in an accelerated reference frame is not symmetric. We find an asymmetry in the twins experiment, also due to the presence of a non-inertial reference frame.

We have studied the case of the same proper times. We can also look at simultaneous times:

$$\text{if } \tau=\frac{c}{g}\ln 2 \text{ then } t=\frac{c}{g}\left(1-e^{-\frac{g\tau}{c}}\right)=\frac{c}{2g}$$

$$\text{and } \lambda_R=\lambda\left(1+\frac{gt}{c}\right)=\frac{3}{2}\lambda=600\, nm$$

3. Head-to-head

Exercise p131

a - By replacing in the formulas, we find:

$$v(x)=c\sqrt{1-\frac{1}{\left(1+\frac{g\,x}{c^2}\right)^2}}\ ,\quad \text{then} \quad \frac{v(D/2)}{c}\simeq 95\,\%$$

relative speed: $\beta_{rel}(D/2)=\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}=\frac{2\beta}{1+\beta^2}\simeq 99.85\,\%$

b - $\frac{v(D/4)}{c}\simeq 87\,\%$ and $\beta_{rel}(D/4)\simeq 99\,\%$.

c - Let us propose the Doppler effect. If each of the vessels continuously emits a monochromatic light beam of known frequency *f* with the help of a lamp, we can deduce the relative speed from the received frequency f_r:

$$f_r=\sqrt{\frac{1+\beta_r}{1-\beta_r}}f \quad \text{then} \quad \beta_r=\frac{\left(\frac{f_r}{f}\right)^2-1}{\left(\frac{f_r}{f}\right)^2+1}$$

d - $v=\frac{\frac{g\,t}{c}}{\sqrt{1+\frac{g^2t^2}{c^2}}}$ & $\frac{g\,t}{c}=sh\left(\frac{g\,\tau}{c}\right)$ => $\frac{v}{c}=th\left(\frac{g\,\tau}{c}\right)$

e - We can consider the two inertial reference frames that coincide at an instant *t* with the vessels. According to the law of composition of velocities:

$$\frac{v_r}{c}=\frac{2\,th\xi}{1+th^2\xi} \quad \text{with} \quad \xi=\frac{g\,\tau}{c}$$

Experimentally, we note that the Doppler effect depends only on the instantaneous velocities of the transmitter and the receiver, and not on their accelerations. We would thus find the same result in the proper reference frame of the vessel which is accelerated and not inertial.

f - $a_r = \dfrac{d v_r}{d\tau}$, $a_r = g\dfrac{d v_r/c}{d\xi}$ & $a_r = \dfrac{2g}{ch^4\xi\left(1+th^2\xi\right)^2}$

At the start: $\tau=0$, $\xi=0$, $ch\xi=1$, $th\xi=0$ and $a_r=2g$.

At the quarter: $\xi=g\tau/c\simeq 0.90$ and $a_r=0.208g$.

At the halfway point: $\xi=g\tau/c\simeq 1.80$ and $a_r=6\times10^{-3}g$.

The acceleration varies with the proper time: the relative motion of the vessels is not uniformly accelerated.

In Newtonian mechanics:

$x_1 = \frac{1}{2}gt^2$, $a_2=-g$, $v_2=-gt$ & $x_2=-\frac{1}{2}gt^2+D$.

=> $x_r = x_2 - x_1 = D - gt^2$, $v_r = -2gt$ & $a_r = -2g$.

In this case the relative motion is also uniformly accelerated with a double acceleration. We well find this result in the classical limit $\xi=0$.

6.1. Euclidean metric

Exercise p159

- Translation: $\begin{cases} x'=x+a \\ y'=y+b \\ z'=z+b \end{cases}$ $\quad dx'=dx+0 \ \ldots \ dl'=dl.$

- Rotation: case of rotation in the plane (Oxy)

By projections: $\begin{cases} \vec{i}=\cos\theta\,\vec{i}\,'-\sin\theta\,\vec{j}\,' \\ \vec{j}=\sin\theta\,\vec{i}\,'+\cos\theta\,\vec{j}\,' \\ \vec{k}=\vec{k}\,' \end{cases}$

$$\vec{r}=x\,\vec{i}+y\,\vec{j}=\vec{r}\,'=x'\,\vec{i}\,'+y'\,\vec{j}\,'$$

$$\vec{r}=x'\,\vec{i}\,'+y'\,\vec{j}\,'=(x\cos\theta+y\sin\theta)\,\vec{i}\,'+(-x\sin\theta+y\cos\theta)\,\vec{j}\,'$$

$$\begin{cases} x'=\ \ x\cos\theta+y\sin\theta \\ y'=-x\sin\theta+y\cos\theta \\ z'=z \end{cases}$$

$$dl'^2=dx'^2+dy'^2+dz'^2=$$

$$(dx\cos\theta+dy\sin\theta)^2+(-dx\sin\theta+dy\cos\theta)^2+dz^2$$

$$dl'^2=(\cos^2\theta+\sin^2\theta)(dx^2+dy^2)$$

$$+(2\cos\theta\sin\theta-2\cos\theta\sin\theta)\,dx\,dy+dz^2$$

$$dl'=dl$$

- Galilean transformation: $\begin{cases} x'=x-vt \\ y'=y \\ z'=z \end{cases}$

The measurement of the position of both ends of a ruler is done at the same time, and the term *vt* is therefore constant :

$$\Delta x'=(x_2-vt)-(x_1-vt)=\Delta x$$

then $\quad dl'=dl$

2. Rapidity *Exercise p159*

1 - With $\gamma = ch\varphi \geqslant 1$ and $-\beta\gamma = sh\varphi$,
we well have $ch^2\varphi - sh^2\varphi = \gamma^2(1-\beta^2) = 1$ & $\varphi = argch\gamma$.
We can verify the invariance of the interval:

$$ds'^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

$$= c^2 dt^2 ch^2\varphi + 2c\,dt\,dx\,ch\varphi\,sh\varphi + dx^2 sh^2\varphi$$

$$-c^2 dt^2 sh^2\varphi - 2c\,dt\,dx\,ch\varphi\,sh\varphi - dx^2 ch^2\varphi - dy^2 - dz^2 = ds^2$$

2 - $\begin{cases} ct' = ct\,ch\varphi_1 + x\,sh\varphi_1 \\ x' = ct\,sh\varphi_1 + x\,ch\varphi_1 \end{cases}$ $\begin{cases} ct'' = ct'\,ch\varphi_2 + x'\,sh\varphi_2 \\ x'' = ct'\,sh\varphi_2 + x'\,ch\varphi_2 \end{cases}$

$$ct'' = (ct\,ch\varphi_1 + x\,sh\varphi_1)ch\varphi_2 + (ct\,sh\varphi_1 + x\,ch\varphi_1)sh\varphi_2$$
$$= ct\,(ch\varphi_1 ch\varphi_2 + sh\varphi_1 sh\varphi_2) + x\,(ch\varphi_1 sh\varphi_2 + sh\varphi_1 ch\varphi_2)$$
$$= ct\,ch(\varphi_1 + \varphi_2) + x\,sh(\varphi_1 + \varphi_2)$$

$x'' = ct\,sh(\varphi_1 + \varphi_2) + x\,ch(\varphi_1 + \varphi_2)$ then $\varphi = \varphi_1 + \varphi_2$

The rapidity, like the covariant velocity, has the advantage of varying from -∞ to +∞. The artifact of the limit speed disappears. Moreover, the rapidity is additive, unlike the covariant and classical velocities which have more complex composition laws in relativity.

3. Rindler metric *Exercise p159*

1 - We have invariance by rotation in the (y,z) plane.
We do not have invariance by rotation in the (r,y) and (r,z) planes. For example:

$$\begin{cases} r' = \;\; r\cos\theta + y\sin\theta \\ y' = -r\sin\theta + y\cos\theta \end{cases}$$

& $ds'^2 = (r\cos\theta + y\sin\theta)^2 d\tau^2 - dr^2 - dy^2 - dz^2 \neq ds^2$

Lorentz transformation: $\begin{cases} \tau' = \tau\,ch\varphi + r\,sh\varphi \\ r' = \tau\,sh\varphi + r\,ch\varphi \end{cases}$ ... $ds'^2 \neq ds^2$

Therefore, the reference frame is not inertial.

2 - For the uniformly accelerated reference frame:

$$ds^2=\left(1+\frac{gx}{c^2}\right)^2 c^2 dt^2-dx^2-dy^2-dz^2$$

It is enough to set $r=c^2/g+x$ and $\tau=gt/c$. This change of origin and units allows us to find the Rindler metric. This metric corresponds to a uniformly accelerated frame of reference.

3 - $ds^2=c^2dt^2-dx^2=(dr\, sh\tau+r\, ch\tau\, d\tau)^2-(dr\, ch\tau+r\, sh\tau\, d\tau)^2$

$$=dr^2 sh^2\tau+2r\, sh\tau\, ch\tau\, dr\, d\tau+r^2 ch^2\tau\, d\tau^2$$

$$-dr^2 ch^2\tau-2r\, sh\tau\, ch\tau\, dr\, d\tau-r^2 sh^2\tau\, d\tau^2$$

$=r^2 d\tau^2-dr^2$ This was to be demonstrated.

Then:

$$\begin{cases} ct'=\left(x+\dfrac{c^2}{g}\right)sh\left(\dfrac{gt}{c}\right) \\ x'=\left(x+\dfrac{c^2}{g}\right)ch\left(\dfrac{gt}{c}\right)-\dfrac{c^2}{g} \end{cases}$$

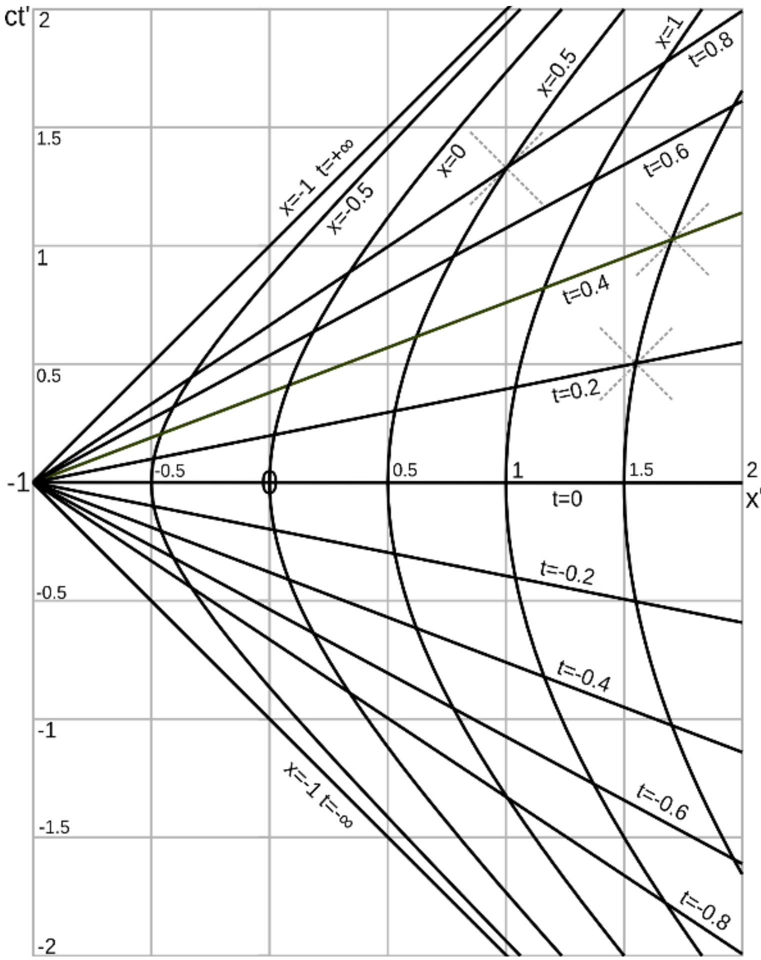
We have simply replaced and changed the origin for *x'* in order to resume the chosen initial conditions (invariance by translation). Thus for the coordinate lines of *x*, we have a grid of hyperbolas centered on ($-c^2/g$, 0):

$$\left(x'+\frac{c^2}{g}\right)^2-c^2t'^2=\left(x+\frac{c^2}{g}\right)^2$$

The coordinate lines of *t* are straight lines that pass through the center of the hyperbola:

$$ct'=th\left(\frac{gt}{c}\right)\left(x'+\frac{c^2}{g}\right)$$

It follows that the coordinate lines of *x* are orthogonal to those of *t*.



4. Free fall in the rocket *Exercise p160*

1 - Let's imagine that we can use a superpower that allows us to reach a speed very close to the speed limit very quickly. To win, we can reach a level as high as we want almost instantly and return just as fast. For example, a level where the time runs twice as fast, and we would come back with 2 min displayed on our clock. Except that it is not possible to come back in time, indeed, from the point of view of the clock at rest, even if you go at the speed limit, one second passes every 300,000 km traveled and you will not be able to rise more than 9 million km in one minute, and the place where the clocks turn twice as fast is nearly one light-year away!

Two opposite effects are at work here, a static effect that makes time go faster as you get higher, and, a dynamic effect that, on the contrary, slows down the clock as you gain speed.

To win, you must find a compromise between elevation and velocity in order to maximize your proper time. A free particle has the maximum proper time. For your clock to be free, it must be in weightlessness and therefore in free fall. The winner will not even need to accompany his clock, he just needs to throw it upwards with the right speed so that it falls back down after one minute.
The same goes for the variant with a bell curve. In this case, there are two fixed clocks at the same level and previously synchronized (possible here, because same *x*).

2-a For the path of maximum proper time:

$$\int \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \delta v \right) dt = 0$$

Link between δx and δv :

$$v_{C'} = \frac{d x_{C'}}{dt} = \frac{d x_C}{dt} + \frac{d \delta x}{dt} = v_C + \delta v \quad \text{and} \quad \delta v = \frac{d}{dt} \delta x$$

Follow an integration by parts:

$$\int \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \frac{d}{dt} \delta x \right) dt = \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt} \frac{\partial L}{\partial v} \right) \delta x \, dt + \left[\frac{\partial L}{\partial v} \delta x \right]_{E_i}^{E_f}$$

The ends of the path are fixed: $\delta x_{E_i} = 0$ and $\delta x_{E_f} = 0$.

$$\Rightarrow \quad \int \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} \right) \delta x \, dt = 0$$

This relationship must be true for all δx tested around the extremal path, hence:

Lagrange's equation : $$\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial v} = 0}$$

$$\frac{\partial L}{\partial x} = \frac{1}{2L}\frac{\partial g}{\partial x} = \frac{g'}{2L} \quad \text{with} \quad g' = \frac{2a}{c^2}\left(1 + \frac{ax}{c^2}\right) \quad \& \quad \frac{\partial L}{\partial v} = -\frac{v}{c^2 L}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v} = -\frac{dv/dt}{c^2 L} + \frac{v}{c^2 L^2}\frac{dL}{dt}$$

$$dL = \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial v}dv \qquad \frac{dL}{dt} = \frac{\partial L}{\partial x}v + \frac{\partial L}{\partial v}\dot{v}$$

$$\frac{d}{dt}\frac{\partial L}{\partial v} = -\frac{\dot{v}}{c^2 L} + \frac{v}{c^2 L^2}\left(\frac{g' v}{2L} - \frac{v \dot{v}}{c^2 L}\right) = -\frac{\dot{v}}{c^2 L}\left(1 + \frac{v^2}{c^2 L^2}\right) + \frac{g' v^2}{2 c^2 L^3}$$

$$\frac{\dot{v}}{c^2}(c^2 L^2 + v^2) = \frac{g'}{2}(v^2 - c^2 L^2)$$

Differential equation of motion: $$\boxed{\frac{\dot{v}}{c^2} = \frac{g'}{g}\left(\frac{v^2}{c^2} - \frac{g}{2}\right)}$$

For a Minkowskian metric we find the rectilinear motion:

$$g = 1 \,, \quad g' = 0 \quad \text{and} \quad \dot{v} = \ddot{x} = 0.$$

For *x* small in front of a light-year:

$$g(0) = 1 \,, \quad g'(0) = \frac{2a}{c^2} \,, \quad \frac{v}{c} \ll 1 \quad \text{and} \quad \dot{v} = -a$$

We find the Newtonian equation of uniformly accelerated motion. After a few months the trajectory will significantly differ from this parabolic trajectory.

This free fall equation corresponds to the trajectory that makes the proper time maximum. So to win, it's very simple, just throw your clock upwards (the front of the ship) so that it falls back 60 seconds later. The difficulty is to measure the initial speed for a return at the right time. The throw is vertical or according to a bell curve for the variant of the game.

Numerical resolution: The duration of the experiment is short compared to t_H=c/a, so we can use the classical parabolic curve and make series expansions:

$$\ddot{x}=-g\ ,\ \dot{x}=-g\,t+v_0\ ,\ x=-\frac{1}{2}g\,t^2+v_0 t,\ v_0=g\frac{\Delta t}{2}\ \ \&\ h=\frac{v_0^2}{2g}$$

$$\tau=\int\sqrt{\left(1+\frac{g\,x}{c^2}\right)^2-\frac{v^2}{c^2}}\,dt\simeq\int\sqrt{1+\frac{2g\,x}{c^2}-\frac{(v_0-g\,t)^2}{c^2}}\,dt$$

$$\tau\simeq\int\sqrt{1+\frac{2g\,t}{c^2}\left(v_0-\frac{1}{2}g\,t\right)-\frac{(v_0^2-2\,v_0\,g\,t+g^2t^2)}{c^2}}\,dt\quad\ldots$$

$$\tau-\Delta t\simeq\frac{1}{c^2}\int_{t=0}^{t=\Delta t}\left(2\,v_0 g\,t-g^2t^2-\frac{v_0^2}{2}\right)dt\quad\ldots$$

after calculation: $$\tau-\Delta t\simeq\frac{g^2\Delta t^3}{24\,c^2}$$

To win: v_0=300 m/s, h=4500 m and your clock will be 10 picoseconds ahead. Well done, you can't do better!

In the frame of reference of the vessel the trajectory is curved, while in the galactic frame of reference the trajectory of the object is rectilinear.

Note that the trajectory of a photon will also be curved in the accelerated reference frame. But to understand what happens, we must differentiate the coordinated velocity *v* that we have defined in the coordinate system of the spacecraft, with the velocity measured in a Minkowskian reference frame defined locally in space and time at the level of the particle. It is this velocity which cannot exceed the limit velocity *c* and which is equal *c* for a photon. The coordinated velocity has no such constraint.

Here, we are in flat space-time and this does not prevent the astronauts from lobbing their playing partners. By a similar reasoning, in the framework of the Schwarzschild metric in curved space-time, we would also find the curved trajectories that we observe when we play ball on the beach. We could abandon our vision of a force of gravity and embrace that of a free particle maximizing its proper time in a non-Minkowskian metric. If we leave the local analogy, there will be notable differences between the accelerated rocket and the approach of a massive star.

2-b $$\frac{dL}{dt}=\frac{\partial L}{\partial x}v+\frac{\partial L}{\partial v}\dot{v}=\left(\frac{\partial L}{\partial x}-\frac{d}{dt}\left(\frac{\partial L}{\partial v}\right)\right)v+\frac{d}{dt}\left(\frac{\partial L}{\partial v}v\right)$$

we recognize in the first term on the right the equation of motion from which:

$$\frac{d}{dt}\left(L-\frac{\partial L}{\partial v}v\right)=0$$

Conservation equation $$\boxed{L-\frac{\partial L}{\partial v}v=cst}$$

The object released in free fall starts from $x=0$ and joins the event horizon at $x_H=-\frac{c^2}{a}$. In the same time $g(x)$ varies from 1 to zero: $g\in]0,1]$.

Expression of the position: $\boxed{x = x_H(1-\sqrt{g})}$

Calculation of the speed: $L + \frac{\beta^2}{L} = cst$ with $\beta = \frac{v}{c}$

Determination of the constant for the initial conditions of a release: at *t=0*, $x=0$, $\dot{x}=0$, $L(0,0)=1$ and $cst=1$.

Then: $L^2+\beta^2 = L$ and $L^4+2\beta^2 L^2+\beta^4 = L^2$

$$g^2-2g\beta^2+\beta^4+2g\beta^2-2\beta^4+\beta^4 = g-\beta^2 \text{ and } \beta = -\sqrt{g(1-g)}$$

Expression of the speed: $\boxed{v = -c\sqrt{g(1-g)}}$

Calculation of the acceleration:

$$\frac{dv}{dt} = \frac{dv}{dg}\frac{dg}{dx}\frac{dx}{dt} = -c\frac{1-2g}{2\sqrt{g(1-g)}}\frac{2a\sqrt{g}}{c^2}v$$

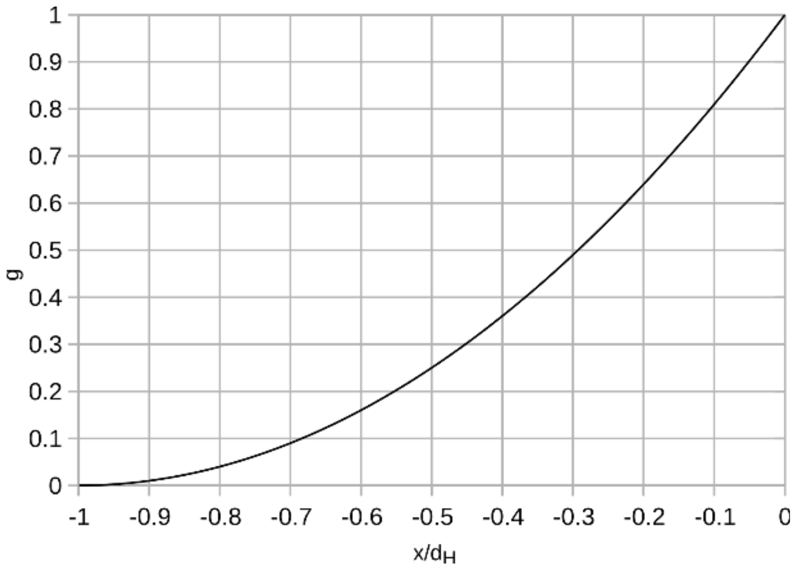
Expression of the acceleration: $\boxed{\frac{dv}{dt} = -a(2g-1)\sqrt{g}}$

The acceleration $\dot{v}$ is zero for $g=1/2$. As expected, since the speed is zero at the beginning, for *t=0*, and tends to zero when *t* tends to infinity, it passes through a maximum:

$$\boxed{v_{max} = \frac{c}{2}} \quad \text{at} \quad x(v_{max}) = \left(1-\frac{1}{\sqrt{2}}\right)x_H \simeq 0.3\,x_H$$

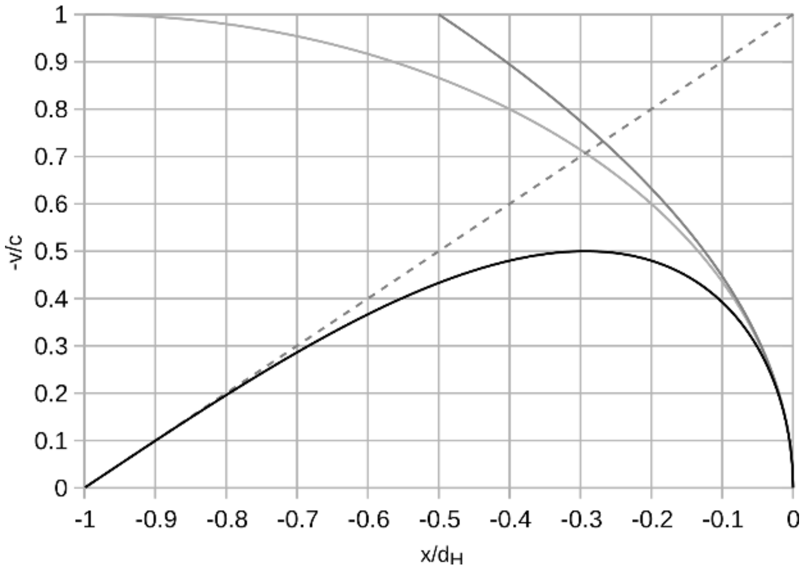
As we will see later, the speed of light in $x(v_{max})$ is c/√2 and $v_{max} = v_{light}/\sqrt{2} \simeq 71\,\% v_{light}$. In agreement with a local Minkowskian observer for whom $v_{mink}(x_{max}) = c/\sqrt{2} \simeq 71\% c$.

Metric factor g(x) :



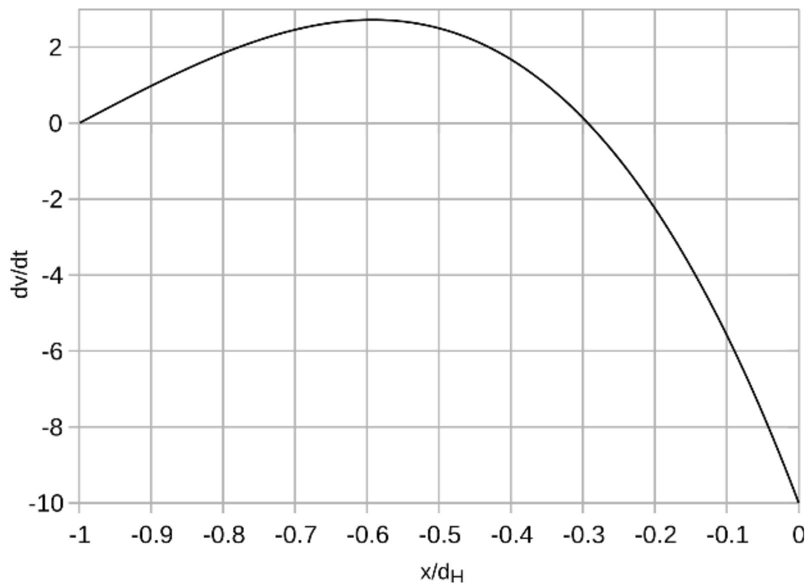
The metric of the reference frame of the uniformly accelerated rocket shows the temporal factor with $g(x)=(1+x/d_H)^2$ *with* $d_H=c^2/a$. $g(x)$ *varies from zero to infinity, when, x varies from* $-d_H$ *, the horizon, to infinity in front of the rocket.*

Velocity of fall of the object v(x) :



Velocity versus position of the fall of an object released without initial speed from x=0. For an observer of the accelerated rocket, the object starts to fall according to a classical parabolic motion (dark gray curve), to then reach the maximum speed c/2 and to become zero on the horizon. The maximum speed is independent of the acceleration of the spacecraft. We have traced in dotted line the coordinated speed of light in this non-inertial reference frame. Indeed, the accelerated rocket frame of reference is not Minkowskian, and the speed of light is not fixed at ±c. For a photon dτ*=0, which gives in the rocket* $v_{light}(x) = \pm|1+x/d_H|c$. *For* $-1<x<0$ $|v_{light}|<c$*, and, for* $x>0$ $|v_{light}|>c$*. For other initial conditions, such as* $x(t=0)=3d_H$*, we find* $v_{max}=-2c$*. We verify that at any point the speed of the falling object is much lower than the speed of light, except on the horizon where the two speeds equalize.*

Acceleration of the object a(x) :



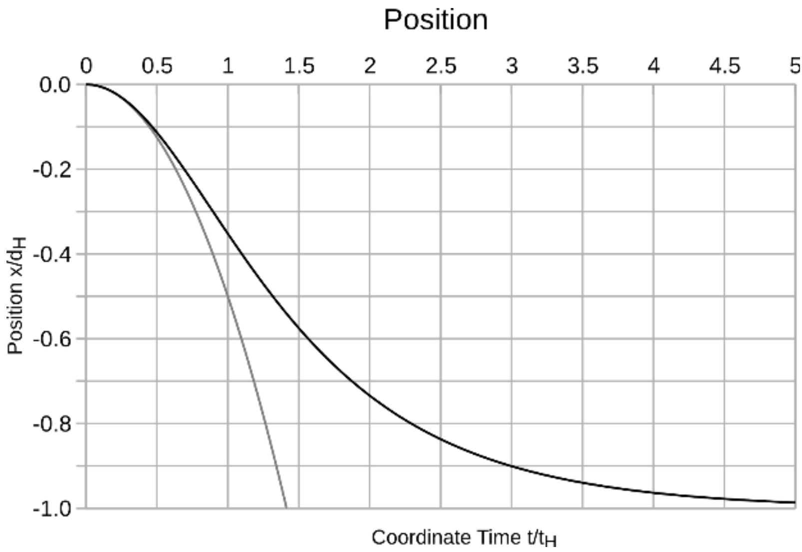
Object in free fall in the rocket: the acceleration is zero when the speed is maximum. The acceleration then changes of sign and the object decelerates to the horizon.

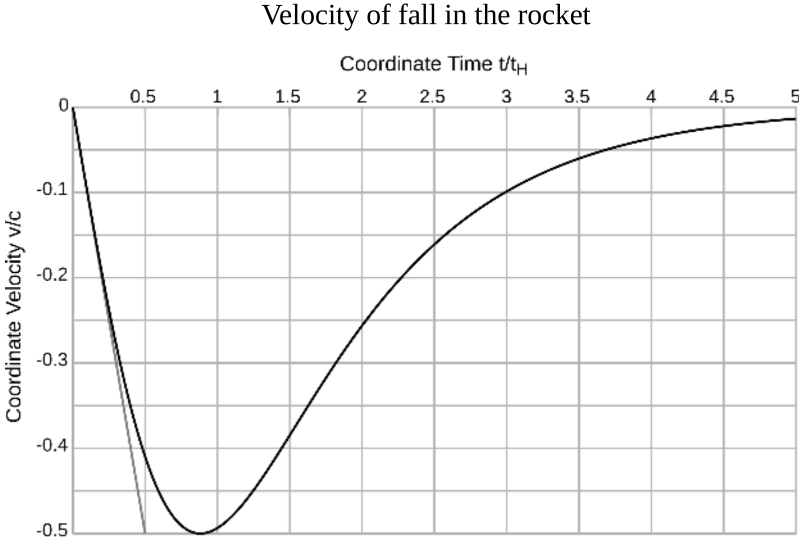
c- To solve the differential equation we have performed numerical simulations in Runge-Kutta 4 for the time evolution. We could also follow an analytical approach and perform a direct calculation using the coordinate change given in the exercise *Rindler metric*. All the equations are expressed in terms of dimensionless quantities:

$$X = x/d_H,\quad d_H = c^2/a = -x_H,\quad T = t/t_H,\quad t_H = c/a = d_H/c$$

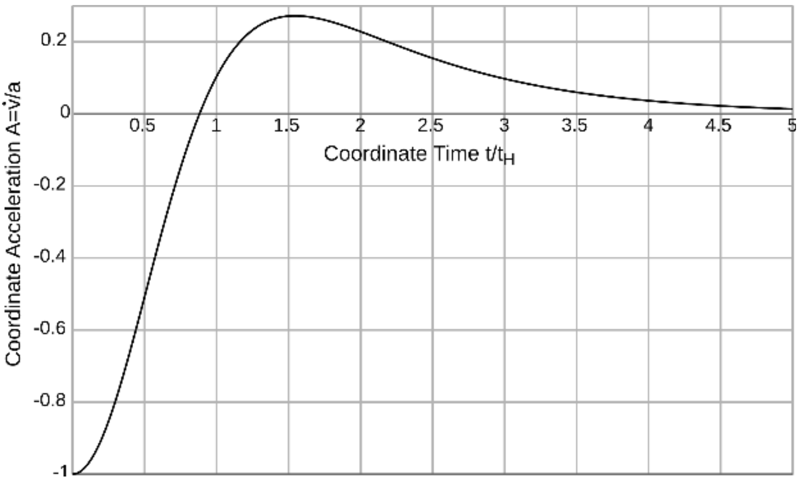
$$\beta = \frac{dX}{dT} = Y,\qquad \frac{\ddot{x}}{a} = \frac{dY}{dT} = [1 - 2(1+X)^2](1+X)$$

The maximum velocity is reached for T≃0.88 (t≃10 months). The horizon is reached only asymptotically when *t* tends to infinity. The classical mechanics curves are plotted as gray lines.





Acceleration

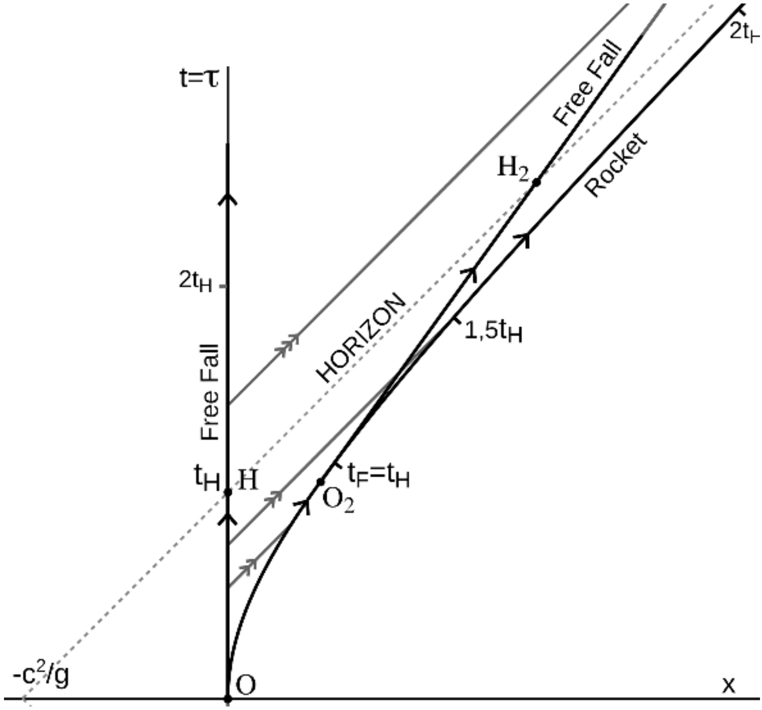


d- ***Proper time***: $\tau = \int \sqrt{g-\beta^2}\,dt = \int g\,dt$

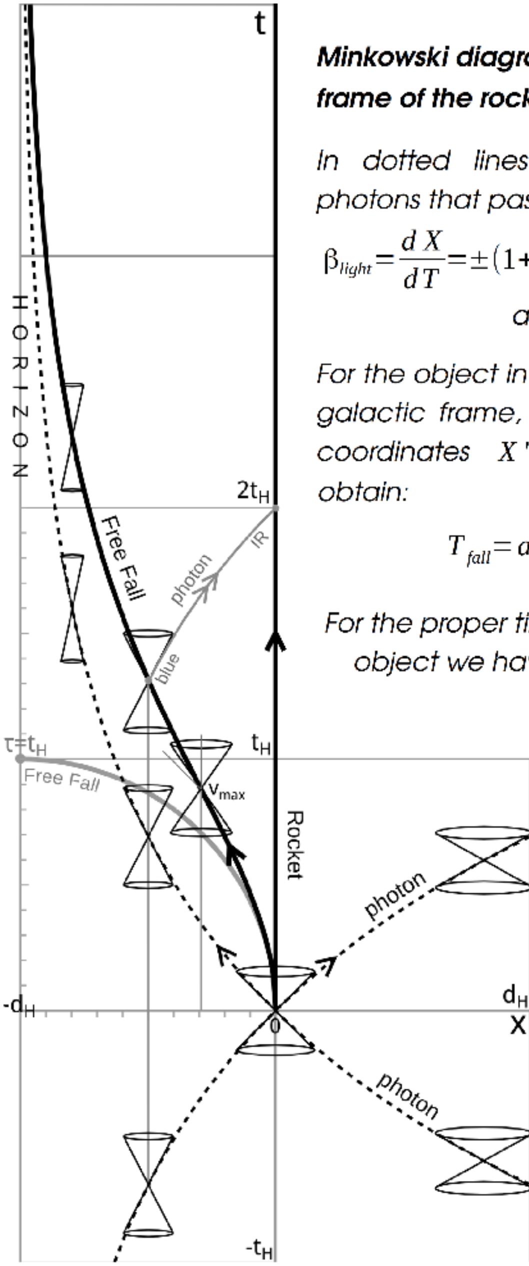
$$\tau = t_H \int (1+X)^2 \frac{dX}{\beta} = -t_H \int \frac{(1+X)}{\sqrt{1-(1+X)^2}}\,dX$$

with $\sin\theta = 1+X$ we find $\tau(X) = t_H\sqrt{1-(1+X)^2}$

So $\tau(x_H) = t_H$, for the observer in free fall the horizon is reached in a time t_H and nothing special happens. The person crosses the horizon without realizing it and his time continues, of course, to flow. On the other hand, the observers of the rocket will see the time of the falling person freeze at t_H and as long as they wait they will never know what happens next. A difference, however, for the person in free fall, before and after the horizon: before he can still stop his fall to join the mother ship with a very fast rocket, after it is impossible, even if his rocket went at the speed of light.



Let us represent the situation on the Minkowski diagram of the inertial reference frame of the object in free fall. The object is dropped, without initial speed, by the astronauts of the accelerated rocket at t=0 and x=0 (event O). The occupants of the spacecraft see the object falling (world lines of the photons with the ↠) and the last photon seen will come from H. Thus the age of the falling object will seem to them to freeze, as if its time stopped after having aged of t_H. But, from the point of view of the object, the time continues to pass and no horizon wall exists. Simply, the horizon defines the place where the causal link between the object and the rocket is broken. Even a photon sent towards the rocket beyond the horizon will not be able to join it (⋙). At O_2 the spacemen let fall a second object. This one has a constant velocity in the reference frame of inertia of the first object and $\tau_{OH}=\tau_{O_2H_2}=t_H$. The first object that the travelers see falling in free fall is the Earth itself that they "dropped" at their departure.



Minkowski diagram in the non-inertial frame of the rocket:

In dotted lines, the worldlines of photons that pass through O:

$$\beta_{light}=\frac{dX}{dT}=\pm(1+X)$$

and $T_{light}=\pm\ln(1+X)$.

For the object in free fall, $X'=0$ *in the galactic frame, with the change of coordinates* $X'=(1+X)chT-1$, *we obtain:*

$$T_{fall}=argch\left(\frac{1}{1+X}\right)$$

For the proper time of the free falling object we have a quarter circle.

e- Minkowskian local observer:

$$dt_{Mink}=(1+X)dt$$

$$\beta_{Mink}=\frac{\beta}{(1+X)}$$

$$=\frac{\beta}{|\beta_{lum}|}$$

$$=-\sqrt{1-(1+X)^2}$$

In the local reference frame of inertia the velocity of the object increases continuously from zero to *c*. The curve has been drawn in light gray on the velocity curve page 391 (arc of circle).

3 - *Analogy with the fall into a black hole:*

a- $L(r,v)=\sqrt{g-\frac{1}{g}\frac{v^2}{c^2}}$ $\quad L-\frac{\partial L}{\partial v}v=cst \quad \frac{\partial L}{\partial v}=-\frac{v}{g L c^2}$

Velocity: $L+\frac{\beta^2}{g L}=cst$ with $\beta=\frac{v}{c}$ and $g(r)=1-\frac{r_S}{r}$

Determination of the constant for the initial conditions of a release: at *t*=0, $r\rightarrow+\infty$, $\dot{r}=0$, $L(+\infty,0)=1$ & $cst=1$.

Then: $g L^2+\beta^2=g L$ and $g^2 L^4+2 g \beta^2 L^2+\beta^4=g^2 L^2$

$g^4-2 g^2\beta^2+\beta^4+2 g^2\beta^2-2\beta^4+\beta^4=g^3-g\beta^2$ and $\beta=-g\sqrt{1-g}$

Expression of the velocity: $\boxed{v=-c\sqrt{g^2(1-g)}}$

Calculation of the acceleration:

$$\frac{dv}{dt}=\frac{dv}{dg}\frac{dg}{dr}\frac{dr}{dt}=-c\frac{2g-3g^2}{2\sqrt{g^2(1-g)}}\frac{(1-g)^2}{r_S}v$$

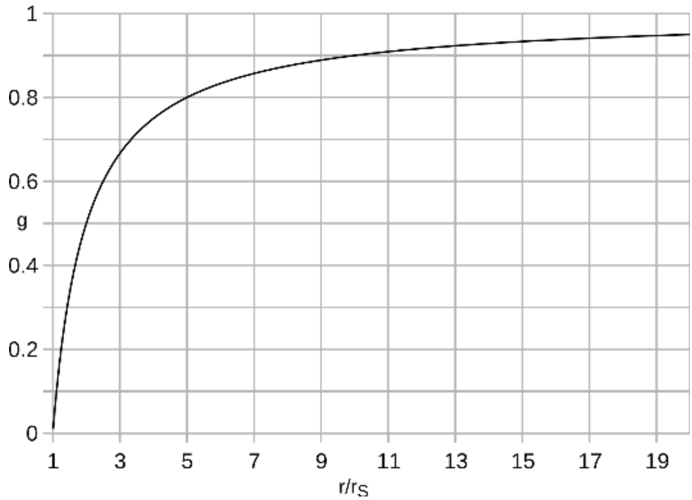
Acceleration: $\boxed{\frac{dv}{dt}=\frac{c^4}{4GM}g(2-3g)(1-g)^2}$

The acceleration $\dot{v}$ is null when $g=2/3$. As expected, since the velocity is zero at the start, at t=0, and tends to zero when *t* tends to infinity, it passes through a maximum:

$\boxed{v_{max}=\frac{2}{3\sqrt{3}}c\simeq 38\%\ c}$ at $r(v_{max})=3 r_S$

The speed af light in $r(v_{max})$ is 2/3 c and $v_{max}=v_{light}/\sqrt{3}$. Speed of the falling object for a local Minkowskian observer: $v_{mink}(r_{max})=c/\sqrt{3}\simeq 58\% c$.

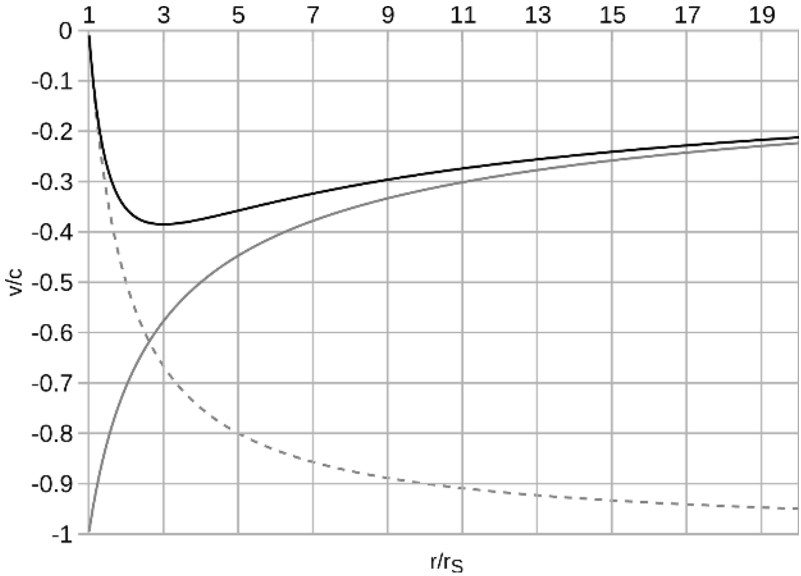
Metric factor g(r) :



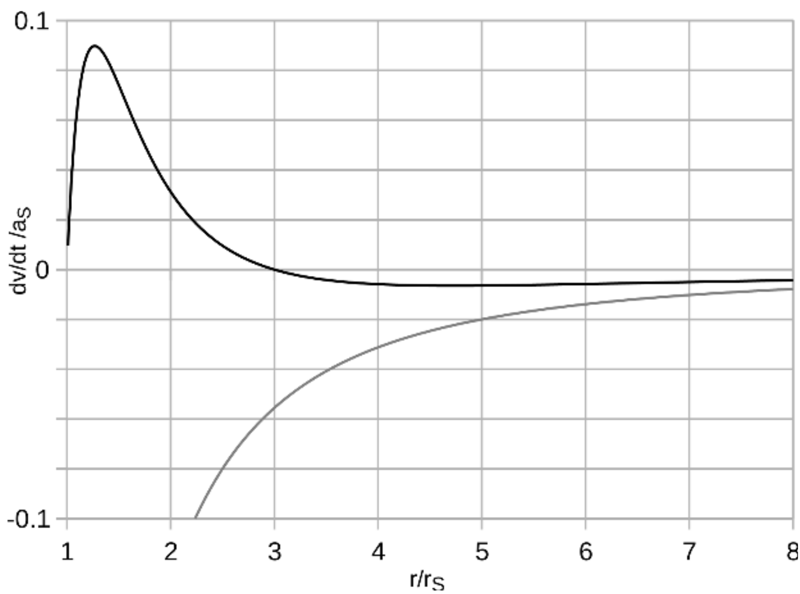
The metric of the Schwarzschild reference system is expressed with the factor $g(r)=1-r_S/r$ *.*

Velocity of fall of the object v(r) :

Curve of the falling velocity of an object released at rest from infinity of a massive star. For an outside observer, the object starts to fall according to a classical motion (grayed curve) to then reach its maximum speed and have a zero speed on the horizon in the case of a black hole (star of radius less than r_S*). The maximum speed is reached if the star has a radius less than 3* r_S*. This speed is the same for any star and does not depend on its mass. We have traced in dotted line the coordinated velocity of light in this non-inertial reference frame. We have in the Schwarzschild coordinate system* $v_{light}(r)=\pm(1-r_S/r)c$*. Here* $|v_{light}|<c$ *and the speed of light cancels on the horizon. The speed of the object is less than the speed of light and equals it at* r_S *.*



Acceleration of the object a(r) :



The acceleration is zero at 3 r_S. In gray the acceleration in Newtonian gravitation.

We introduce dimensionless quantities:

$$R=\frac{r}{r_S},\quad T=\frac{t}{t_S},\quad A=\frac{d\beta}{dT}=\frac{\ddot{r}}{a_S}\quad \text{with}\quad t_S=\frac{r_S}{c}\quad \text{and}\quad a_S=\frac{c^2}{r_S}.$$

$$\beta=-\left(1-\frac{1}{R}\right)\frac{1}{\sqrt{R}}\quad \text{and}\quad A=\frac{1}{2R^2}\left(1-\frac{1}{R}\right)\left(\frac{3}{R}-1\right)$$

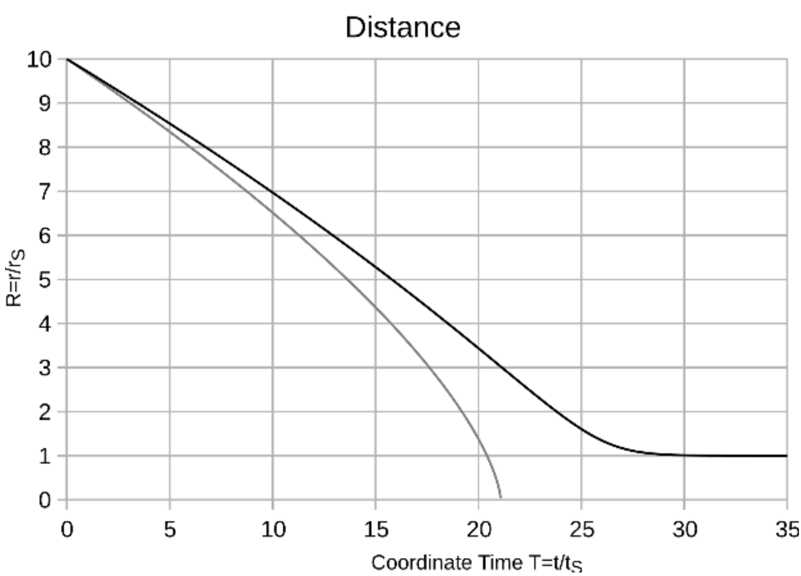
In Newton's gravitation:

$$\frac{1}{2}mv^2-\frac{GMm}{r}=0,\quad v=-\sqrt{\frac{2GM}{r}}\quad \text{and}\quad \beta=-\frac{1}{\sqrt{R}},$$

$$m\ddot{r}\vec{u}_r=\vec{F}=-\frac{GMm}{r^2}\vec{u}_r,\quad \ddot{r}=-\frac{GM}{r^2}\quad \text{and}\quad \frac{\ddot{r}}{a_S}=-\frac{1}{2R^2}.$$

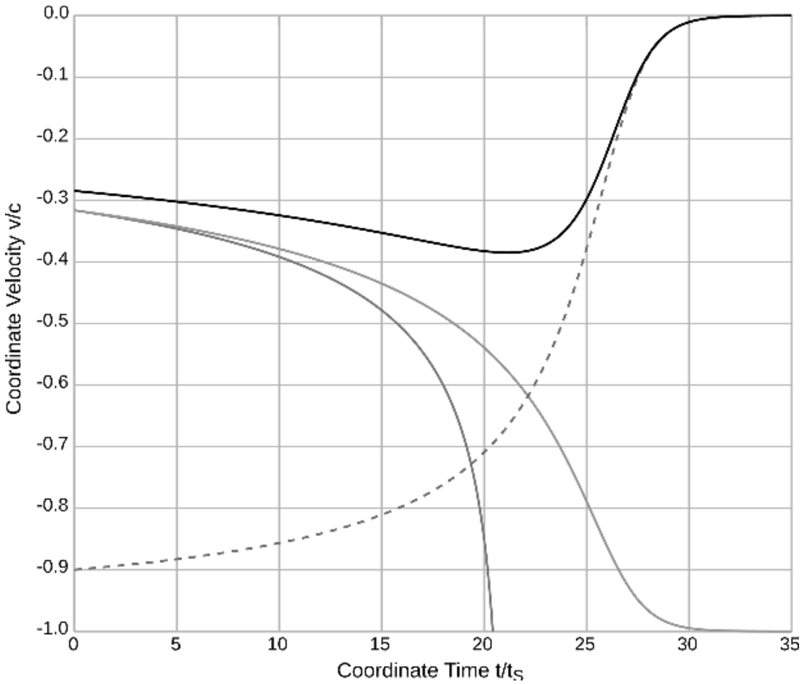
b- Equations for numerical resolution:

$$\beta=\frac{dR}{dT}=Y,\qquad \frac{\ddot{r}}{a_S}=\frac{dY}{dT}=F(R)$$



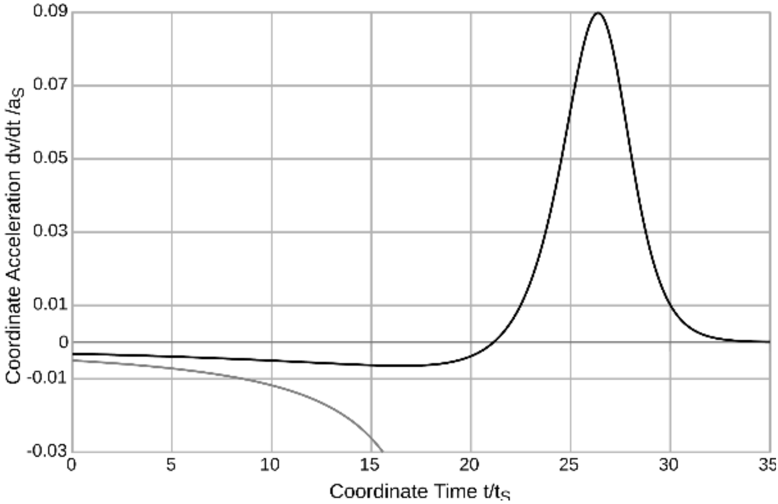
From the external Schwarzschild point of view the falling object takes an infinite time to reach the horizon in r=r_S. In gray the classical curve.

Velocity of fall v(t):



The body gains speed during its fall up to 3 r_S. From a distance of 10 r_S, it has then elapsed approximately 21 t_S before reaching this maximum. This maximum of 38% of c reached, the speed then decreases until it cancels on the horizon. The maximum velocity is independent of the size of the black hole, it is not the case of distances and times. For a super-massive black hole of 40 million solar masses, t_S is about 6 minutes and 35 seconds. In dark gray the velocity curve according to Newton's laws. In dotted line the speed of light in the Schwarzschild coordinate system.

Acceleration a(t):



The Schwarzschild coordinate acceleration becomes zero, then changes sign, forms a peak and tends towards zero at the horizon. In gray the classical curve.

For Newton: $\frac{dr}{dt}=-\sqrt{\frac{2GM}{r}}$, $\sqrt{r}\,dr=-\sqrt{2GM}\,dt$

$$\int_{r_0}^{r}\sqrt{r}\,dr=-\sqrt{2GM}\,t \quad \text{and} \quad R=\left(R_0^{\frac{3}{2}}-\frac{3}{2}T\right)^{\frac{2}{3}}$$

c- *Proper time*: $\tau=\int\sqrt{g-\frac{\beta^2}{g}}dt=\int g\,dt$

$$\tau=t_S\int\left(1-\frac{1}{R}\right)\frac{dR}{\beta}=-t_S\int_{R_0}^{1}\sqrt{R}\,dR=\frac{2}{3}\left(R_0^{3/2}-1\right)t_S$$

For example, from the maximum speed, at r=3 r_S, to the horizon in r=r_S, a proper time of $\tau\simeq 2.8\,t_S$ elapses. It is interesting to note that the singularity on the horizon has disappeared. We can calculate the proper time needed to reach the black hole center: $\tau=-t_S\int_1^0\sqrt{R}\,dR=\frac{2}{3}t_S$, about 4 min 20 s for the super-massive black hole of 40 million solar masses. The person in free fall does not realize that she crosses the horizon, but beyond that point she cannot exit the black hole and her causal link to the outside world is broken. The outside observers will see the free falling person slowly stop on the horizon and his time freeze at the proper time of passage.

d- *Local observer*: $dt_{Mink}=\sqrt{g}\,dt$, $dr_{Mink}=dr/\sqrt{g}$

$$\beta_{Mink}=\frac{dr_{Mink}}{dt_{Mink}}=\frac{\beta}{g}=\frac{\beta}{|\beta_{light}|}=-\frac{1}{\sqrt{R}} \quad \text{and} \quad v_{Mink}(r_S)=-c$$

In the local and instantaneous inertial frame the speed of the object reaches *c* on the horizon. The curve has been drawn in light gray on the velocity curve page 401.

e- *Comparison to experimental data*:

<u>Theory:</u> for a static black hole and a free fall from infinity without initial velocity $\beta=\left(1-\frac{1}{R}\right)\frac{1}{\sqrt{R}}$.

<u>Experiment:</u>

For R=20 : $\beta\simeq 0.21$ to compare with 0.3 measured.

For R=200 : $\beta\simeq 0.07$ to compare with 0.1 measured.

The black hole concerned is rotating and the matter in free fall can have an initial velocity. The results are consistent for the order of magnitude and even in value. There is an uncertainty on the radius and in theory v_{max} can reach 38 %*c* which is consistent with the experimental 30 %. If we consider the speed of light variations between the non-inertial Schwarzschild frame, and the local Minkowskian one, the differences are not significant, because the *r*'s are large compared to r_S.

5. Fall of a blue ball *Exercise p167*

The first blue ball released in free fall is the planet Earth at the departure of the rocket. The behavior will be the same for later releases. Our planet, observed from the spacecraft, will reach the half of the speed *c* after 10 months at 0.3 l.y., when the time tends towards infinity it will stop and freeze at one light-year.
Its color is given by the formula of the Doppler effect established in the exercise *Reality show* on page 129:

$$\lambda_R = \lambda_E e^{\frac{at}{c}}$$

Our beautiful blue planet will be perceived as red after 8 months of travel. If, instead of releasing the blue ball in free fall, we hang it from a rope, it will also redden with the length of rope unwound, but in a different manner.

6. Trajectory of a ray of light in the Einstein's Elevator *Exercise p167*

1 - Position *x* of the box in the Galilean reference system:

$$\ddot{x} = a \quad \dot{x} = at \quad x = \frac{1}{2} a t^2 \quad t_F = L/c \quad \Delta x = -\frac{a L^2}{2 c^2}$$

$$y = ct \quad x(y) = -\frac{a}{2c^2} y^2 \quad y(x) = \sqrt{-\frac{2c^2 x}{a}} \quad v(x) = \sqrt{c^2 - 2ax}$$

2 - *Special Relativity.* Change of coordinates between an inertial reference frame (ct', x', y') and a uniformly accelerating reference frame (ct, x, y) (exercise on the Rindler metric):

$$\begin{cases} y'=ct'=y=\left(x+\dfrac{c^2}{a}\right)sh\left(\dfrac{at}{c}\right) \\ x'=0=\left(x+\dfrac{c^2}{a}\right)ch\left(\dfrac{at}{c}\right)-\dfrac{c^2}{a} \end{cases}$$

$$ch(at/c)=\frac{c^2/a}{x+c^2/a} \qquad sh(at/c)=\frac{y}{x+c^2/a}$$

$$c^4/a^2-y^2=(x+c^2/a)^2 \qquad y(x)=\sqrt{c^4/a^2-(x+c^2/a)^2}$$

$$x(y)=\sqrt{c^4/a^2-y^2}-c^2/a \qquad \Delta x=\sqrt{c^4/a^2-L^2}-c^2/a$$

$$\Delta x=\frac{c^2}{a}\left(\sqrt{1-\frac{a^2L^2}{c^4}}-1\right) \qquad \lim_{L\ll d_H}(\Delta x)_{SR}=(\Delta x)_{Newton}$$

For the coordinate velocity of light in *R* non-Minkowskian, we can also directly use the metric:

$$d\tau=0 \quad \text{and} \quad v(x)=\frac{dl}{dt}=c\left(1+\frac{ax}{c^2}\right)$$

Direct calc.: $$\begin{cases} y(t)=\dfrac{c^2}{a}th\left(\dfrac{at}{c}\right) \\ x(t)=\dfrac{c^2}{a}\dfrac{1}{ch\left(\dfrac{at}{c}\right)}-\dfrac{c^2}{a} \end{cases} \qquad \begin{cases} \dot{y}=c\dfrac{1}{ch^2} \\ \dot{x}=-c\dfrac{sh}{ch^2} \end{cases}$$

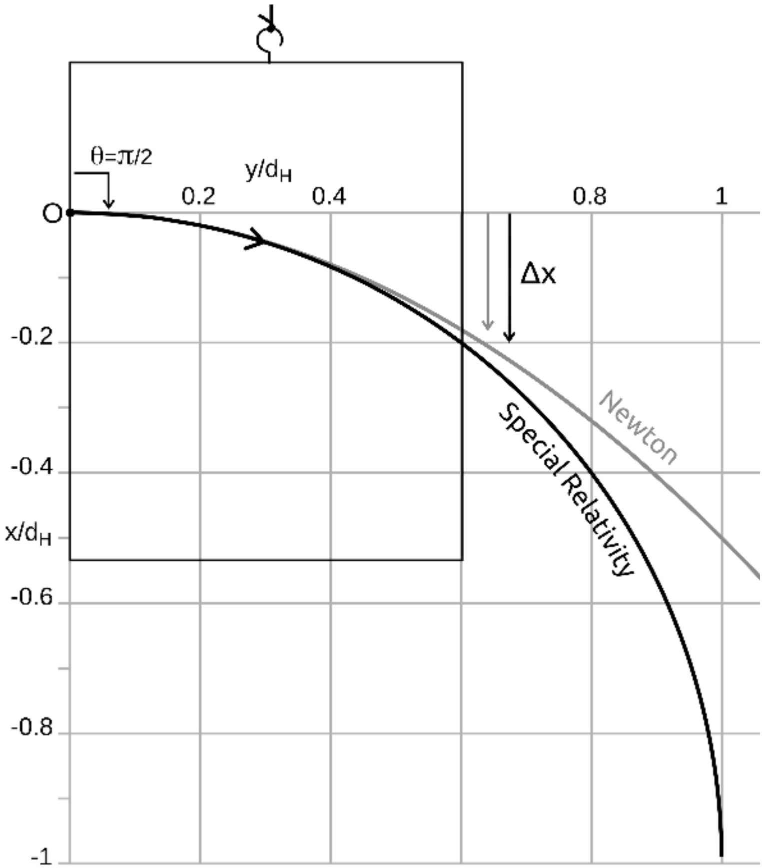
$$v^2=\dot{x}^2+\dot{y}^2=\frac{c^2}{ch^4}(1+sh^2)=c^2\left(1+\frac{ax}{c^2}\right)^2$$

3 - Dimensionless quantities: $d_H=\dfrac{c^2}{a} \quad X=\dfrac{x}{d_H} \quad Y=\dfrac{y}{d_H}$

Newton : $X=-\dfrac{Y^2}{2} \quad \beta=\dfrac{v}{c}=\sqrt{1-2X}$

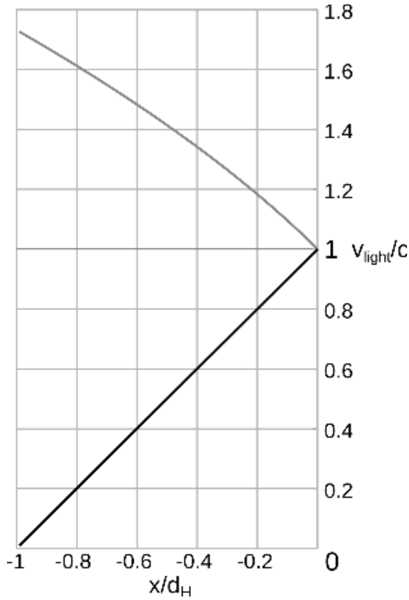
Special Relativity: $(X+1)^2+Y^2=1 \quad \beta=1+X$

Trajectory of a ray of light



The ray follows a quarter circle of radius d_H, horizon distance, and center (-d_H, 0). The velocity of the photon decreases until zero, and, in an infinite time, the photon reaches the position (-d_H, d_H). In the Newtonian approximation the trajectory is parabolic, the deviation is twice as small at d_H, and the speed tends to infinity.

Velocity of the photons in the box



The velocity of the photons initially equal to c decreases linearly until it becomes zero on the horizon. On the contrary, in Newton's case in gray, the velocity increases by addition of the velocities and tends towards infinity like that of the box with respect to the Galilean reference frame.

7. Spherical coordinate system *Exercise p 169*

In addition to giving useful tools for physics and astronomy, we introduce the notion of solid angle — a very physical approach that is rarely explained. We often limit ourselves to plane angles whereas the world is in 3D.

1 - $\vec{r}=\begin{cases} x=r\sin\theta\cos\varphi \\ y=r\sin\theta\sin\varphi \\ z=r\cos\theta \end{cases} \quad \vec{u}_r=\dfrac{\dfrac{\partial\vec{r}}{\partial r}}{\left\|\dfrac{\partial\vec{r}}{\partial r}\right\|} \quad \vec{u}_\theta=\dfrac{\dfrac{\partial\vec{r}}{\partial\theta}}{\left\|\dfrac{\partial\vec{r}}{\partial\theta}\right\|} \quad \vec{u}_\varphi=\dfrac{\dfrac{\partial\vec{r}}{\partial\varphi}}{\left\|\dfrac{\partial\vec{r}}{\partial\varphi}\right\|}$

$$\begin{cases} \vec{u}_r = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \vec{u}_\theta = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} \\ \vec{u}_\varphi = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} \end{cases}$$

$$\begin{cases} \vec{i} = \sin\theta\cos\varphi\,\vec{u}_r + \cos\theta\cos\varphi\,\vec{u}_\theta - \sin\varphi\,\vec{u}_\varphi \\ \vec{j} = \sin\theta\sin\varphi\,\vec{u}_r + \cos\theta\sin\varphi\,\vec{u}_\theta + \cos\varphi\,\vec{u}_\varphi \\ \vec{k} = \cos\theta\,\vec{u}_r - \sin\theta\,\vec{u}_\theta \end{cases}$$

$$\vec{A} = A_x\,\vec{i} + A_y\,\vec{j} + A_z\,\vec{k} = A_r\vec{u}_r + A_\theta\vec{u}_\theta + A_\varphi\vec{u}_\varphi$$

$$\begin{cases} A_x = A_r\sin\theta\cos\varphi + A_\theta\cos\theta\cos\varphi - A_\varphi\sin\varphi \\ A_y = A_r\sin\theta\sin\varphi + A_\theta\cos\theta\sin\varphi + A_\varphi\cos\varphi \\ A_z = A_r\cos\theta - A_\theta\sin\theta \end{cases}$$

$$\begin{cases} A_r = A_x\sin\theta\cos\varphi + A_y\sin\theta\sin\varphi + A_z\cos\theta \\ A_\theta = A_x\cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\theta \\ A_\varphi = -A_x\sin\varphi + A_y\cos\varphi \end{cases}$$

2 - $\vec{r} = \overline{OM} = r\,\vec{u}_r$, $\vec{u}_r(\theta,\varphi)$, $\vec{dr} = dr\,\vec{u}_r + r\,d\vec{u}_r$.

$$d\vec{u}_r = \frac{\partial\vec{u}_r}{\partial\theta}d\theta + \frac{\partial\vec{u}_r}{\partial\varphi}d\varphi = \vec{u}_\varphi \wedge \vec{u}_r\,d\theta + \vec{u}_z \wedge \vec{u}_r\,d\varphi$$

then $\vec{dr} = dr\,\vec{u}_r + r\,d\theta\,\vec{u}_\theta + r\sin\theta\,d\varphi\,\vec{u}_\varphi$.

3 - $S = \int dS = \iint R\,d\theta\,R\sin\theta\,d\varphi = R^2 \int_{\theta=0}^{\pi} d\theta\sin\theta \int_{\varphi=0}^{2\pi} d\varphi = 4\pi R^2$

$$V = \int dV = \iiint dr\,r\,d\theta\,r\sin\theta\,d\varphi = \int_{r=0}^{R} r^2\,dr \times 4\pi = \frac{4}{3}\pi R^3$$

4-a-b All of space is viewed under an angle of 4π steradians (surface of a unit sphere). One hemisphere under 2π steradians. From one corner, one eighth is perceived hence $\pi/2$ steradians. Under an angle 2α:

$$S_{R=1} = \int_{\theta=0}^{\alpha} \sin\theta\,d\theta \int_{\varphi=0}^{2\pi} d\varphi = 2\pi(1-\cos\alpha)$$

c- Probability for an isotropic distribution of stars:

$$p=1-\frac{2\times 2\pi(1-\cos\alpha)}{4\pi}=\cos\alpha \qquad p(\alpha=80°)\simeq 1/6$$

7 .1. Change of basis

Exercise p243

$$\tilde{e}_\mu=\Lambda^\nu{}_\mu\tilde{e}'_\nu \quad \text{and} \quad \Lambda^\mu{}_\nu=\frac{\partial x'^\mu}{\partial x^\nu}$$

1 - Rocket: $\begin{cases} ct'=\left(x+\frac{c^2}{g}\right)sh\left(\frac{gt}{c}\right) \\ x'=\left(x+\frac{c^2}{g}\right)ch\left(\frac{gt}{c}\right)-\frac{c^2}{g} \end{cases}$ & $\begin{cases} T'=(1+X)\,sh\,T \\ X'=(1+X)\,ch\,T-1 \end{cases}$

$$\text{with} \quad X=\frac{g\,x}{c^2} \quad \text{and} \quad T=\frac{g\,t}{c}$$

$$\Lambda^0{}_0=\frac{\partial ct'}{\partial ct}=(1+X)\,ch\,T, \qquad \Lambda^1{}_1=\frac{\partial x'}{\partial x}=ch\,T \quad \ldots$$

$$\text{and} \quad \Lambda^\mu{}_\nu=\begin{pmatrix}(1+X)\,ch\,T & sh\,T \\ (1+X)\,shT & chT\end{pmatrix}$$

$$\tilde{e}_0=\Lambda^0{}_0\tilde{e}'_0+\Lambda^1{}_0\tilde{e}'_1 \quad \ldots \qquad \begin{cases} \tilde{e}_0=(1+X)(chT\,\tilde{e}'_0+shT\,\tilde{e}'_1) \\ \tilde{e}_1=shT\,\tilde{e}'_0+chT\,\tilde{e}'_1 \end{cases}$$

The basis is orthogonal and we find the components of the metric tensor:

$$\tilde{e}_0\cdot\tilde{e}_0=(1+X)^2(chT^2g'_{00}+shT^2g'_{11}+0+0)=(1+X)^2=g(x)=g_{00}$$

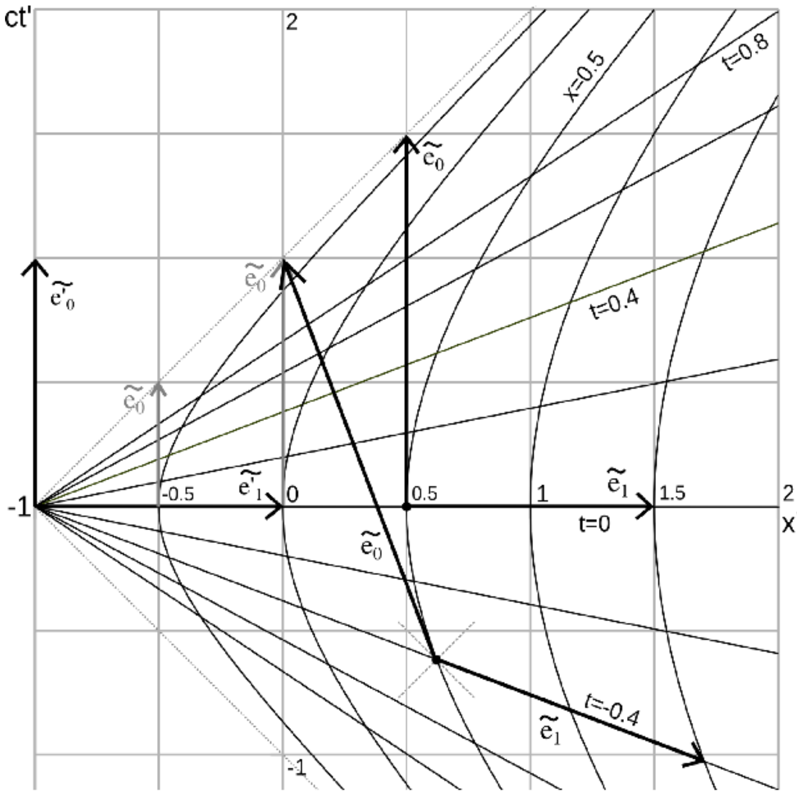
$$\tilde{e}_0\cdot\tilde{e}_1=(1+X)(shT\,chT-shT\,chT)=0=g_{01}=g_{10}$$

$$\tilde{e}_1\cdot\tilde{e}_1=shT^2g'_{00}+chT^2g'_{11}+2\,shT\,chT\,g'_{01}=-1=g_{11}$$

For T=-0.4 and X=0.5 : $\begin{cases} \tilde{e}_0 \simeq \;\; 1.62\,\tilde{e}\,'_0 - 0.62\,\tilde{e}\,'_1 \\ \tilde{e}_1 \simeq -0.41\,\tilde{e}\,'_0 + 1.08\,\tilde{e}\,'_1 \end{cases}$

Lengths and angles on the euclidean sheet:

$$\begin{cases} \|\tilde{e}_0\|_{Euclide} \simeq 1.73 \\ \|\tilde{e}_1\|_{Euclide} \simeq 1.16 \end{cases} \quad \text{and} \quad \begin{cases} \widehat{(\tilde{e}_0, \tilde{e}\,'_0)} \simeq +20.8° \\ \widehat{(\tilde{e}_1, \tilde{e}\,'_1)} \simeq -20.8° \end{cases}$$



The basis vectors associated with the time coordinate of each reference frame are tangent to the worldlines of the particles at rest. They are time-like and point to the future. The reference system is synchronous: $g_{0i}=0$ with $i=1, 2$ or 3.

2 - <u>Disk</u>: $\begin{cases} ct' = ct \\ \rho' = \rho \\ \theta' = \theta + \omega t \end{cases}$, $\Lambda^0{}_1 = \dfrac{\partial ct'}{\partial \rho} = 0,\;\; \Lambda^2{}_0 = \dfrac{\partial \theta'}{\partial ct} = \dfrac{\omega}{c}$ ...

then $\Lambda^{\mu}{}_{\nu}=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ \omega/c & 0 & 1\end{pmatrix}$ and $\begin{cases}\tilde{e}_0=\tilde{e}'_0+\frac{\omega}{c}\tilde{e}'_2\\ \tilde{e}_1=\tilde{e}'_1\\ \tilde{e}_2=\tilde{e}'_2\end{cases}$

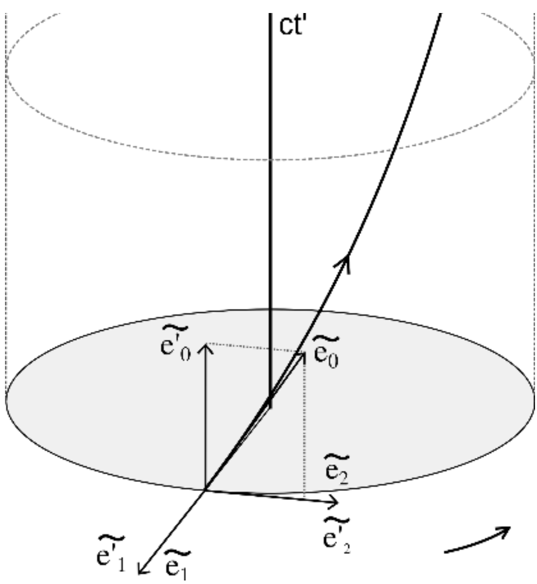
The basis is not orthogonal and the reference system is not synchronous because:

$$\tilde{e}_0\cdot\tilde{e}_2=g_{02}=\frac{\omega}{c}g'_{22}=-\frac{\rho^2\omega}{c}\neq 0$$

We find the components of the metric tensor:

$$\tilde{e}_0\cdot\tilde{e}_0=g'_{00}+2\frac{\omega}{c}g'_{02}+\frac{\omega^2}{c^2}g'_{22}=1-\frac{\rho^2\omega^2}{c^2}=g_{00}\ \ldots$$

In the Minkowski reference frame (ct, ρ, θ) of the inertial observer, the worldlines of the particles at rest in the rotating disk reference frame form ascending helices, with constant pitch and radius ρ. $\tilde{e}_0$ is tangent to the worldlines and oriented according to increasing proper time. Here, the vectors of the spatial basis $(\tilde{e}_{1,}\tilde{e}_{2,}\tilde{e}_3)$ are not all orthogonal to the worldlines of the particles at rest that define the disk.



2. Riemann curvature tensor *Exercise p243*

1 -a- Rocket: $g_{\mu\nu}$ is here diagonal, thus $g^{\mu\nu}g_{\nu\alpha}=\delta^{\mu}_{\ \alpha}$ becomes $g^{\mu\mu}g_{\mu\mu}=\delta^{\mu}_{\ \mu}=1$ and $g^{\mu\mu}=1/g_{\mu\mu}$.

$$g_{00}=g,\quad g_{11}=g_{22}=g_{33}=-1,\quad g^{00}=1/g,\quad g^{11}=g^{22}=g^{33}=-1.$$

b- The only non-zero derivative term that can appear in the connections is $\partial_1 g_{00}=g'$. Moreover, since the metric is diagonal, we have for the first factor $g^{\alpha\beta}=g^{\alpha\alpha}$. The only non-zero connections have two 0's and one 1 as indices. As there is symmetry on the last two indices there are only 2 possible cases :

$$\Gamma^{1}_{\ 00}=\frac{1}{2}g^{11}(\partial_0 g_{10}+\partial_0 g_{10}-\partial_1 g_{00})=-\frac{1}{2}g^{11}\partial_1 g_{00}=\frac{g'}{2}$$

$$\Gamma^{0}_{\ 01}=\Gamma^{0}_{\ 10}=\frac{1}{2}g^{00}(\partial_1 g_{00}+\partial_0 g_{01}-\partial_0 g_{01})=\frac{1}{2}g^{00}\partial_1 g_{00}=\frac{g'}{2g}$$

c- Antisymmetry: $R^{\alpha}_{\ \beta\gamma\gamma}=0$ and $R^{\alpha}_{\ \beta\gamma\delta}=-R^{\alpha}_{\ \beta\delta\gamma}$.

No indices 2 or 3, otherwise the tensor component is zero: Zero connection coefficient or zero derivative (no dependence in θ or *z*). Only indices 0 or 1.

It remains: $R^{0}_{\ 001}$, $R^{1}_{\ 101}$, $R^{1}_{\ 001}$ and $R^{0}_{\ 101}$.

$$R^{\alpha}_{\ \beta 01}=\Gamma^{\alpha}_{\ \beta 1,0}-\Gamma^{\alpha}_{\ \beta 0,1}+\Gamma^{\alpha}_{\ \sigma 0}\Gamma^{\sigma}_{\ \beta 1}-\Gamma^{\alpha}_{\ \sigma 1}\Gamma^{\sigma}_{\ \beta 0}$$

the first term is zero $\Gamma^{\alpha}_{\ \beta 1,0}=\frac{\partial}{\partial ct}\Gamma^{\alpha}_{\ \beta 1}=0$

$$R^{0}_{\ 001}=0-\Gamma^{0}_{\ 00,1}+\Gamma^{0}_{\ \sigma 0}\Gamma^{\sigma}_{\ 01}-\Gamma^{0}_{\ \sigma 1}\Gamma^{\sigma}_{\ 00}$$

the second term is zero,

3rd: $\Gamma^{0}_{\ \sigma 0}\Gamma^{\sigma}_{\ 01}=\Gamma^{0}_{\ 00}\Gamma^{0}_{\ 01}+\Gamma^{0}_{\ 10}\Gamma^{1}_{\ 01}+\Gamma^{0}_{\ 20}\Gamma^{2}_{\ 01}+\Gamma^{0}_{\ 30}\Gamma^{3}_{\ 01}$
$=0+0+0+0$,

last: $\Gamma^{0}_{\ \sigma 1}\Gamma^{\sigma}_{\ 00}=\Gamma^{0}_{\ 01}\Gamma^{0}_{\ 00}+\Gamma^{0}_{\ 11}\Gamma^{1}_{\ 00}=0$,

$$\Rightarrow \quad R^0{}_{001}=0.$$

$$R^1{}_{101}=0-\Gamma^1{}_{10,1}+\Gamma^1{}_{\sigma 0}\Gamma^\sigma{}_{11}-\Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{10}=0-0+*\times 0-0\times *$$

$$\Rightarrow \quad R^1{}_{101}=0.$$

$$R^1{}_{001}=0-\Gamma^1{}_{00,1}+\Gamma^1{}_{\sigma 0}\Gamma^\sigma{}_{01}-\Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{00}=-\frac{g''}{2}+\frac{g'}{2}\times\frac{g'}{2g}$$

$$R^1{}_{001}=\frac{1}{2}\left(-g''+\frac{g'^2}{2g}\right)=\frac{1}{2}\left(-\frac{2a^2}{c^4}+\frac{4a^2}{c^4}\frac{\left(1+\frac{ax}{c^2}\right)^2}{2\left(1+\frac{ax}{c^2}\right)^2}\right)$$

$$\Rightarrow \quad R^1{}_{001}=0.$$

$$R^0{}_{101}=0-\Gamma^0{}_{10,1}+\Gamma^0{}_{\sigma 0}\Gamma^\sigma{}_{11}-\Gamma^0{}_{\sigma 1}\Gamma^\sigma{}_{10}=-\frac{1}{2}\left(\frac{g''g-g'^2}{g^2}\right)+0-\frac{g'^2}{4g^2}$$

$$\Rightarrow \quad R^0{}_{101}=0.$$

Conclusion: in the reference frame of the uniformly accelerated rocket, the curvature tensor is identically zero because all its components are zero. This is logical, because if a tensor is identically zero in one reference frame, it is zero in all reference frames (whatever the changes of coordinates made). Indeed, we pass from the galactic inertial reference frame to the rocket reference frame by a change of coordinates (given in the exercise on the Rindler coordinate system), and, in an inertial reference frame the curvature is zero (all the components of the metric tensor are independent of the coordinates). Finally, the spacetime of the rocket is flat, which does not prevent the clocks from being out of sync with each other, and the photons from having curved trajectories. Let's say it !

2 -a Disk:

$$g_{00}=1-\frac{\rho^2\omega^2}{c^2},\ g_{02}=-\frac{\rho^2\omega}{c},\ g_{22}=-\rho^2 \text{ and } g_{11}=g_{33}=-1.$$

The metric is diagonal for 1 and 3: $g^{11}=g^{33}=-1.$

$$g^{\mu\nu}g_{\nu\alpha}=\delta^{\mu}{}_{\alpha} \qquad \begin{cases} g^{00}g_{00}+g^{02}g_{20}=\delta^0{}_0=1 \\ g^{00}g_{02}+g^{02}g_{22}=\delta^0{}_2=0 \end{cases} \qquad g^{20}g_{02}+g^{22}g_{22}=1$$

$$\begin{cases} g^{00}\left(1-\frac{\rho^2\omega^2}{c^2}\right)-g^{02}\frac{\rho^2\omega}{c}=1 \\ -g^{00}\frac{\rho^2\omega}{c}-g^{02}\rho^2=0 \end{cases} \qquad \begin{cases} g^{00}=1 \\ g^{02}=-\frac{\omega}{c} \end{cases} \qquad g^{22}=\frac{\omega^2}{c^2}-\frac{1}{\rho^2}$$

$$g^{\mu\nu}=\begin{pmatrix} 1 & 0 & -\frac{\omega}{c} & 0 \\ 0 & -1 & 0 & 0 \\ -\frac{\omega}{c} & 0 & \frac{\omega^2}{c^2}-\frac{1}{\rho^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

b- $\partial_1 g_{00}=-2\frac{\rho\omega^2}{c^2},\quad \partial_1 g_{02}=-2\frac{\rho\omega}{c}$ and $\partial_1 g_{22}=-2\rho.$

18 connections with 0, 1 and 2:

$$\Gamma^0{}_{\mu\nu}=\frac{1}{2}g^{00}(\partial_\nu g_{0\mu}+\partial_\mu g_{0\nu}-\partial_0 g_{\mu\nu})+\frac{1}{2}g^{02}(\partial_\nu g_{2\mu}+\partial_\mu g_{2\nu}-\partial_2 g_{\mu\nu})$$

$$\Gamma^0{}_{00}=0 \qquad \Gamma^0{}_{01}=\frac{1}{2}g^{00}\partial_1 g_{00}+\frac{1}{2}g^{02}\partial_1 g_{20}=-\frac{\rho\omega^2}{c^2}+\frac{\omega}{c}\frac{\rho\omega}{c}=0$$

$$\Gamma^0{}_{02}=0 \qquad \Gamma^0{}_{11}=0 \qquad \Gamma^0{}_{22}=0$$

$$\Gamma^0{}_{12}=\frac{1}{2}g^{00}\partial_1 g_{02}+\frac{1}{2}g^{02}\partial_1 g_{22}=-\frac{\rho\omega}{c}+\frac{\omega}{c}\rho=0$$

$$\Gamma^{1}{}_{\mu\nu}=\frac{1}{2}g^{11}(\partial_{\nu}g_{1\mu}+\partial_{\mu}g_{1\nu}-\partial_{1}g_{\mu\nu})\qquad \Gamma^{1}{}_{11}=0\qquad \Gamma^{1}{}_{12}=0$$

$$\Gamma^{1}{}_{00}=-\frac{1}{2}g^{11}\partial_{1}g_{00}=-\frac{\rho\omega^{2}}{c^{2}}\qquad \Gamma^{1}{}_{01}=0$$

$$\Gamma^{1}{}_{02}=-\frac{1}{2}g^{11}\partial_{1}g_{02}=-\frac{\rho\omega}{c}\qquad \Gamma^{1}{}_{22}=-\frac{1}{2}g^{11}\partial_{1}g_{22}=-\rho$$

$$\Gamma^{2}{}_{\mu\nu}=\frac{1}{2}g^{20}(\partial_{\nu}g_{0\mu}+\partial_{\mu}g_{0\nu}-\partial_{0}g_{\mu\nu})+\frac{1}{2}g^{22}(\partial_{\nu}g_{2\mu}+\partial_{\mu}g_{2\nu}-\partial_{2}g_{\mu\nu})$$

$$\Gamma^{2}{}_{00}=0\qquad \Gamma^{2}{}_{22}=0\qquad \Gamma^{2}{}_{11}=0$$

$$\Gamma^{2}{}_{10}=\frac{1}{2}g^{20}\partial_{1}g_{00}+\frac{1}{2}g^{22}\partial_{1}g_{20}=\frac{\rho\omega^{3}}{c^{3}}-\left(\frac{\omega^{2}}{c^{2}}-\frac{1}{\rho^{2}}\right)\frac{\rho\omega}{c}=\frac{\omega}{\rho c}$$

$$\Gamma^{2}{}_{12}=\frac{1}{2}g^{20}\partial_{1}g_{02}+\frac{1}{2}g^{22}\partial_{1}g_{22}=\frac{\rho\omega^{2}}{c^{2}}-\left(\frac{\omega^{2}}{c^{2}}-\frac{1}{\rho^{2}}\right)\rho=\frac{1}{\rho}$$

<u>Results</u>:

$$\Gamma^{1}{}_{00}=-\frac{\rho\omega^{2}}{c^{2}}\quad \Gamma^{1}{}_{02}=-\frac{\rho\omega}{c}\quad \Gamma^{1}{}_{22}=-\rho\quad \Gamma^{2}{}_{10}=\frac{\omega}{\rho c}\quad \Gamma^{2}{}_{12}=\frac{1}{\rho}$$

c- The tensor components with an index 3 are zero.
No connection with the first index zero: $R^{0}{}_{\beta\gamma\delta}=0$.

2x3x3=18 components to test.

$$R^{1}{}_{001}=0-\Gamma^{1}{}_{00,1}+\Gamma^{1}{}_{\sigma 0}\Gamma^{\sigma}{}_{01}-0=\frac{\omega^{2}}{c^{2}}-\rho\frac{\omega}{c}\frac{\omega}{\rho c}=0$$

$$R^{1}{}_{101}=0-\Gamma^{1}{}_{10,1}+\Gamma^{1}{}_{\sigma 0}\Gamma^{\sigma}{}_{11}-\Gamma^{1}{}_{\sigma 1}\Gamma^{\sigma}{}_{10}=0$$

$$R^{1}{}_{002}=0-0+\Gamma^{1}{}_{\sigma 0}\Gamma^{\sigma}{}_{02}-\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{00}=0$$

$$R^{1}{}_{202}=0-0+\Gamma^{1}{}_{\sigma 0}\Gamma^{\sigma}{}_{22}-\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{20}=0$$

$$R^{1}{}_{112}=\Gamma^{1}{}_{12,1}-0+\Gamma^{1}{}_{\sigma 1}\Gamma^{\sigma}{}_{12}-\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{11}=0$$

$$R^{1}{}_{212}=\Gamma^{1}{}_{22,1}-0+0-\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{21}=-1+\rho\frac{1}{\rho}=0$$

$$R^1{}_{120}=0-0+\Gamma^1{}_{\sigma 2}\Gamma^\sigma{}_{10}-\Gamma^1{}_{\sigma 0}\Gamma^\sigma{}_{12}=-\frac{\omega}{c}+\frac{\omega}{c}=0$$

$$R^1{}_{012}=\Gamma^1{}_{02,1}-0+\Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{02}-\Gamma^1{}_{\sigma 2}\Gamma^\sigma{}_{01}=-\frac{\omega}{c}+\frac{\omega}{c}=0$$

$$R^1{}_{210}=\Gamma^1{}_{20,1}-0+\Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{20}-\Gamma^1{}_{\sigma 0}\Gamma^\sigma{}_{21}=-\frac{\omega}{c}+\frac{\omega}{c}=0$$

$$R^2{}_{001}=0-\Gamma^2{}_{00,1}+\Gamma^2{}_{\sigma 0}\Gamma^\sigma{}_{01}-\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{00}=0$$

$$R^2{}_{101}=0-\Gamma^2{}_{10,1}+0-\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{10}=\frac{\omega}{\rho^2 c}-\frac{1}{\rho}\frac{\omega}{\rho c}=0$$

$$R^2{}_{002}=0-0+\Gamma^2{}_{\sigma 0}\Gamma^\sigma{}_{02}-\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{00}=-\frac{\omega}{\rho c}\frac{\rho\omega}{c}+\frac{1}{\rho}\frac{\rho\omega^2}{c^2}=0$$

$$R^2{}_{202}=0-0+\Gamma^2{}_{\sigma 0}\Gamma^\sigma{}_{22}-\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{20}=\ldots=0$$

$$R^2{}_{112}=\Gamma^2{}_{12,1}-0+\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{12}-\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{11}=\ldots=0$$

$$R^2{}_{212}=\Gamma^2{}_{22,1}-0+\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{22}-\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{21}=0$$

$$R^2{}_{120}=0-0+\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{10}-\Gamma^2{}_{\sigma 0}\Gamma^\sigma{}_{12}=0$$

$$R^2{}_{012}=\Gamma^2{}_{02,1}-0+\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{02}-\Gamma^2{}_{\sigma 2}\Gamma^\sigma{}_{01}=0$$

$$R^2{}_{210}=\Gamma^2{}_{20,1}-0+\Gamma^2{}_{\sigma 1}\Gamma^\sigma{}_{20}-\Gamma^2{}_{\sigma 0}\Gamma^\sigma{}_{21}=0$$

Conclusion: in the proper frame of reference of the uniformly rotating disk the space-time curvature is zero (as expected). In a next exercise, we will see that this is not the case for the spatial curvature.

2 -a Spherical body: metric diagonal

$$g_{00}=g=e^f,\quad g_{11}=-1/g=-e^{-f}\quad g_{22}=-r^2,\quad g_{33}=-r^2\sin^2\theta,$$

$$g^{00}=1/g=e^{-f},\quad g^{11}=-e^f,\quad g^{22}=-1/r^2,\quad g^{33}=-1/(r^2\sin^2\theta).$$

b- $g'=f'g$ and $g''=(f''+f'^2)g=-2r_S/r^3$ . ... .

Connections: $\Gamma^0{}_{10}=f'/2$ $\quad\Gamma^1{}_{11}=-f'/2$

$$\Gamma^1{}_{00}=e^{2f}f'/2\qquad \Gamma^2{}_{21}=\Gamma^3{}_{31}=1/r\qquad \Gamma^1{}_{22}=-rg$$

$$\Gamma^1{}_{33}=-rg\sin^2\theta\qquad \Gamma^2{}_{33}=-\sin\theta\cos\theta\qquad \Gamma^3{}_{32}=1/\tan\theta$$

c- $R^0{}_{101}=\Gamma^0{}_{11,0}-\Gamma^0{}_{10,1}+\Gamma^0{}_{\sigma 0}\Gamma^\sigma{}_{11}-\Gamma^0{}_{\sigma 1}\Gamma^\sigma{}_{10}$

$$R^0{}_{101}=0-f''/2-(f'/2)^2-(f'/2)^2=-(f''+f'^2)/2=-g''/2g$$

$$R^0{}_{101}=\frac{r_S}{gr^3} \quad \text{and} \quad R_{0101}=g_{0\mu}R^\mu{}_{101}=g_{00}R^0{}_{101}=\frac{r_S}{r^3}$$

Conclusion: there is a non-zero component of the curvature tensor, so spacetime is curved for a spherical body. Curvature is an intrinsic property of every spacetime. The space-time described by the Schwarzschild metric will be curved whatever the reference frame of observation.
Nevertheless, in case of non-nullity of the set of components, the expression of the components of a tensor depends on the coordinate system. We can obtain an invariant quantity by forming a scalar. We show that:

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}=12r_S^2/r^6 \quad \text{(Kretschmann scalar).}$$

We see that the singularity in r_S does not appear, on the other hand, the singularity in r=0 is visible. The central singularity is essential because it is present in all observation frames of reference.

3.1- A non-uniformly rotating Disk *Exercise p245*

$$g_{\mu\nu}=\begin{pmatrix} 1-\rho^2\frac{\dot\lambda^2}{c^2} & 0 & -\rho^2\frac{\dot\lambda}{c} \\ 0 & -1 & 0 \\ -\rho^2\frac{\dot\lambda}{c} & 0 & -\rho^2 \end{pmatrix} \qquad g^{\mu\nu}=\begin{pmatrix} 1 & 0 & -\frac{\dot\lambda}{c} \\ 0 & -1 & 0 \\ -\frac{\dot\lambda}{c} & 0 & \frac{\dot\lambda^2}{c^2}-\frac{1}{\rho^2} \end{pmatrix}$$

Always $\partial_1 g_{00}=-2\frac{\rho\dot\lambda^2}{c^2}$, $\partial_1 g_{02}=-2\frac{\rho\dot\lambda}{c}$ and $\partial_1 g_{22}=-2\rho$.

To which we add: $\partial_0 g_{00} = -2\rho^2 \frac{\dot{\lambda}\ddot{\lambda}}{c^2}$ and $\partial_0 g_{02} = -\rho^2 \frac{\ddot{\lambda}}{c}$.

We check, one by one, the previously calculated connections and we notice that they are not modified:

$$\Gamma^1{}_{00} = -\frac{\rho\dot{\lambda}^2}{c^2} \quad \Gamma^1{}_{02} = -\frac{\rho\dot{\lambda}}{c} \quad \Gamma^1{}_{22} = -\rho \quad \Gamma^2{}_{10} = \frac{\dot{\lambda}}{\rho c} \quad \Gamma^2{}_{12} = \frac{1}{\rho}$$

$$\Gamma^0{}_{00} = \frac{1}{2} g^{00} (\partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00}) + \frac{1}{2} g^{02} (\partial_0 g_{20} + \partial_0 g_{20} - \partial_2 g_{00})$$

$$\Gamma^0{}_{00} = \frac{1}{2} g^{00} \partial_0 g_{00} + g^{02} \partial_0 g_{20} = -\rho^2 \frac{\dot{\lambda}\ddot{\lambda}}{c^2} + \frac{\dot{\lambda}}{c} \rho^2 \frac{\ddot{\lambda}}{c} = 0 \quad \dots \dots$$

$$\Gamma^2{}_{00} = \frac{1}{2} g^{20} \partial_0 g_{00} + g^{22} \partial_0 g_{20} = \frac{\dot{\lambda}}{c} \rho^2 \frac{\dot{\lambda}\ddot{\lambda}}{c^2} - \left(\frac{\dot{\lambda}^2}{c^2} - \frac{1}{\rho^2} \right) \rho^2 \frac{\ddot{\lambda}}{c} = \frac{\ddot{\lambda}}{c}$$

... ... only one new non-zero connection: $\Gamma^2{}_{00} = \frac{\ddot{\lambda}}{c}$.

2 - Similarly, we do not transcribe here all the calculations, but all the components of the curvature tensor are indeed zero.

3 - Whatever the change of coordinates, if all the components of a tensor are zero in a reference frame *R*, they are zero in all reference frames *R'*:

$$R'^{\alpha}{}_{\beta\gamma\delta} = \Lambda^{\alpha}{}_{\mu} \Lambda_{\beta}{}^{\nu} \Lambda_{\gamma}{}^{\rho} \Lambda_{\delta}{}^{\lambda} R^{\mu}{}_{\nu\rho\lambda} \qquad \Gamma'^{\alpha}{}_{\beta\gamma} \neq \Lambda^{\alpha}{}_{\mu} \Lambda_{\beta}{}^{\nu} \Lambda_{\gamma}{}^{\rho} \Gamma^{\mu}{}_{\nu\rho}$$

So as expected the tensor is null, since we start from a Minkowskian frame of reference. We take this opportunity to point out that not every object with indices is a tensor. For example, the connection is not a tensor. It is null for all its components in Cartesian coordinates in *R*, and non-null in *R'* in polar coordinates.

4. Spatial cuvatures *Exercise p246*

1 - <u>Rocket</u>: The reference system is synchronous – no cross terms in the metric between t and the space coordinates x, y and z. Space is disjoint from time, and we directly recognize the Euclidean metric:

$$dl^2 = dx^2 + dy^2 + dz^2$$

Flat space.

2 - <u>Spherical body</u>: Here again, the system is synchronous.

Spatial metric: $d l^2 = \frac{1}{g} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$.

$$\gamma_{ij} = \begin{pmatrix} 1/g & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad \gamma^{ij} = \begin{pmatrix} g & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1/(r^2\sin^2\theta) \end{pmatrix}$$

Calculation: $R^1{}_{212} = \Gamma^1{}_{22,1} - \Gamma^1{}_{21,2} + \Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{22} - \Gamma^1{}_{\sigma 2}\Gamma^\sigma{}_{21}$

$$\Gamma^1{}_{ij} = \frac{1}{2}\gamma^{11}(\partial_j \gamma_{1i} + \partial_i \gamma_{1j} - \partial_1 \gamma_{ij}) \qquad \Gamma^1{}_{22} = -\frac{1}{2} g \partial_1 \gamma_{22} = -rg$$

$$\Gamma^1{}_{11} = \frac{1}{2} g \partial_1 \gamma_{11} = -\frac{g'}{2g} \qquad \Gamma^2{}_{22} = 0 \qquad \Gamma^3{}_{22} = 0 \qquad \Gamma^1{}_{12} = 0$$

$$\Gamma^2{}_{21} = \frac{1}{2}\gamma^{22}\partial_1 \gamma_{22} = \frac{1}{2r^2} \times 2r = \frac{1}{r} \qquad \Gamma^3{}_{21} = 0$$

$$\Rightarrow \quad R^1{}_{212} = -g - rg' - 0 + rg'/2 + g = -rg'/2 = -\frac{r_S}{2r} \neq 0.$$

Curved space.

3 -a- <u>Disk</u>: The reference system is not synchronous: $g_{02} \neq 0$

b- $\gamma_{11} = -g_{11} = 1 \quad \gamma_{12} = \frac{g_{01} g_{02}}{g_{00}} = 0 \quad \gamma_{13} = 0 \quad \gamma_{23} = 0 \quad \gamma_{33} = 1$

$$\gamma_{22}=-g_{22}+\frac{g_{02}^2}{g_{00}}=\rho^2+\frac{\frac{\rho^4\omega^2}{c^2}}{1-\frac{\rho^2\omega^2}{c^2}}=\gamma^2\rho^2$$

with the Lorentz factor: $\gamma=\frac{1}{\sqrt{1-\frac{\rho^2\omega^2}{c^2}}}=\frac{1}{\sqrt{1-\beta^2}}$

Spatial metric: $\gamma_{ij}=\begin{pmatrix}1 & 0 & 0\\ 0 & \gamma^2\rho^2 & 0\\ 0 & 0 & 1\end{pmatrix}$

c- The reference system is stationary.

$$\frac{P}{D}=\frac{\int_{\theta=0}^{\theta=2\pi}\sqrt{\gamma_{22}}\,d\theta}{2\int_{\rho=0}^{\rho}\sqrt{\gamma_{11}}\,d\rho}=\frac{\gamma\rho\int_{\theta=0}^{\theta=2\pi}d\theta}{2\int_{\rho=0}^{\rho}d\rho}=\gamma\pi \quad \text{(at t fixed } d\theta=d\theta'\text{)}$$

$\frac{P}{D}>\pi$: the space is non-Euclidean, it is curved.

We find again the intuition of Albert Einstein. Because of the contraction of the lengths due to the speed on the edge of the disk, the experimenter on the disk must transfer his unit ruler more times than the inertial experimenter to measure the perimeter. On the other hand, there is no contraction along the radius and the two measurements are in this case equal. Ehrenfest's "paradox" is solved.

Curved space.

d- $$R^1{}_{212}=\Gamma^1{}_{22,1}-\Gamma^1{}_{21,2}+\Gamma^1{}_{\sigma 1}\Gamma^\sigma{}_{22}-\Gamma^1{}_{\sigma 2}\Gamma^\sigma{}_{21}$$

$$\gamma'=\frac{1}{\rho}\beta^2\gamma^3 \qquad \partial_1\gamma_{22}=2\gamma^2\rho+2\gamma\gamma'\rho^2=2\rho\gamma^4$$

$$\Gamma^1{}_{ij}=\frac{1}{2}\gamma^{11}(\partial_j\gamma_{1i}+\partial_i\gamma_{1j}-\partial_1\gamma_{ij})\quad \Gamma^1{}_{11}=0\quad \Gamma^2{}_{22}=0\quad \Gamma^3{}_{22}=0$$

$$\Gamma^1{}_{22}=-\frac{1}{2}\gamma^{11}\partial_1\gamma_{22}=-\rho\gamma^4\qquad \Gamma^1{}_{12}=0$$

$$\Gamma^2{}_{21}=\frac{1}{2}\gamma^{22}\partial_1\gamma_{22}=\frac{\gamma^2}{\rho}\qquad \Gamma^3{}_{12}=0$$

$$R^1{}_{212}=-\gamma^4-4\rho\gamma^3\frac{1}{\rho}\beta^2\gamma^3+\gamma^6=-\gamma^4(1+4\beta^2\gamma^2-\gamma^2)=-3\beta^2\gamma^6$$

It is the only non-zero component taking into account the symmetries. The curvature tensor is zero at the rotation axis.

e- $$R_{1212}=\gamma_{11}R^1{}_{212}=-3\beta^2\gamma^6$$

$$K=\frac{1}{R_1R_2}=\frac{R_{1212}}{\gamma_{11}\gamma_{22}}=-3\frac{\beta^2\gamma^4}{\rho^2}=-3\frac{\omega^2}{c^2}\gamma^4<0$$

The radii of curvature are therefore of opposite signs (as on a mountain col, a horse's saddle or the inside of a torus). The curvature increases away from the axis of rotation as γ^4 (in absolute value).

On the surface of a sphere, the curvatures on two perpendicular directions, meanwhile, have the same signs:

$$dl^2=g(x,y)\,dx^2+g(x,y)\,dy^2\qquad g(x,y)=\frac{1}{\left(1+\frac{x^2+y^2}{4R^2}\right)^2}=\frac{1}{h^2}$$

$$\gamma_{ij}=\begin{pmatrix} g & 0\\ 0 & g\end{pmatrix}\quad \partial_1\gamma_{11}=\partial_1\gamma_{22}=-\frac{x}{R^2h^3}\quad \partial_2\gamma_{11}=\partial_2\gamma_{22}=-\frac{y}{R^2h^3}$$

$$\Gamma^1{}_{11}=-\frac{x}{2R^2h}\quad \Gamma^2{}_{22}=-\frac{y}{2R^2h}\quad \Gamma^1{}_{22}=\frac{x}{2R^2h}\quad \Gamma^2{}_{11}=\frac{y}{2R^2h}$$

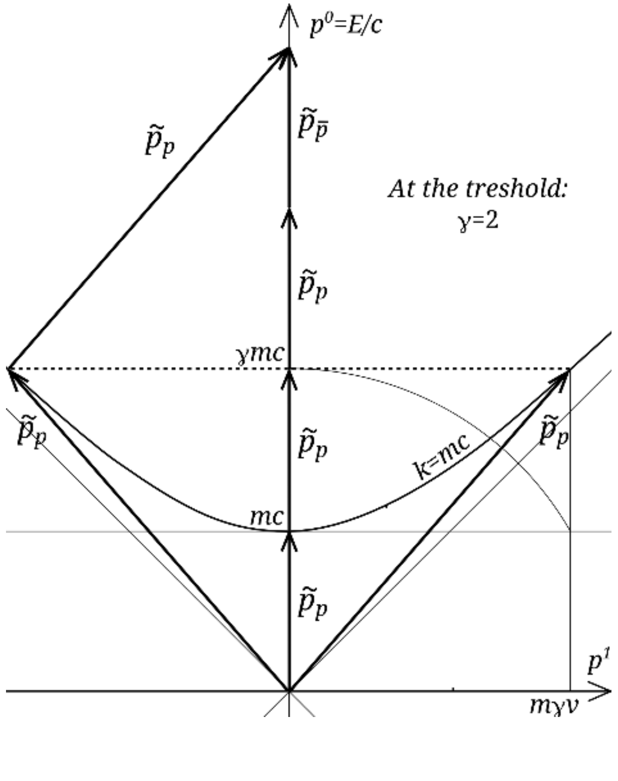
$$\Gamma^1{}_{12}=-\frac{y}{2R^2h}\qquad \Gamma^2{}_{12}=-\frac{x}{2R^2h}$$

$$R^{1}{}_{212}=\Gamma^{1}{}_{22,1}-\Gamma^{1}{}_{21,2}+\Gamma^{1}{}_{\sigma 1}\Gamma^{\sigma}{}_{22}-\Gamma^{1}{}_{\sigma 2}\Gamma^{\sigma}{}_{21}=\frac{1}{R^2 h^2}$$

$$R_{1212}=\gamma_{11}R^{1}{}_{212}=\frac{1}{R^2 h^4} \qquad K=\frac{1}{R_1 R_2}=\frac{1}{R^2}>0$$

5. Pair production

Exercise p248.



6. Wave equation

Exercise p249.

1 -

$$d\varphi'=\frac{\partial\varphi'}{\partial x'}dx'+\frac{\partial\varphi'}{\partial t'}dt'=\frac{\partial\varphi'}{\partial x'}(dx-v\,dt)+\frac{\partial\varphi'}{\partial t'}dt$$

$$d\varphi'=\left[\frac{\partial}{\partial x'}dx+\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)dt\right]\varphi'$$

$$d\varphi=\left[\frac{\partial}{\partial x}dx+\frac{\partial}{\partial t}dt\right]\varphi$$

then $\dfrac{\partial}{\partial x}=\dfrac{\partial}{\partial x'}$, $\dfrac{\partial}{\partial t}=\dfrac{\partial}{\partial t'}-v\dfrac{\partial}{\partial x'}$ and $\varphi=\varphi'$.

$$\frac{\partial^2\varphi}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2}=\frac{\partial^2\varphi'}{\partial x'^2}-\frac{1}{c^2}\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)^2\varphi'=0$$

$$\left(1-\frac{v^2}{c^2}\right)\frac{\partial^2\varphi'}{\partial x'^2}+\frac{2v}{c^2}\frac{\partial^2\varphi'}{\partial t'\partial x'}-\frac{1}{c^2}\frac{\partial^2\varphi'}{\partial t'^2}=0$$

This equation differs completely from the d'Alembert equation. The propagation equation is established in the reference frame where the propagation medium is at rest. For example, the d'Alembert equation of the sound wave is valid in the reference frame where the relative wind is zero.

2 - $$d\ \ =\frac{\partial}{\partial x}dx+\frac{\partial}{\partial ct}dct=\frac{\partial}{\partial x'}dx'+\frac{\partial}{\partial ct'}dct'$$

$$d\ \ =\frac{\partial}{\partial x'}\gamma(dx-\beta\, dct)+\frac{\partial}{\partial t'}\gamma(dct-\beta\, dx)$$

$$d\ \ =\gamma\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial t'}\right)dx+\gamma\left(\frac{\partial}{\partial ct'}-\beta\frac{\partial}{\partial x'}\right)dct$$

$$\frac{\partial}{\partial ct}=\gamma\left(\frac{\partial}{\partial ct'}-\beta\frac{\partial}{\partial x'}\right)\qquad\frac{\partial}{\partial x}=\gamma\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)$$

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial y'}\qquad\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}$$

$$\square E_x=0$$

$$\left[\gamma^2\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)^2+\frac{\partial^2}{\partial y'^2}+\frac{\partial^2}{\partial z'^2}-\gamma^2\left(\frac{\partial}{\partial ct'}-\beta\frac{\partial}{\partial x'}\right)^2\right]E_{x'}=0$$

$\gamma^2(1-\beta^2)=1$ and double products are eliminated:

$$\square' E_{x'}=0.$$

For the components along *y* and *z*:

$$\Box E_y = 0 \text{ and } \Box B_z = 0$$

$$\left[\gamma^2\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)^2-\gamma^2\left(\frac{\partial}{\partial ct'}-\beta\frac{\partial}{\partial x'}\right)^2\right]\gamma(E_{y'}+\beta c B_{z'})=0$$

$$\left[\gamma^2\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)^2-\gamma^2\left(\frac{\partial}{\partial ct'}-\beta\frac{\partial}{\partial x'}\right)^2\right]\gamma(B_{z'}+\beta E_{y'}/c)=0$$

(1)-βc(2) gives $\Box' E_{y'}=0$ and so on for the six equations. In this case we have invariance of the wave equation, the speed of light in vacuum is the same in all reference frames of inertia. The aether, the supposed medium for the propagation of light, does not exist.

7. Schrödinger equation *Exercise p250.*

1 - Let us start with the Schrödinger equation in *R'* and show that it is always verified in *R*:

$$i\hbar\frac{\partial\Psi'}{\partial t'}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi'}{\partial x'^2} \text{ with } \frac{\partial}{\partial t'}=\frac{\partial}{\partial t}+v\frac{\partial}{\partial x} \;\&\; \frac{\partial}{\partial x'}=\frac{\partial}{\partial x}$$

$$\frac{\partial\Psi'}{\partial t}=\left(\frac{iE}{\hbar}\Psi+\frac{\partial\Psi}{\partial t}\right)e^{\frac{i}{\hbar}(Et-px)}$$

$$\frac{\partial\Psi'}{\partial x}=\left(-\frac{ip}{\hbar}\Psi+\frac{\partial\Psi}{\partial x}\right)e^{\frac{i}{\hbar}(Et-px)}$$

then $i\hbar\left[\left(\frac{iE}{\hbar}\Psi+\frac{\partial\Psi}{\partial t}\right)+v\left(-\frac{ip}{\hbar}\Psi+\frac{\partial\Psi}{\partial x}\right)\right]$

$$=-\frac{\hbar^2}{2m}\left[\left(-\frac{ip}{\hbar}\frac{\partial\Psi}{\partial x}+\frac{\partial^2\Psi}{\partial x^2}\right)-\frac{ip}{\hbar}\left(-\frac{ip}{\hbar}\Psi+\frac{\partial\Psi}{\partial x}\right)\right]$$

and $-E\Psi+i\hbar\frac{\partial\Psi}{\partial t}+vp\Psi+i\hbar v\frac{\partial\Psi}{\partial x}$

$$=\frac{1}{2}i\hbar v\frac{\partial\Psi}{\partial x}-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}+\frac{p^2}{2m}\Psi+\frac{1}{2}i\hbar v\frac{\partial\Psi}{\partial x}$$

Eventually we well have: $i\hbar\frac{\partial\Psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2}$.

2 - Lorentz transformation of the coordinates:

$$i\hbar\gamma\left(\frac{\partial}{\partial t}+c\beta\frac{\partial}{\partial x}\right)\Psi'=-\frac{\hbar^2}{2m}\gamma^2\left(\frac{\partial}{\partial x}+\beta\frac{\partial}{\partial ct}\right)^2\Psi'$$

In the left member we have only first derivatives, and in the right member we will have only one term with a second time derivative, which cannot cancel, whatever the choice for Ψ'. The Schrödinger equation does not work for relativistic particles.

8. The electromagnetic field

Exercise p252.

1 - Temporal component:

$$\frac{dp^0}{d\tau}=F^{0\nu}j_\nu=F^{00}j_0+F^{01}j_1+F^{02}j_2+F^{03}j_3$$

$$\tilde{j}=j^\mu=(q\gamma c,q\vec{u})\quad j_0=j^0\quad j_i=-j^i$$

$$\frac{dE/c}{d\tau}=0+\left(-\frac{E_x}{c}\right)(-qu^x)+\frac{E_y}{c}qu^y+qE_zu^z/c$$

$$\frac{dE}{d\tau}=q\vec{E}\cdot\vec{u}\quad\text{or}\quad\frac{dE}{dt}=q\vec{E}\cdot\vec{v}$$

We find the power of the electric force. The magnetic force does not work.

Spatial components: $\frac{dp^i}{d\tau}=F^{i\nu}j_\nu$

$$\frac{dp^1}{d\tau}=F^{10}j_0+F^{11}j_1+F^{12}j_2+F^{13}j_3=q\gamma E_x-B_z(-qu^y)-B_yqu^z$$

$$\vec{g}=\frac{d\vec{p}}{d\tau}=\gamma q\vec{E}+q\vec{u}\wedge\vec{B}=\gamma q(\vec{E}+\vec{v}\wedge\vec{B})$$

2 - $$F'^{\mu\nu}=\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

- $$\frac{E'_x}{c}=F'^{10}=\Lambda^{1}{}_{1}\Lambda^{0}{}_{0}F^{10}+\Lambda^{1}{}_{0}\Lambda^{0}{}_{1}F^{01}$$

(only non-zero components)

$$\frac{E'_x}{c}=\gamma^2\frac{E_x}{c}+\beta^2\gamma^2\left(-\frac{E_x}{c}\right) \quad \text{then} \quad E'_x=E_x$$

- $$\frac{E'_y}{c}=F'^{20}=\Lambda^{2}{}_{2}\Lambda^{0}{}_{0}F^{20}+\Lambda^{2}{}_{2}\Lambda^{0}{}_{1}F^{21}$$

$$\frac{E'_y}{c}=\gamma\frac{E_y}{c}-\beta\gamma B_z \quad \text{then} \quad E'_y=\gamma(E_y-vB_z)$$

- $$\frac{E'_z}{c}=F'^{30}=\Lambda^{3}{}_{3}\Lambda^{0}{}_{0}F^{30}+\Lambda^{3}{}_{3}\Lambda^{0}{}_{1}F^{31}$$

$$\frac{E'_z}{c}=\gamma\frac{E_z}{c}-\beta\gamma(-B_y) \quad \text{then} \quad E'_z=\gamma(E_z+vB_y)$$

- $$B'_x=F'^{32}=\Lambda^{3}{}_{3}\Lambda^{2}{}_{2}F^{32} \quad \text{then} \quad B'_x=B_x$$

- $$B'_y=F'^{13}=\Lambda^{1}{}_{0}\Lambda^{3}{}_{3}F^{03}+\Lambda^{1}{}_{1}\Lambda^{3}{}_{3}F^{13}$$

$$B'_y=-\beta\gamma\left(-\frac{E_z}{c}\right)+\gamma B_y \quad \text{then} \quad B'_y=\gamma\left(B_y+\beta\frac{E_z}{c}\right)$$

- $$B'_z=F'^{21}=\Lambda^{2}{}_{2}\Lambda^{1}{}_{0}F^{20}+\Lambda^{2}{}_{2}\Lambda^{1}{}_{1}F^{21}$$

$$B'_z=-\beta\gamma\frac{E_y}{c}+\gamma B_z \quad \text{then} \quad B'_z=\gamma\left(B_z-\beta\frac{E_y}{c}\right)$$

The transformations of $\vec{E}$ and $\vec{B}$ are very different from the Lorentz transformation.

Transformation of the electromagnetic field	$$\begin{cases} E'_x = E_x \\ E'_y = \gamma\left(E_y - \beta c B_z\right) \\ E'_z = \gamma\left(E_z + \beta c B_y\right) \\ B'_x = B_x \\ B'_y = \gamma\left(B_y + \beta E_z / c\right) \\ B'_z = \gamma\left(B_z - \beta E_y / c\right) \end{cases}$$

3 - $F_{\mu\nu} = g_{\mu\alpha} g_{\nu\beta} F^{\alpha\beta}$. Only the diagonal terms of the metric are nonzero, hence: $F_{\mu\nu} = g_{\mu\mu} g_{\nu\nu} F^{\mu\nu}$. They will therefore differ, at most, by one sign. The tensor remains antisymmetric and the diagonal elements zero. The magnetic block (3x3 submatrix) remains the same:

$$F_{ij} = g_{ii} g_{jj} F^{ij} = (-1)(-1) F^{ij} = F^{ij}$$

The electric blocks change sign:

$$F_{0j} = g_{00} g_{jj} F^{0j} = (+1)(-1) F^{0j} = -F^{0j}$$

$$\boldsymbol{F} = F_{\mu\nu} = \begin{pmatrix} 0 & \frac{E_x}{c} & \frac{E_y}{c} & \frac{E_z}{c} \\ -\frac{E_x}{c} & 0 & -B_z & B_y \\ -\frac{E_y}{c} & B_z & 0 & -B_x \\ -\frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

4 - • Invariant $F^{\mu\nu}F_{\mu\nu}$: 16 components including 4 nulls, remaining 12 and 2 groups of 6 alike.

$F^{\mu\nu}F_{\mu\nu}=$

$$F^{01}F_{01}+F^{10}F_{10}+F^{02}F_{02}+F^{20}F_{20}+F^{03}F_{03}+F^{20}F_{30}$$

$$+F^{12}F_{12}+F^{21}F_{21}+F^{13}F_{13}+F^{31}F_{31}+F^{23}F_{23}+F^{32}F_{32}$$

$$F^{\mu\nu}F_{\mu\nu}=-\frac{{E_x}^2}{c^2}\times 2-2\frac{{E_y}^2}{c^2}-2\frac{{E_z}^2}{c^2}+2\left({B_x}^2+{B_y}^2+{B_z}^2\right)$$

Invariant: $\vec{B}^2-\frac{\vec{E}^2}{c^2}$.

• Invariant $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$: the tensor $\epsilon^{\mu\nu\alpha\beta}$ has 4^4 components of witch 4!=24 non-zero.

$$\begin{aligned}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}= \\ &\epsilon^{0123}F_{01}F_{23}+\epsilon^{0132}F_{01}F_{32}+\epsilon^{0213}F_{02}F_{13} \\ &+\epsilon^{0231}F_{02}F_{31}+\epsilon^{0312}F_{03}F_{12}+\epsilon^{0321}F_{03}F_{21} \\ &+\epsilon^{1023}F_{10}F_{23}+\epsilon^{1032}F_{10}F_{32}+\epsilon^{1203}F_{12}F_{03} \\ &+\epsilon^{1230}F_{12}F_{30}+\epsilon^{1302}F_{13}F_{02}+\epsilon^{1320}F_{13}F_{20} \\ &+\epsilon^{2103}F_{21}F_{03}+\epsilon^{2130}F_{21}F_{30}+\epsilon^{2013}F_{20}F_{13} \\ &+\epsilon^{2031}F_{20}F_{31}+\epsilon^{2310}F_{23}F_{10}+\epsilon^{2301}F_{23}F_{01} \\ &+\epsilon^{3120}F_{31}F_{20}+\epsilon^{3102}F_{31}F_{02}+\epsilon^{3210}F_{32}F_{10} \\ &+\epsilon^{3201}F_{32}F_{01}+\epsilon^{3012}F_{30}F_{12}+\epsilon^{3021}F_{30}F_{21}\end{aligned}$$

Each component appears eight times: the antisymmetries on each $F_{\mu\nu}$ compensated in sign by the $\epsilon^{\mu\nu\alpha\beta}$, and the interversion of the two $F_{\mu\nu}$ which corresponds to two permutations in the $\epsilon^{\mu\nu\alpha\beta}$. There are thus only three types of components:

$$\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}=8\left(-E_xB_x-E_yB_y-E_zB_z\right)/c$$

Invariant: $\vec{E}\cdot\vec{B}$.

Lorentz invariants	$\vec{B}^2 - \frac{\vec{E}^2}{c^2}$ $\qquad$ $\vec{E} \cdot \vec{B}$

5 -a- The correction is succinct because the solution is the one given in an electrostatics course.
For an infinite plate of surface density σ we have the electric field at the intersection of two planes of symmetry of the charge distribution and hence along *z*. We choose the usual Gaussian surface to find the field. We apply the principle of superposition with a second plane of opposite charge distant from e. Conclusion: the electric field is zero outside the plates and is equal inside:

$$\vec{E} = -\frac{\sigma}{\epsilon_0} \vec{u}_z$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & \sigma/\epsilon_0 c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\sigma/\epsilon_0 c & 0 & 0 & 0 \end{pmatrix}$$

(zero tensor outside the plates)

b- Appears in *R'* the surface current density $\vec{j}_S = -\sigma \vec{v}$ on the upper plane. The magnetic field is along *y* because it is perpendicular to the plane of symmetry of the current distribution. We first apply Ampere's theorem to the upper plane only. With the usual rectangular contour we find the field. Using the principle of superposition we find a zero magnetic field outside the plates, and inside we have:

$$\vec{B}' = -\mu_0 \sigma v \vec{u}_y$$

$$F'^{\mu\nu}=\begin{pmatrix} 0 & 0 & 0 & \sigma/\epsilon_0 c \\ 0 & 0 & 0 & -\mu_0\sigma v \\ 0 & 0 & 0 & 0 \\ -\sigma/\epsilon_0 c & \mu_0\sigma v & 0 & 0 \end{pmatrix}$$

(zero tensor outside the plates)

c- We use the field transformation formulas that we have previously established:

$$E'_z=\gamma E_z \quad \text{and} \quad B'_y=\gamma\beta\frac{E_z}{c}=\gamma v\frac{\sigma}{\epsilon_0 c^2}=-\gamma\mu_0\sigma v$$

$$F'^{\mu\nu}=\begin{pmatrix} 0 & 0 & 0 & \gamma\sigma/\epsilon_0 c \\ 0 & 0 & 0 & -\gamma\mu_0\sigma v \\ 0 & 0 & 0 & 0 \\ -\gamma\sigma/\epsilon_0 c & \gamma\mu_0\sigma v & 0 & 0 \end{pmatrix}$$

The result differs by a factor γ. For a relativistic observer of *R'* the lengths of *R* are contracted by a factor γ along *x*. Thus the surface elements, which contain charges at rest, are contracted and the surface density is multiplied by γ. Similarly for the current density. This explains the expression of the tensor: $\sigma'=\gamma\sigma$.

Lorentz invariants:

$$\vec{B}^2-\frac{\vec{E}^2}{c^2}=0-\left(\frac{\sigma}{\epsilon_0 c}\right)^2=-\frac{\sigma^2}{\epsilon_0^2 c^2}$$

$$\vec{B}'^2-\frac{\vec{E}'^2}{c^2}=\left(\gamma\beta\frac{\sigma}{\epsilon_0 c}\right)^2-\left(\gamma\frac{\sigma}{\epsilon_0 c}\right)^2=-\frac{\sigma^2}{\epsilon_0^2 c^2}\gamma^2(1-\beta^2)=-\frac{\sigma^2}{\epsilon_0^2 c^2}$$

For the second invariant, it is zero in *R* because the magnetic field is zero, and it is zero in *R'* because the fields are orthogonal.

6- In the reference frame *R'* where the charges are at rest, the charge volume density is lower by a factor γ compared to that shown in the lab reference frame where the charges are moving. We apply Gauss' theorem. The electric field in *R'* has the expression outside the charge distribution:

$$\vec{E}\,'=-\frac{n e r^2}{2\gamma\rho\epsilon_0}\vec{u}_\rho \qquad (q=-e) \quad \text{and} \quad \vec{B}\,'=\vec{0}.$$

In the laboratory reference frame *R :* $\vec{E}=-\frac{n e r^2}{2\rho\epsilon_0}\vec{u}_\rho$.

In this frame the charges are in motion and an orthoradial magnetic field appears. $\vec{v}=v\,\vec{u}_z$ and the current is along $-\vec{u}_z$. The magnetic field therefore rotates in the retrograde direction. For the norm of the field we use a Lorentz invariant:

$$\vec{B}^2-\frac{\vec{E}^2}{c^2}=\vec{B}\,'^2-\frac{\vec{E}\,'^2}{c^2}$$

$$\vec{B}^2=\frac{\vec{E}^2}{c^2}-\frac{\vec{E}\,'^2}{c^2}=\left(\frac{n e r^2}{2\rho\epsilon_0 c}\right)^2\left(1-\frac{1}{\gamma^2}\right)=\left(\frac{n e v r^2}{2\gamma\rho\epsilon_0 c^2}\right)^2$$

Finally: $\vec{B}=-\frac{\mu_0 n e v r^2}{2\gamma\rho}\vec{u}_\theta$

9. Maxwell's equation *Exercise p255.*

1 -a- Newton's second law: $m\vec{a}=\vec{F}$. In classical mechanics, mass and force are invariant: $m=m'$ and $\vec{F}=\vec{F}\,'$. The acceleration also does not change, because, for a Galilean transformation, *R'* is in uniform rectilinear translation with respect to *R*: $\vec{v}_{R'/R}=\vec{cst}$ and $\vec{a}=\vec{a}\,'$. Thus, in the new reference frame of inertia *R'*, Newton's law is also verified: the force exerted on an object gives it an acceleration equal to the force vector divided by the object's mass. Note that this is not the case in a non-galilean frame of reference, where, for example, an object can set itself in motion without any forces being exerted on it (through what are sometimes called fictitious forces of inertia).

b- The magnetic force is not invariant under the Galilean transformation: $\vec{v} \neq \vec{v}\,'$. We then consider the sum of the electric and magnetic forces, called the Lorentz force, which we believe to be Galilean invariant. In return, the fields depend on the reference frame:

$$R: \ \vec{F} = q\vec{E} + q\vec{v} \wedge \vec{B} \qquad\qquad R': \ \vec{F}\,' = q\vec{E}\,' + q\vec{v}\,' \wedge \vec{B}\,'$$

but $\vec{v} = \vec{v}_e + \vec{v}\,'$ and $\vec{F} = \vec{F}\,'$, then:

$$\vec{E}\,' + \vec{v}\,' \wedge \vec{B}\,' = \vec{E} + (\vec{v}_e + \vec{v}\,') \wedge \vec{B} = (\vec{E} + \vec{v}_e \wedge \vec{B}) + \vec{v}\,' \wedge (\vec{B})$$

$$\begin{cases} \vec{E}\,' = \vec{E} + \vec{v}_e \wedge \vec{B} \\ \vec{B}\,' = \vec{B} \end{cases} \quad \text{and} \quad \begin{cases} E'_x = E_x \\ E'_y = E_y - v B_z \\ E'_z = E_z + v B_y \\ B'_x = B_x \\ B'_y = B_y \\ B'_z = B_z \end{cases}$$

On the left are the transformation laws in vector form which are general and apply to all Galilean transformations. Those on the right correspond to a standardl transformation to which we can always return by a suitable choice of axes. From the relativistic field transformation laws given on page 427 we find the right expressions by making *c* tend to infinity.

c- Second equation: the divergence of the magnetic field is zero, which means that there are no magnetic monopoles.
The magnetic flux is conserved on a field tube.
It is clearly Galilean invariant because the nabla operator and the magnetic field are invariant:

Conclusion: $\vec{\nabla}'\cdot\vec{B}'=\vec{\nabla}\cdot\vec{B}=0$.

First equation: $\vec{\nabla}'\wedge\vec{E}'=-\dfrac{\partial\vec{B}'}{\partial t'}$

$$\vec{\nabla}\wedge(\vec{E}+\vec{v}_e\wedge\vec{B})=-\left(\frac{\partial}{\partial t}+\vec{v}_e\cdot\vec{\nabla}\right)\vec{B}$$

$$\vec{\nabla}\wedge\vec{E}+\vec{\nabla}\wedge(\vec{v}_e\wedge\vec{B})=-\frac{\partial\vec{B}}{\partial t}-(\vec{v}_e\cdot\vec{\nabla})\vec{B}$$

$$\vec{\nabla}\wedge\vec{E}+\vec{v}_e(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{v}_e)+(\vec{B}\cdot\vec{\nabla})\vec{v}_e-(\vec{v}_e\cdot\vec{\nabla})\vec{B}=-\frac{\partial\vec{B}}{\partial t}-(\vec{v}_e\cdot\vec{\nabla})\vec{B}$$

The curl of the cross product gives 4 terms: the first is zero ($\vec{\nabla}\cdot\vec{B}=0$) and so are the next two because $\vec{v}_e$ is a constant vector (all derivatives are zero).

Conclusion: $\vec{\nabla}\wedge\vec{E}=-\dfrac{\partial\vec{B}}{\partial t}$.

d- Third equation: local expression of the Gauss' theorem. The divergence of the electric field is zero in the absence of charge, the electric flux is then conserved.

$$\vec{\nabla}'\cdot\vec{E}'=\vec{\nabla}\cdot(\vec{E}+\vec{v}_e\wedge\vec{B})=0$$

$$\vec{\nabla}\cdot\vec{E}+\vec{\nabla}\cdot(\vec{v}_e\wedge\vec{B})=\vec{\nabla}\cdot\vec{E}+\vec{B}\cdot(\vec{\nabla}\wedge\vec{v}_e)-\vec{v}_e\cdot(\vec{\nabla}\wedge\vec{B})=0$$

$\vec{\nabla}\wedge\vec{v}_e=\vec{0}$ and $\vec{\nabla}\wedge\vec{B}=\vec{\nabla}'\wedge\vec{B}'=\mu_0\epsilon_0\dfrac{\partial\vec{E}'}{\partial t}$

Conclusion: $\vec{\nabla}\cdot\vec{E}=\mu_0\epsilon_0\vec{v}_e\cdot\dfrac{\partial\vec{E}'}{\partial t}\neq 0$

Fourth equation: local expression of Ampere's theorem.

$$\vec{\nabla}'\wedge\vec{B}'=\mu_0\epsilon_0\frac{\partial\vec{E}'}{\partial t'}$$

$$\vec{\nabla}\wedge\vec{B}=\mu_0\epsilon_0\left(\frac{\partial}{\partial t}+\vec{v}_e\cdot\vec{\nabla}\right)(\vec{E}+\vec{v}_e\wedge\vec{B})$$

$$\vec{\nabla}\wedge\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}+\mu_0\epsilon_0\left(\vec{v}_e\wedge\frac{\partial\vec{B}}{\partial t}+(\vec{v}_e\cdot\vec{\nabla})\vec{E}+(\vec{v}_e\cdot\vec{\nabla})(\vec{v}_e\wedge\vec{B})\right)$$

We have three extra terms. To prove that their sum is not identically zero, we just need to find a special situation where this is the case. Let us consider a standard transformation, in this case $(\vec{v}_e\wedge\quad)_x=0$ and:

$$(\vec{\nabla}\wedge\vec{B})_x=\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)_x+\mu_0\epsilon_0 v\frac{\partial E_x}{\partial x}.$$

The last term has no reason to be identically zero.

Conclusion: $\vec{\nabla}\wedge\vec{B}\neq\mu_0\epsilon_0\dfrac{\partial\vec{E}}{\partial t}.$

2 -a- We can consider the standard Lorentz transform without losing generality.

- $\vec{\nabla}\cdot\vec{B}=\dfrac{\partial B_x}{\partial x}+\dfrac{\partial B_y}{\partial y}+\dfrac{\partial B_z}{\partial z}$

so $\dfrac{\partial}{\partial ct}=\gamma\left(\dfrac{\partial}{\partial ct'}-\beta\dfrac{\partial}{\partial x'}\right)$, $\quad\dfrac{\partial}{\partial x}=\gamma\left(\dfrac{\partial}{\partial x'}-\beta\dfrac{\partial}{\partial ct'}\right)$,

$$\frac{\partial}{\partial y}=\frac{\partial}{\partial y'}\quad\text{and}\quad\frac{\partial}{\partial z}=\frac{\partial}{\partial z'}.$$

$$\vec{\nabla}\cdot\vec{B}=\gamma\left(\frac{\partial B'_x}{\partial x'}-\beta\frac{\partial B'_x}{\partial ct'}\right)+\gamma\frac{\partial(B'_y-\beta E'_z/c)}{\partial y'}+\gamma\frac{\partial(B'_z+\beta E'_y/c)}{\partial z'}$$

$$\vec{\nabla}\cdot\vec{B}=\gamma\vec{\nabla}'\cdot\vec{B}'-v\gamma\left(\frac{\partial B'_x}{\partial t'}+\left[\frac{\partial E'_z}{\partial y'}-\frac{\partial E'_y}{\partial z'}\right]\right)$$

The first term on the right is zero and the second term also because it is the x component of the first Maxwell equation.

Conclusion: $\vec{\nabla}\cdot\vec{B}=0$.

- $$\left(\vec{\nabla}\wedge\vec{E}+\frac{\partial\vec{B}}{\partial t}\right)_x=\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z}+\frac{\partial B_x}{\partial t}$$

$$=\gamma\frac{\partial\left(E'_z-\beta c B'_y\right)}{\partial y'}-\gamma\frac{\partial\left(E'_y+\beta c B'_z\right)}{\partial z'}+\gamma\left(\frac{\partial B'_x}{\partial t'}-v\frac{\partial B'_x}{\partial x'}\right)$$

$$=-\gamma v\vec{\nabla}'\cdot\vec{B}'+\gamma\left(\vec{\nabla}'\wedge\vec{E}'+\frac{\partial\vec{B}'}{\partial t'}\right)_{x'}=0$$

$$\left(\vec{\nabla}\wedge\vec{E}+\frac{\partial\vec{B}}{\partial t}\right)_y=\frac{\partial E_x}{\partial z}-\frac{\partial E_z}{\partial x}+\frac{\partial B_y}{\partial t}$$

$$=\frac{\partial E'_x}{\partial z'}-\gamma^2\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)\left(E'_z-\beta c B'_y\right)$$

$$+\gamma^2\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)\left(B'_y-\beta E'_z/c\right)$$

$$=\frac{\partial E'_x}{\partial z'}-\left(1-\beta^2\right)\gamma^2\frac{\partial E'_z}{\partial x'}+\left(1-\beta^2\right)\gamma^2\frac{\partial B'_y}{\partial t'}+0+0$$

$$=\left(\vec{\nabla}'\wedge\vec{E}'+\frac{\partial\vec{B}'}{\partial t'}\right)_{y'}=0$$

$$\left(\vec{\nabla}\wedge\vec{E}+\frac{\partial\vec{B}}{\partial t}\right)_z=\frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}+\frac{\partial B_z}{\partial t}$$

$$=\gamma^2\left(\frac{\partial}{\partial x'}-\beta\frac{\partial}{\partial ct'}\right)\left(E'_y+\beta c B'_z\right)-\frac{\partial E'_x}{\partial y'}$$

$$+\gamma^2\left(\frac{\partial}{\partial t'}-v\frac{\partial}{\partial x'}\right)\left(B'_z+\beta E'_y/c\right)$$

$$=\left(1-\beta^2\right)\gamma^2\frac{\partial E'_y}{\partial x'}-\frac{\partial E'_x}{\partial y'}+\left(1-\beta^2\right)\gamma^2\frac{\partial B'_z}{\partial t'}+0+0$$

$$=\left(\vec{\nabla}'\wedge\vec{E}'+\frac{\partial\vec{B}'}{\partial t'}\right)_{z'}=0$$

Conclusion: $\vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}$.

• $\vec{\nabla}\cdot\vec{E}=\frac{\partial E_x}{\partial x}+\frac{\partial E_y}{\partial y}+\frac{\partial E_z}{\partial z}$

$$\vec{\nabla}\cdot\vec{E}=\gamma\left(\frac{\partial E'_x}{\partial x'}-\beta\frac{\partial E'_x}{\partial ct'}\right)+\gamma\frac{\partial\left(E'_y+\beta c B'_z\right)}{\partial y'}+\gamma\frac{\partial\left(E'_z-\beta c B'_y\right)}{\partial z'}$$

$$\vec{\nabla}\cdot\vec{E}=\gamma\vec{\nabla}'\cdot\vec{E}'-v\gamma\left(\frac{1}{c^2}\frac{\partial E'_x}{\partial t'}-\left[\frac{\partial B'_z}{\partial y'}-\frac{\partial B'_y}{\partial z'}\right]\right)$$

The first term on the right-hand side is zero and the second term is also zero because it is the *x* component of the fourth Maxwell equation.

Conclusion: $\vec{\nabla}\cdot\vec{E}=0$.

• The verification of the Lorentz invariance of the fourth Maxwell equation is left to the insight of the reader.

b- Conservation of charge equation: $\partial_\mu j^\mu=0$.

with $\tilde{j}=\rho_p\tilde{u}=(\rho c,\rho\vec{v})$ and $\rho=\gamma\rho_p$.

Demonstration: $\partial_\mu j^\mu=\frac{\partial\rho c}{\partial ct}+\vec{\nabla}\cdot(\rho\vec{v})=\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{j}=0$.

c- Lorentz condition: $\partial_\mu A^\mu=0$.

Demonstration: $\partial_\mu A^\mu=\frac{\partial V/c}{\partial ct}+\vec{\nabla}\cdot\vec{A}=\frac{1}{c^2}\frac{\partial V}{\partial t}+\vec{\nabla}\cdot\vec{A}=0$.

Let us propose the following antisymmetric tensor:

$$\boxed{F^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu}$$

Demonstration:

• $F^{01}=\partial^0 A^1-\partial^1 A^0=\frac{\partial A^x}{\partial ct}+\frac{\partial V/c}{\partial x}$

moreover $\vec{E}=-\vec{\nabla}V-\frac{\partial\vec{A}}{\partial t}$ then $F^{01}=-\frac{E_x}{c}$.

- $F^{12}=\partial^1 A^2-\partial^2 A^1=-\frac{\partial A^y}{\partial x}+\frac{\partial A^x}{\partial y}=-(\vec{\nabla}\wedge\vec{A})_z=-B_z$.

and so on

d- • $\partial_\mu F^{\mu 0}=\mu_0 j^0=\mu_0\rho c=\partial_1 F^{10}+\partial_2 F^{20}+\partial_3 F^{30}$

$\partial_x E_x/c+\partial_y E_y/c+\partial_z E_z/c=\rho/\epsilon_0 c$ and $\vec{\nabla}\cdot\vec{E}=\rho/\epsilon_0$.

• $\partial_\mu F^{\mu i}=\mu_0 j^i$

$$\partial_\mu F^{\mu 1}=\mu_0 j^x=\partial_0 F^{01}+\partial_2 F^{21}+\partial_3 F^{31}$$

$$-\partial_{ct}E_x/c+\partial_y B_z-\partial_z B_y=\mu_0 j^x$$

$$-\frac{1}{c^2}\frac{\partial E_x}{\partial t}+(\vec{\nabla}\wedge\vec{B})_x=\mu_0 j^x$$

,,,

$$\vec{\nabla}\wedge\vec{B}=\mu_0\vec{j}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}$$

e- • $\partial^i F^{j0}+\partial^j F^{0i}+\partial^0 F^{ij}=0$ & $i\neq j$

$$\partial^1 F^{20}+\partial^2 F^{01}+\partial^0 F^{12}=0=\partial^x E_y/c-\partial^y E_x/c-\partial_{ct}B_z$$

$$0=-(\vec{\nabla}\wedge\vec{E})_z-\frac{\partial B_z}{\partial t} \quad \cdots \quad \vec{\nabla}\wedge\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

• $\partial^1 F^{23}+\partial^2 F^{31}+\partial^3 F^{12}=0$

$$-\partial^x B_x-\partial^y B_y-\partial^z B_z=\vec{\nabla}\cdot\vec{B}=0$$

f- $\partial_\mu F^{\mu\nu}=\partial_\mu(\partial^\mu A^\nu-\partial^\nu A^\mu)=\partial_\mu\partial^\mu A^\nu-\partial^\nu\partial_\mu A^\mu=\mu_0 j^\nu$

$\partial_\mu\partial^\mu A^0-\partial^0\partial_\mu A^\mu=\mu_0 j^0$ and $\partial_\mu\partial^\mu A^i-\partial^i\partial_\mu A^\mu=\mu_0 j^i$

Then: $\Box V-\frac{\partial}{\partial t}\left(\frac{1}{c^2}\frac{\partial V}{\partial t}+\vec{\nabla}\cdot\vec{A}\right)=\frac{\rho}{\epsilon_0}$

And: $\square\vec{A}+\vec{\nabla}\left(\frac{1}{c^2}\frac{\partial V}{\partial t}+\vec{\nabla}\cdot\vec{A}\right)=\mu_0\vec{j}$

With the Lorentz condition: $\square V=\frac{\rho}{\epsilon_0}$, $\square\vec{A}=\mu_0\vec{j}$.

Covariant form: $\square\widetilde{A}=\mu_0\widetilde{j}$ and $\partial_\mu\partial^\mu A^\nu=\mu_0 j^\nu$.

3 - Gauge: $\forall f \quad A'_\mu=A_\mu+\partial_\mu f$

Then:

$$F'^{\mu\nu}=\partial^\mu A'^\nu-\partial^\nu A'^\mu=\partial^\mu\left(A^\nu+\partial^\nu f\right)-\partial^\nu\left(A^\mu+\partial^\mu f\right)$$

$$F'^{\mu\nu}=\partial^\mu A^\nu-\partial^\nu A^\mu+\partial^\mu\partial^\nu f-\partial^\nu\partial^\mu f=F^{\mu\nu}$$

The field tensor is not modified.

The Lorentz condition $\partial_\mu A^\mu=0$ gives:

$$\partial_\mu A'^\mu=\partial_\mu\left(A^\mu+\partial^\mu f\right)=\partial_\mu A^\mu+\partial_\mu\partial^\mu f \quad \text{and} \quad \square f=0.$$

8 1. Units

Exercise page 273

Units of P : $[P]=W=\frac{J}{s}$

$$\left[\frac{2e^2}{3c^3}a^2\right]=\frac{C^2 s^3 N^2}{m^3 kg^2}=\frac{C^2 s^4 N}{m^4 kg^2}\frac{J}{s}=\frac{C^2}{m^2 N}\frac{J}{s}$$

with $N.m=J$ and $\frac{kg.m}{s^2}=N$. Moreover $\left[\frac{1}{4\pi\epsilon_0}\right]=\frac{N m^2}{C^2}$

then we obtain the good expression:

$$P=\frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}a^2=\frac{e^2 a^2}{6\pi\epsilon_0 c^3}$$

2. Relativistic equation of motion

Exercise p273

$$f_x=\frac{d p_x}{dt}=\frac{d}{dt}\left(m\frac{v_x}{\sqrt{1-\frac{v_x^2}{c^2}}}\right) \quad \text{and} \quad a_x=\frac{f_x}{m}\left(1-\frac{v_x^2}{c^2}\right)^{\frac{3}{2}}$$

3. Radiation damping 4-force

Exercise page 274

1- $\left(\frac{d w^\mu}{d\tau}-\frac{u^\mu u^\nu}{c^2}\frac{d w_\nu}{d\tau}\right)u_\mu=\frac{d w^\mu}{d\tau}u_\mu-\frac{u^\mu u_\mu u^\nu}{c^2}\frac{d w_\nu}{d\tau}=0$

because $u^\mu u_\mu=\tilde{u}\cdot\tilde{u}=c^2$ and $a^\mu b_\mu=g_{\mu\nu}a^\mu b^\nu=b^\nu a_\nu$.

2- $g^1=\frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\left(\frac{d w^1}{d\tau}-\frac{u^1 u^0}{c^2}\frac{d w_0}{d\tau}-\frac{u^1 u^1}{c^2}\frac{d w_1}{d\tau}\right) \quad \tilde{u}=\gamma(c,v)$

$$\frac{d w^1}{d\tau}-\gamma^2\frac{v}{c}\frac{d w_0}{d\tau}-\gamma^2\frac{v^2}{c^2}\frac{d w_1}{d\tau}=\frac{d w^1}{d\tau}(1+\gamma^2\beta^2)-\gamma^2\beta\frac{d w^0}{d\tau}$$

$$w_1=g_{11}w^1=-w^1 \qquad w_0=g_{00}w^0=w^0 \qquad 1+\gamma^2\beta^2=\gamma^2$$

$$\tilde{w}=\frac{d\tilde{u}}{d\tau} \qquad \tilde{u}=\frac{d\tilde{x}}{d\tau}=\gamma\frac{d\tilde{x}}{dt} \qquad u^1=\gamma v \qquad \frac{d\gamma}{dt}=\frac{v a}{c^2}\gamma^3$$

$$w^1 = \gamma \frac{d(\gamma v)}{dt} = \gamma\left(\frac{v^2 a}{c^2}\gamma^3 + \gamma a\right) = \gamma^2 a(\beta^2\gamma^2 + 1) = \gamma^4 a$$

$$\frac{dw^1}{d\tau} = \gamma \frac{d(\gamma^4 a)}{dt} = \gamma\left(4\gamma^3 \frac{va}{c^2}\gamma^3 a + \gamma^4 \frac{da}{dt}\right) = \gamma^5\left(\dot{a} + 4\gamma^2 \frac{va^2}{c^2}\right)$$

$$u^0 = \gamma c \qquad w^0 = \gamma \frac{d(\gamma c)}{dt} = \gamma\left(\frac{va}{c}\gamma^3\right) = \gamma^4 \beta a$$

$$\frac{dw^0}{d\tau} = \gamma \frac{d(\gamma^4 \beta a)}{dt} = \gamma\left(4\gamma^3 \frac{v^2 a^2}{c^3}\gamma^3 + \gamma^4 \frac{a^2}{c} + \gamma^4 \beta \dot{a}\right)$$

$$= \gamma^5\left((1 + 4\gamma^2\beta^2)\frac{a^2}{c} + \beta\dot{a}\right)$$

$$\frac{dw^1}{d\tau} - \frac{u^1 u^\nu}{c^2}\frac{dw_\nu}{d\tau} =$$

$$= \gamma^5\left[\left(\dot{a} + 4\gamma^2 \frac{va^2}{c^2}\right)(1 + \gamma^2\beta^2) - \gamma^2\beta\left((1 + 4\gamma^2\beta^2)\frac{a^2}{c} + \beta\dot{a}\right)\right]$$

$$= \gamma^5\left(\dot{a} + 3\gamma^2 \frac{va^2}{c^2}\right)$$

$$g = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\gamma^5\left(\dot{a} + 3\gamma^2\frac{va^2}{c^2}\right) = \gamma f$$

4. Four-potential magnitude

E. p274

In the Minkowski plan with the signs that correspond to the example of the course:

$$\tilde{r} = r(1,-1) \quad \tilde{u} = \gamma c(1,-\beta) \quad \tilde{r}\cdot\tilde{u} = r\gamma c(1-\beta) \quad \tilde{u}\cdot\tilde{u} = c^2$$

$$\tilde{A}\cdot\tilde{A} = \left(\frac{q}{4\pi\epsilon_0}\right)^2 \frac{(1+\beta)}{r^2(1-\beta)} \qquad k_A = \frac{|q|}{4\pi\epsilon_0}\frac{1}{r}\sqrt{\frac{1+\beta}{1-\beta}}$$

9.1. Figures

Exercise page 305

- v=61,000 km/h, d=4 ly. t=d/v

$$t_a=\frac{d_{ly}}{\beta}=4\times3.10^8\frac{3.6}{61000}\simeq70820\,yrs$$

- $$E=15.10^9\times42.10^9=63.10^{19}\,J/yr$$

$$m=\frac{1}{2}\frac{E}{c^2}=\frac{1}{2}\frac{63.10^{19}}{9.10^{16}}=3500\,kg/yr$$

- The interplanetary antiprotonic flux is the same in the whole solar system. R is the radius of influence of the magnetosphere of the star:

$$\frac{\Phi_{heliosphere}}{4\pi R_{heliosphere}^2}=\frac{\Phi_{planet}}{4\pi R_{planet}^2}$$

For a heliopause at 100 astronomical units:

$$\Phi_h=\Phi_J\left(\frac{R_h}{45\,R_J}\right)^2\simeq9.1\left(\frac{100\times150.10^6}{45\times69911}\right)^2 \qquad \Phi_h\simeq207.10^6\,kg/yr$$

Outside the heliosphere the cosmic radiation is more important, because it is not repelled by the Sun. If we evaluate towards the maximum of the curves to 2 Gev:

$$\varphi_{inside(max)}\simeq0.015\,(m^2.s.sr.GeV)^{-1}$$

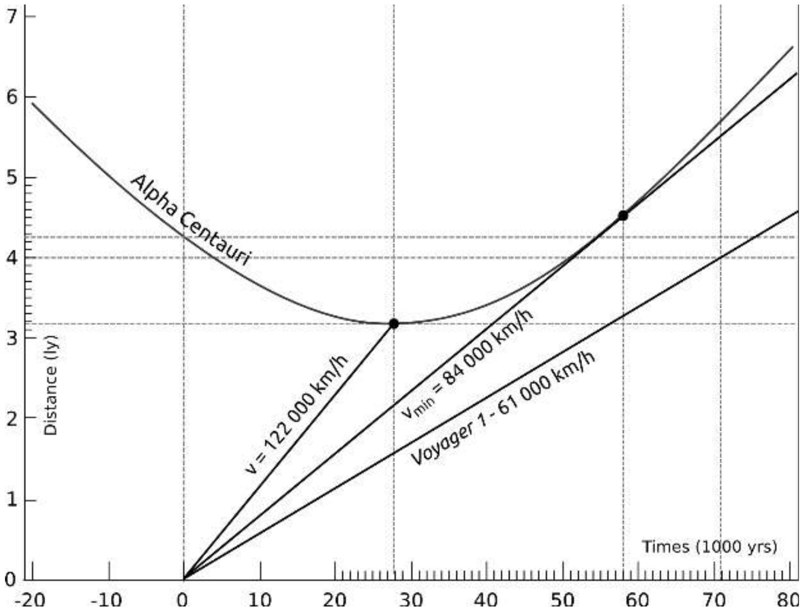
$$\varphi_{inside(min)}\simeq0.022\,(m^2.s.sr.GeV)^{-1}$$

$$\varphi_{inside(moy)}\simeq0.019\,(m^2.s.sr.GeV)^{-1}$$

$$\varphi_{outside}\simeq0.034\,(m^2.s.sr.GeV)^{-1}$$

$\dfrac{\varphi_{outside}}{\varphi_{inside}}\simeq1.8$ and $\Phi_{out}\simeq 370\ 000\ t/yr$.

2. The distances of stars over time *Exercise p309*



$v \simeq 34\,km/s$ $\quad v_{min} \simeq 23\,km/s$ $\quad v_{V1} \simeq 17\,km/s$

3. Sling effect *Exercise page 309.*

a - Composition of velocities: $\vec{v}_a = \vec{v}_e + \vec{v}_r$

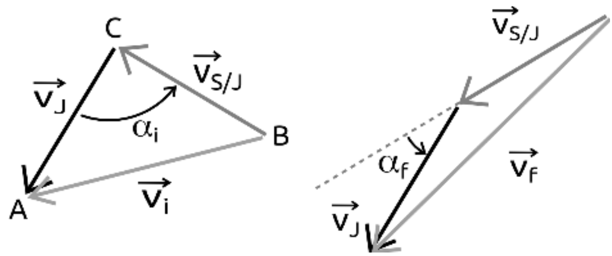
$\vec{v}_a = \vec{v}_R(M)$: "absolute" velocity.

$\vec{v}_e = \vec{v}_R(0') + \vec{\Omega} \wedge \overrightarrow{O'M} = \vec{v}_R(0')$: coinciding velocity (the planetocentric reference frame is in circular translation with respect to the heliocentric reference frame).

$\vec{v}_r = \vec{v}_{R'}(M)$: relative velocity.

Hence the expression of the heliocentric velocity of the spacecraft: $\vec{v} = \vec{v}_J + \vec{v}_{S/J}$.

On the graph provided by NASA, the trajectory of the probe is represented in the reference frame which has for origin Jupiter and directions of distant stars supposed fixed. This trajectory is hyperbolic, symmetrical and the planetocentric velocity of the probe $\vec{v}_{S/J}$ is tangent to this trajectory. The time of the deviation, the heliocentric velocity of Jupiter can be considered as a constant vector:

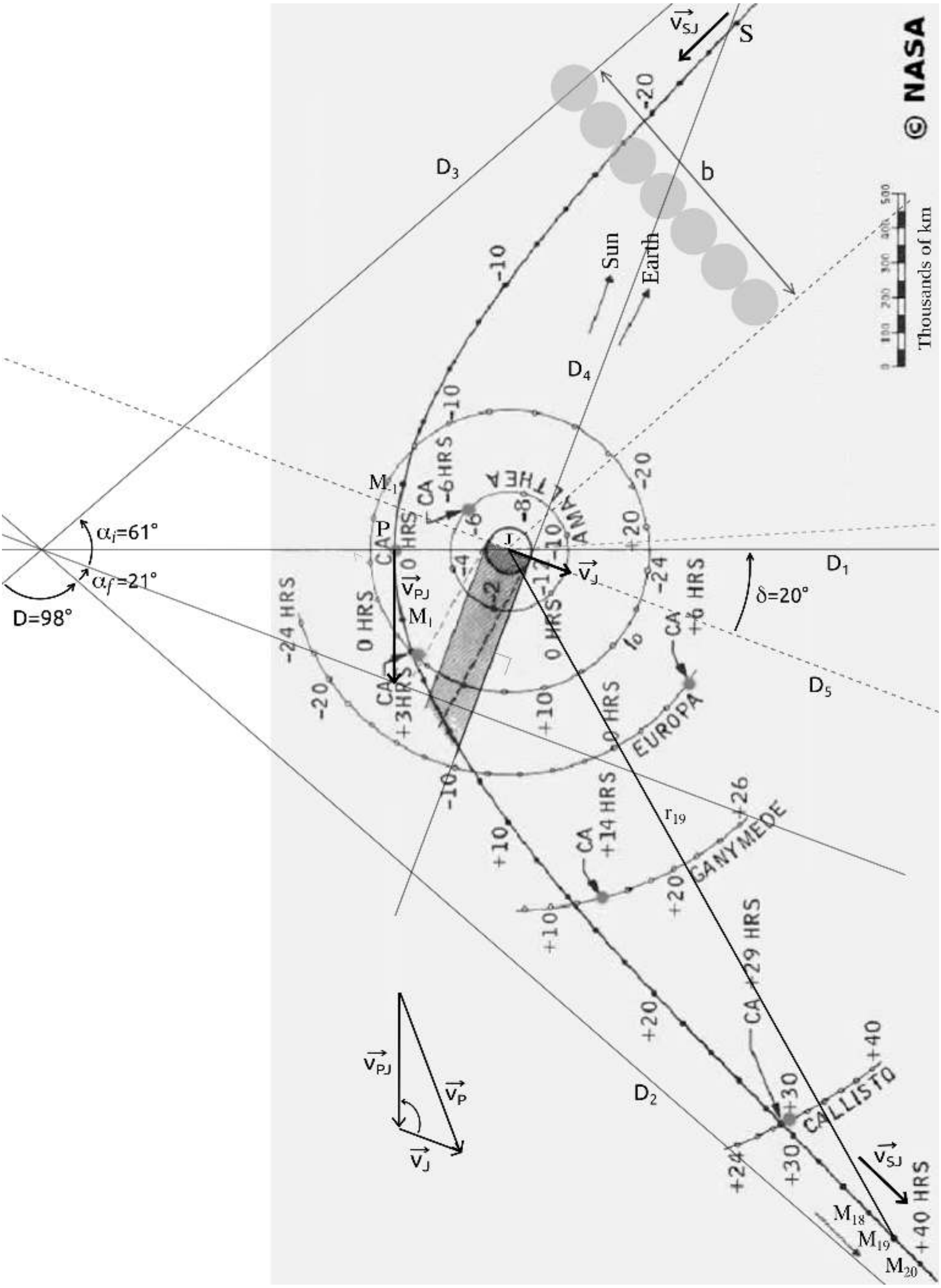


We estimate α_i and α_f on the graph. The straight line D_2 is placed along the asymptote estimated in $+\infty$. The straight line D_1 is the central axis and the second asymptote D_3 is obtained by symmetry. The angle between D_2 and D_1 is estimated at 41°. The deflection D is therefore about 98°.

The heliocentric motion of Jupiter is supposed to be circular, so the velocity of Jupiter is orthoradial along D_5 and perpendicular to the line D_4 aligned with the shadow of the Sun at large distance.

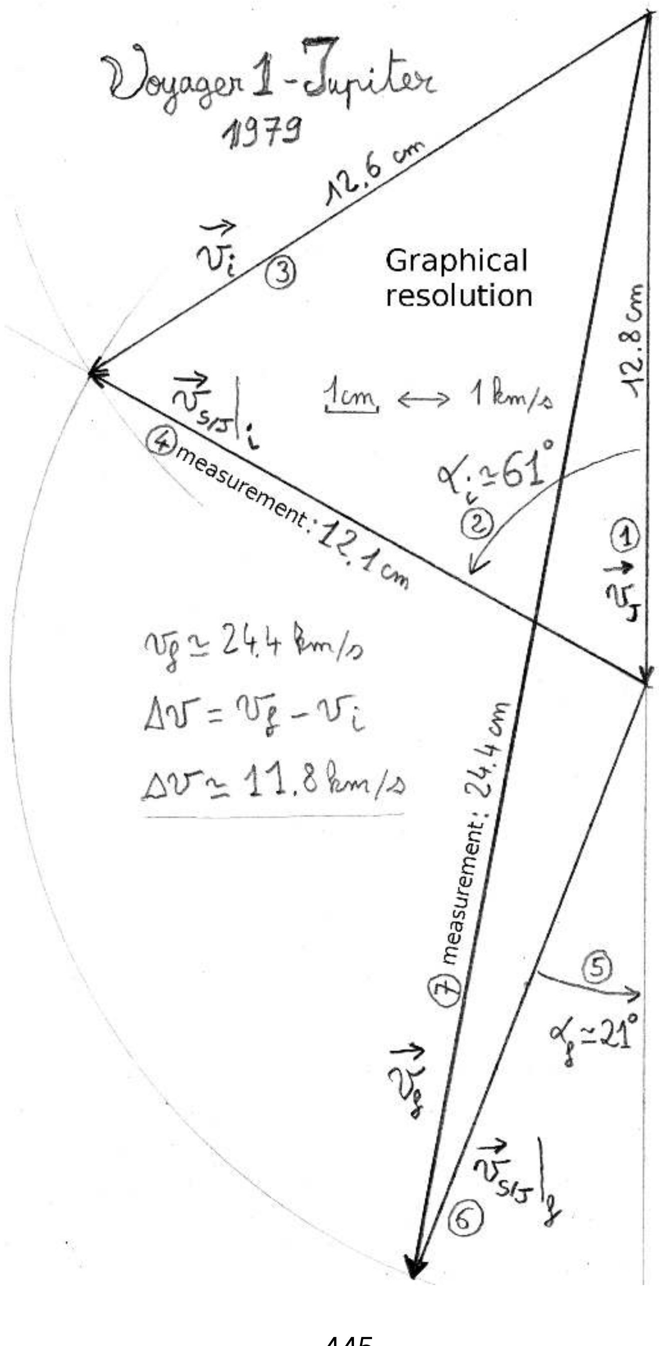
Angle between D_5 and D_3 : $\alpha_i \simeq 61°$.

Angle between D_2 and D_5 : $\alpha_f \simeq 21°$.



<u>Geometric resolution:</u>

We will carry out a graphical construction with a graduated ruler, compass and protractor. The results are shown on the drawing.



Analytical resolution:

Let's apply the trigonometric properties of triangles:
$\vec{AB} \cdot \vec{AB} = \vec{AC} \cdot \vec{AC} + 2\,\vec{AC} \cdot \vec{CB} + \vec{CB} \cdot \vec{CB}$

then: $v_i^2 = v_J^2 - 2\,v_J v_{SJ} \cos\alpha_i + v_{SJ}^2$ (1)

and: $v_f^2 = v_J^2 + 2\,v_J v_{SJ} \cos\alpha_f + v_{SJ}^2$ (2)

c_f(1)+c_i(2) : $v_{SJ}^2 = \dfrac{c_f v_i^2 + c_i v_f^2}{c_i + c_f} - v_J^2$

After some calculations, we obtain a quadratic equation with respect to v_f^2 and finally:

$$\boxed{v_f = \sqrt{v_i^2 + 2(c_i + c_f)\,v_J\left[c_i v_J \pm \sqrt{v_i^2 - s_i^2 v_J^2}\right]}}$$

Limit case: $\alpha_i = \alpha_f = 0$

Case of maximum deviation / half turn: $D = \pi$.

We obtain: $\Delta v = v_f - v_i = \pm 2\,v_J$ OK.

The sign changes depending on whether $\vec{v}_i$ is in the opposite direction of $\vec{v}_J$ or in the same direction. When I shoot with a ball on the back of the train when it moves away, the ball is slowed down (the ball reaches the train if $v_i > v_J$).

Limit case: $\alpha_i + \alpha_f = \pi$ (no deviation)

$c_i + c_f = \cos\alpha_i + \cos(\pi - \alpha_i) = 0$ and $v_f = v_i$ OK.

Numerical application:

$$v_i = 12.6\,km/s\ , \quad v_J = 12.8\,km/s$$

then $v_f \simeq 24.4\,km/s$ and $\Delta v \simeq 11.8\,km/s$

Values consistent with those observed on the curve on page 283.

b - We estimate the velocity at the periastron with the average between M_{-1} and M_1 : $v_{PJ} \simeq 28\, km/s$.

In the heliocentric frame of reference:

$$v_P^2 = v_J^2 - 2 v_J v_{PJ} \cos\left(\frac{\pi}{2} + \delta\right) + v_{PJ}^2$$

and $v_P \simeq 37\, km/s$ (as on the speed profile)

Theorem of angular momentum:

$\dfrac{d\vec{\sigma}}{dt} = \vec{r} \wedge \vec{F} = \vec{0}$ (central force) and $\vec{\sigma} = m\vec{r} \wedge \vec{v} = \overrightarrow{cst}$

then: $b v_{SJ} = r_{min} v_{PJ}$ with $v_{SJ} = v_\infty$

We estimate v_{19} on the NASA graph and with the conservation of mechanical energy we find v_∞ and then b:

$$\frac{1}{2} v_{19}^2 - \frac{G M_J}{r_{19}} = \frac{1}{2} v_\infty^2 \quad \text{and} \quad b = \frac{v_{PJ}}{v_{SJ}} r_{min} \simeq 12\, R_J$$

c - $p = \dfrac{r_{min}^2 v_{PJ}^2}{G M_J} \simeq 11.5\, R_J$

At periastron $\theta = 0$, $r_{min} = \dfrac{p}{1+e}$ & $e = \dfrac{p}{r_{min}} - 1 \simeq 1.3 > 1$.

Moreover: $\theta_{max} = arcos\left(-\dfrac{1}{e}\right) \simeq 139\,°$,

also $\alpha_f + \alpha_i = 2\pi - 2\theta_{max} = \pi - D$ and $D \simeq 98\,°$.

d - • From the formula, we see that v_f is maximal for $c_f = 1$, so $\alpha_f = 0$. The calculation then gives: $v_f \simeq 24.8\, km/s$ and $\Delta v \simeq 12.2\, km/s$.

Interstellar speed calculation:

$$\frac{1}{2}m v_f^2 - G\frac{m M_S}{D_{JS}} = \frac{1}{2}m v_\infty^2$$

$$v_\infty = \sqrt{v_f^2 - \frac{2 G M_S}{D_{JS}}} \simeq 16.8\,km/s \simeq 60400\,km/h.$$

Instead of 50,000 km/h for *Voyager 1* (if it had not then taken advantage of Saturn).

• <u>Ellipse</u>: $\frac{1}{2}m v_i^2 - G\frac{m M_S}{D_{JS}} = -G\frac{m M_S}{2a}$

$\frac{1}{2a} = \frac{1}{D_{JS}} - \frac{v_i^2}{2 G M_S}$ then $a \simeq 763.4 \times 10^6\,km$.

Speed on the ellipse at the level of the Earth:

$$v = \sqrt{2 G M_S\left(\frac{1}{D_{TS}} - \frac{1}{2a}\right)} \simeq 40.049\,km/s$$

For the semi-minor axis: $b = \sqrt{p a}$

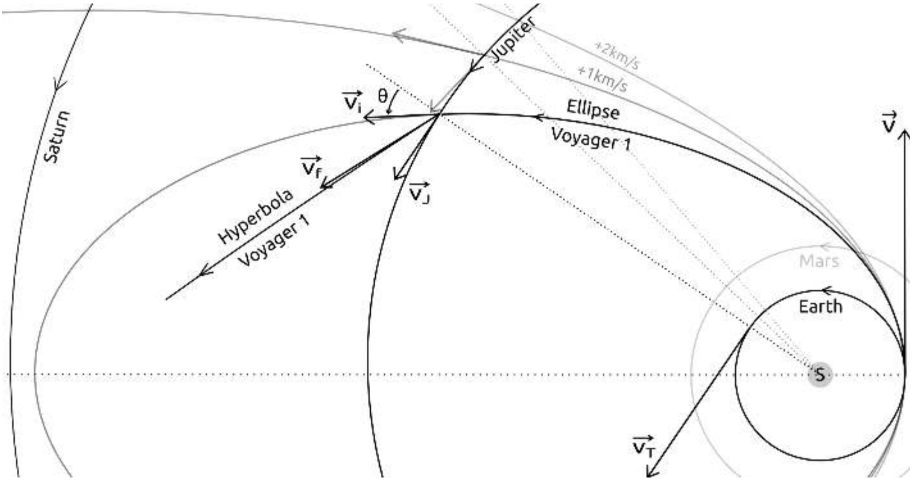
$p = \frac{D_{ST}^2 v^2}{G M_s}$ then $b \simeq 454 \times 10^6\,km$.

For the angle: $cst = L/m = D_{SJ} v_i \sin\theta = D_{ST} v$

$\sin\theta = \frac{D_{ST} v}{D_{SJ} v_i}$ and $\theta \simeq 36.5°$.

At the Earth level: $v_T \simeq 30\,km/s$ then $\Delta v \simeq 10\,km/s$, increase of speed necessary to leave the Earth circular orbit. Kinetic energy provided by the Titan rocket.

To increase the slingshot effect, we can think of decreasing the angle θ. Hence a greater speed given by the rocket to join a wider ellipse (case in gray).



On the other hand, as the speed at the approach to Jupiter is higher, the angle α_i will increase. For example, if, with a larger amount of propellant, we increase the initial Δv by 1 km/s, we gain 4.2 km/s of interstellar speed. The approach seems to be validated, it is much more interesting to use the propellant at the Earth level than after (Oberth effect). We summarize all the results on the next page. We consider the optimal case where $\alpha_f = 0°$.

By further increasing the initial velocity at the departure of the Earth orbit, the trajectory is no longer elliptical and becomes hyperbolic. At the same time, the impact parameter decreases and care must be taken not to collide with Jupiter: $r_{min} > R_J$. The Δv is thus limited to 4.8km/s: the probe already goes much faster than the *Voyager* probes.

+ km /s	a (10^6 km)	b (10^6 km)	r_{min} (R_J)	θ (°)	v_{SJ} (km /s)	α_i (°)	v_f (km /s)	Δv (km /s)	v_{inter} (10^3 m/s)	(10^3 km /h)
0	763	454	2.3	36.5	11.4	62.4	23.5	10.9	15.9	57
1	1424	636	1.8	29.8	14.4	69.2	27.2	11.7	20.1	72
2	16647	1942	1.5	26.0	16.9	73.1	29.7	11.7	23.4	84
3	hyp.	hyp.	1.3	23.6	19.1	75.7	31.9	11.7	26.2	94
4	hyp.	hyp.	1.12	21.8	21.2	77.6	34.0	11.7	28.6	103
4.8	hyp.	hyp.	1.01	20.7	22.7	78.8	35.5	11.7	30.4	110
5	hyp.	hyp.	0.99	<col.	x	x	x	x	x	x

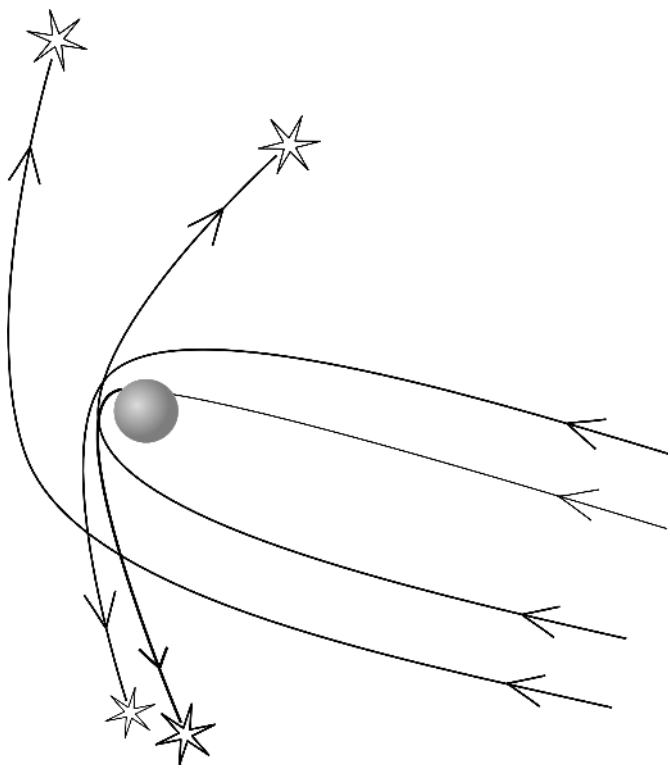
• Mars: The planet has an orbital velocity of 24 km/s and the slingshot Δv could seemingly reach a whopping 48 km/s. On the other hand, the mass of Mars is small compared to the giant planets, and since the mass of the planet is not concentrated in one point, we are limited by the minimum approach distance R_M. The Δv_{sling} is under the best conditions a small 0.6 km/s.

• Modeling *Voyager 1* : The spreadsheet gives us a good correspondence with the historical values. Our simplifying assumptions are thus validated: heliocentric motions of the planets, coplanar orbits, Hohmann orbit (minimal energy to be provided).
File: www.voyagepourproxima.fr/docs/FrondesVoyager1.ods

• Project *Voyager 3* : We have chained the 4 successive slingshot of Jupiter, Saturn, Uranus and Neptune. By optimizing the approach distances, with a surplus of 4.8 km/s at the level of the Earth, we

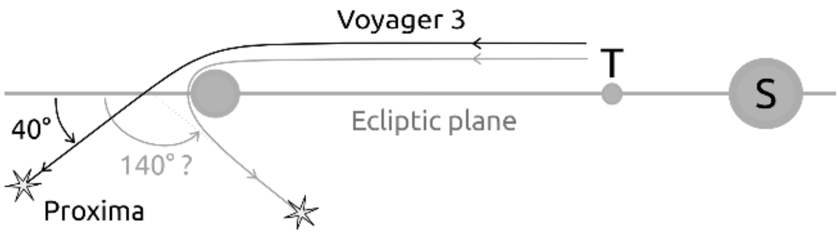
reach 140 000km/h. More than twice the speed of the historical *Voyager* probes.
File: .../docs/FrondesVoyager3.ods

• A simple solution that does not require additional fuel: the last sling is used to deviate the trajectory. By a tiny correction, just after the penultimate slingshot, we can freely choose the future impact parameter, as much in value as in direction. We could thus target a star outside the ecliptic plane:



There is one limitation, however: the minimum approach distance. For the same approach

distance, the faster the probe goes, the smaller the deviation. On the contrary, the deviation increases with the mass of the planet and its density.
To reach Proxima the deviation must be between 40° and 140° depending on the position on the ecliptic at the time of the exit:



At the level of Neptune the speed of the probe is just high, nevertheless Neptune is more massive and dense than Uranus. Let's have a look on our spread-sheet to see what the numerical value is: we obtain a deviation of about 20°. This is not enough to go to Proxima. We give a non-exhaustive list of options:

- *Could we get more slings to increase this deflection?*

- The satellites of Neptune? For example, Triton, the most massive and dense, would give only one tenth of a degree of additional deviation. The same for Pluto, or the asteroids of the Kuiper belt (located after Neptune between 30 and 55 au). The masses of all these bodies are too small. Unless we string together dozens of small slingshots?

- A 2019 study hypothesizes the existence of Trans-Neptunian primordial black holes at 300 au (see exercise p329). These small black holes of 5 Earth masses would give a more than sufficient deflection, up to 145°.

- The hypothetical planet 9? To explain certain anomalies in the trajectories of planets, there is the hypothesis of a planet of five Jovian masses at 8000 au. The deviation could reach 115°.

- *Do we need to change the global pattern of slings and propellant use?*

- To obtain, at the level of Neptune, a higher deviation, we can remove the impulse of 4.8 km/s at the Earth level. The probe arrives more slowly and we obtain then a deviation of 30°. This is better but still insufficient.

- We can limit the slingshot to the Jupiter-Saturn pair. Jupiter for the speed increase and Saturn for the deviation. For Proxima, we must then use propellants to reach 137 000 km/h. The mass of the whole becomes more important.

On page 328, thirteen nearby stars are represented with their characteristics.

• We will investigate the possibility of a gravitational slowdown of the *Voyager 3* probe using the four currently known components:

The star Alpha Centauri A: M_A=1.1M_S, R_A=1.23 R_S.
The star Alpha Centauri B: M_B=0.91M_S, R_B=0.87 R_S.
Distance between the two stars: D_{AB}=23 au.
The star Proxima Centauri: M_C=0.123M_S, R_C=0.141 R_S.
Distance between Proxima and AB: $D_{P\text{-}AB} \simeq$13,000 au.
The planet Proxima Cent. b: $M_{Pb} \simeq$1.27M_T, $R_{Pb} \simeq$1.08 R_T.
Distance: $D_{P\text{-}Pb} \simeq$0.0485 au.

Finally, we have two rather distant subsystems: the A-B pair and the star-planet pair. Let's imagine that the probe goes back and forth between these two pairs to slow down and finally orbit around one of them.

→ A-B: Let us consider, first of all, the two stellar components Alpha Centauri A and B. These have masses similar to our Sun. To simplify, we can model by a system consisting of two stars in circular motion. The two components rotate around their barycenter G middle of the segment [AB].
Kepler's law for the fictitious particle M (formulas page 98):

$$\frac{a^3}{T^2}=\frac{\alpha}{4\pi^2\mu} \quad \text{with} \quad \alpha=GM_AM_B$$

Moreover $\mu=\frac{M_AM_B}{M_A+M_B}$ then $\frac{R^3}{T^2}=\frac{G(M_A+M_B)}{4\pi^2}$

Also $\overrightarrow{AB}=\overrightarrow{GM}$ and $R=GM=D_{AB}$,

$$T=\sqrt{\frac{4\pi^2 {D_{AB}}^3}{G(M_A+M_B)}} \quad \& \quad v=\frac{2\pi R}{T}=\sqrt{\frac{G(M_A+M_B)}{D_{AB}}}$$

v is the speed of the fictitious particle.

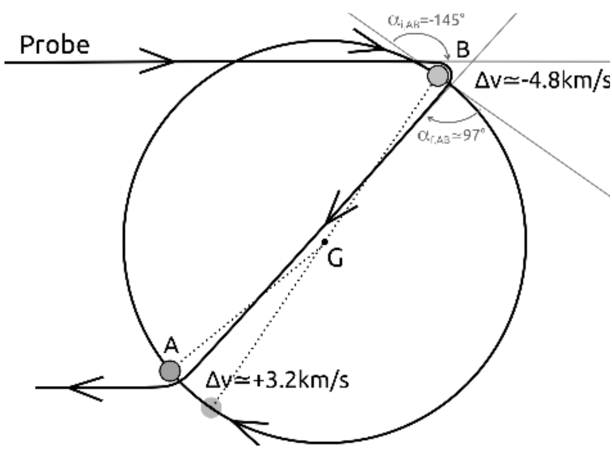
Let's determine v_A and v_B :

$$\overrightarrow{GA}=-\frac{M_B}{M_A+M_B}\overrightarrow{GM} \quad \text{and} \quad v_A=\frac{M_B}{M_A+M_B}v$$

NA: $T\simeq 78\,yrs$, $v\simeq 8.82\,km/s$,

$$v_A\simeq 3.99\,km/s \quad \text{and} \quad v_B\simeq 4.82\,m/s.$$

For a U-turn, one must slow down with one component and accelerate with the other, however, it can be arranged so that deceleration prevails:



Slingshot U-turn using a binary system. With a single star, we can not make a perfect half-turn, it will always lack a few degrees. To form a couple, we can also use a gas giant. Here, the Alpha Centauri A / Alpha Centauri B system for Voyager 3 with an initial speed of 140 000 km/h.

→ <u>P-Pb</u>: Let's determine the velocity of Proxima in the frame of reference of AB. The star Proxima is far from the system AB and has a small mass compared to this system. We can thus consider the stellar system AB punctual and fixed in G=A=B. The formulas are the same as before, replacing D_{AB} by $D_{P\text{-}AB}$:

NA: $T_P \simeq 1$ million years, $v_{P/AB} \simeq 0.37\, km/s$.

Period and velocity of the planet Proxima b in the Proxima reference frame (in this case G=P) :

$$T = \sqrt{\frac{4\pi^2 D_{P-b}{}^3}{G M_P}} \quad \text{and} \quad v = \sqrt{\frac{G M_P}{D_{P-b}}}$$

NA: $T_b \simeq 11\, days$, $v_{b/P} \simeq 47.5\, km/s$.

We can consider the star as fixed. Here, the turn around is not possible, because the planet is not massive enough and the deviation that it gives to the probe is too weak to complete the turn around started with the star.

→ <u>Conclusion</u>: With the known components, the slingshot effect cannot slow down the probe sufficiently. But, there are most probably many Jovian components that will be discovered later and that will allow the probe to be put into orbit using little propellant.

We can broaden our view of the slingshot effect by reversing the direction of time. For example, for the train, by reversing the arrow of time, the ball arrives at 130 km/h on a train that is going away in reverse at 50 km/h and the ball returns to the child's hand at 30 km/h. This is a feasible experiment. This is why the slingshot effect can speed up as well as slow down. If we rewind the movie of *Voyager 3*'s four successive slingshots, it arrives from the interstellar medium to slow down with Neptune, Uranus, Saturn, Jupiter, and finally decelerate by 4.8 km/s using propellants to orbit the Earth. There is a good chance that the probe, when it arrives at a distant star system, will proceed in a similar way.

4. Numerical simulations of the slings

Exercise p313.

1 -a- *Kepler's laws:*

$$p=\frac{r_{min}^2 v_{max}^2}{G M_S}\simeq 149\,507\,901\,890\,m$$

$$e=\frac{p}{r_{min}}-1\simeq 0.016382$$

$$r_{max}=\frac{p}{1-e}-1\simeq 151\,998\,002\,652\,m$$

$$v_{min}=\frac{r_{min}}{r_{max}}v_{max}\simeq 29310.644\,m/s$$

$$a=\frac{r_{max}+r_{min}}{2}\simeq 149\,548\,038\,326\,m$$

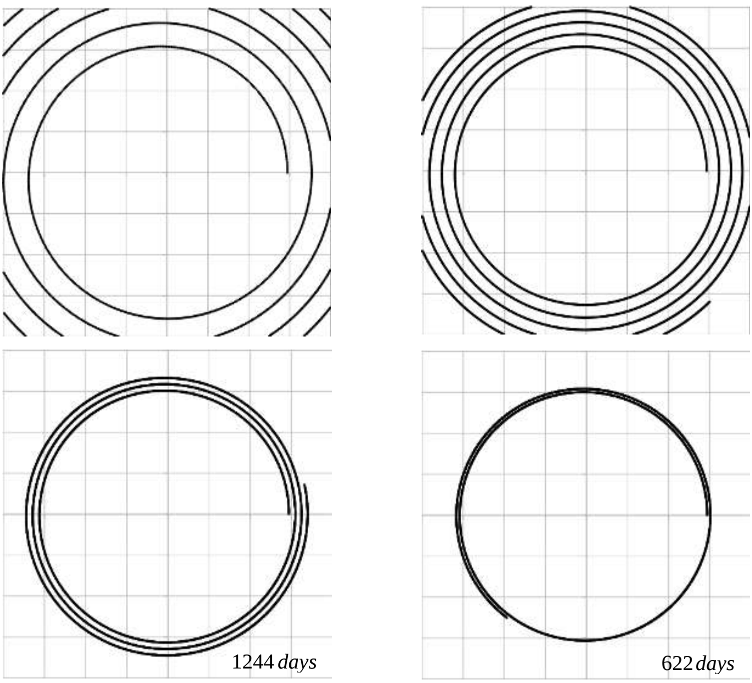
$$T=\frac{2\pi}{\sqrt{G M_S}}a^{3/2}\simeq 365.011\,days$$

Note: these values are not fully consistent with other known values (T=365.256 days, r_{max}=152 097 701 km and v_{min}=29 291 m/s) but we will take them as references to test our numerical methods.

1 -b- *Earth-Sun* : We take, as an indicator of the global error, the distance to the Sun after one revolution. The laws of physics impose to come back to the same point. When the Earth has made a rotation of 360° we obtain the percentage of global error $\% e_r=\frac{r_{\mathrm{sim}}-r_{theo}}{r_{theo}}$ ($r_{theo}\simeq 1.47\times 10^{11}$m):

Step	$r=\sqrt{x^2+y^2}$	%e_r / turn	ΔE / kg / yr
h=1day	1.768×10^{11}m	20.2 %	+68 MJ
h/2	1.624×10^{11}m	10.4 %	+39 MJ
h/4	1.549×10^{11}m	5.3 %	+22 MJ
h/8	1.510×10^{11}m	2.7 %	+11 MJ

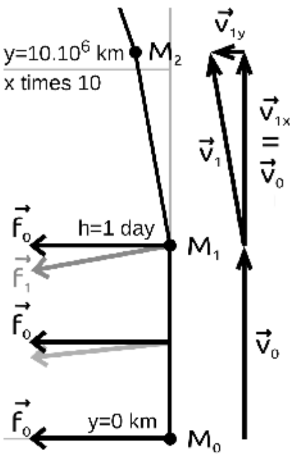
The trajectories for h, h/2, h/4 and h/8:



To answer the first question, it is clear that, for h=1 day, the simulation is absolutely unsatisfactory. We should obtain a closed trajectory which returns exactly on its steps. Everything happens as if the mechanical energy of the system increases instead of remaining constant. Nevertheless the error decreases linearly with the step size.

Let us note *T* the duration of the numerical experiment, here the period of revolution, and *n* the number of steps, we have then: $h=T/n$. *n* corresponds to the necessary numerical work. The global error evolves in $1/n$. This linear variation is characteristic of first order methods.

For h=1/8 day, the error remains consequent: $r_{max}=154.3\times10^6$km, v_{min}=29,083 m/s and the year is about 372 days...



At the first step, we notice that the speed of the Earth increases, which is physically impossible since we start from the perihelion. According to the law of areas, the speed must decrease on each time interval until the aphelion. However, on the first step, the orthoradial velocity has not been modified. It is the force that modifies the velocity vector and we have considered this force to be constant and equal to that at the beginning of the interval. In fact, over a step, the position, as well as the force, vary continuously.

We propose a modification of the method, we take the beginning of the interval to estimate the components of the velocity and the end of the interval for the positions. We could thus globally compensate our errors, because we use, in a loop from one rank to the other, the positions to calculate the velocities and the velocities to calculate the positions:

$$v_{x,i,n+1}=v_{x,i,n}+F_{x,i}(x_{j,n},y_{j,n})\Delta t \quad x_{i,n+1}=x_{i,n}+v_{x,i,n+1}\Delta t$$

Already for h=1 day, the error becomes much smaller:
r_{max}=152.4×10^{6}km, v_{min}=29,240 m/s and the year is about 366 days (anomalies: r_{max} and v_{min} do not occur after a half turn, nor at the same time).
For h=1/8 day: r_{max}=152.004×10^{6}km, v_{min}=29,309 m/s and the year is about 365.0 days.

Improved Euler Method			
Step	$r=\sqrt{x^2+y^2}$	%e_r / turn	ΔE /kg /yr
h=1day	1.47097506026×10^{11}m	-0.00039 %	-149 J
h/2	1.47098004533×10^{11}m	-0.000047 %	-28 J
h/4	1.47098066041×10^{11}m	-0.0000054 %	-8.3 J
h/8	1.47098073350×10^{11}m	-0.00000044 %	-3.4 J

The improved method is impressive, for a tiny modification of the calculation method, we have results, certainly still insufficient, but much better for a numerical work eight times lower! The error does not evolve linearly anymore, we are getting closer to what is called the midpoint method where the error decreases with the square of the numerical work.

In conclusion, the calculation method used at each step appears to be a key element, more important than the raw computing power. We will therefore introduce a higher order numerical method.

File: www.voyagepourproxima.fr/docs/Terre-Soleil-Euler.ods

2 -a- *RK4* :

$$\frac{dx}{dt}=A(x,y,v_x,v_y)=v_x, \qquad \frac{dy}{dt}=B(x,y,v_x,v_y)=v_y$$

$$\frac{dv_x}{dt}=C(x,y,v_x,v_y)=-GM\frac{x}{(x^2+y^2)^{3/2}}$$

$$\frac{dv_y}{dt}=D(x,y,v_x,v_y)=-GM\frac{y}{(x^2+y^2)^{3/2}}$$

$$A_1=A(x_n,y_n,v_{xn},v_{yn}) \qquad B_1=B(x_n,y_n,v_{xn},v_{yn})$$

$$C_1=C(x_n,y_n,v_{xn},v_{yn}) \qquad D_1=D(x_n,y_n,v_{xn},v_{yn})$$

$$A_2=A\left(x_n+\frac{h}{2}A_1,y_n+\frac{h}{2}B_1,v_{xn}+\frac{h}{2}C_1,v_{yn}+\frac{h}{2}D_1\right)$$

... ...

$$A_3=A\left(x_n+\frac{h}{2}A_2,y_n+\frac{h}{2}B_2,v_{xn}+\frac{h}{2}C_2,v_{yn}+\frac{h}{2}D_2\right)$$

... ...

... ...

$$D_4=D(x_n+hA_3,y_n+hB_3,v_{xn}+hC_3,v_{yn}+hD_3)$$

$$x_{n+1}=x_n+\frac{h}{6}(A_1+2A_2+2A_3+A_4)$$

... ...

$$v_{y\,n+1}=v_{y\,n}+\frac{h}{6}(D_1+2D_2+2D_3+D_4)$$

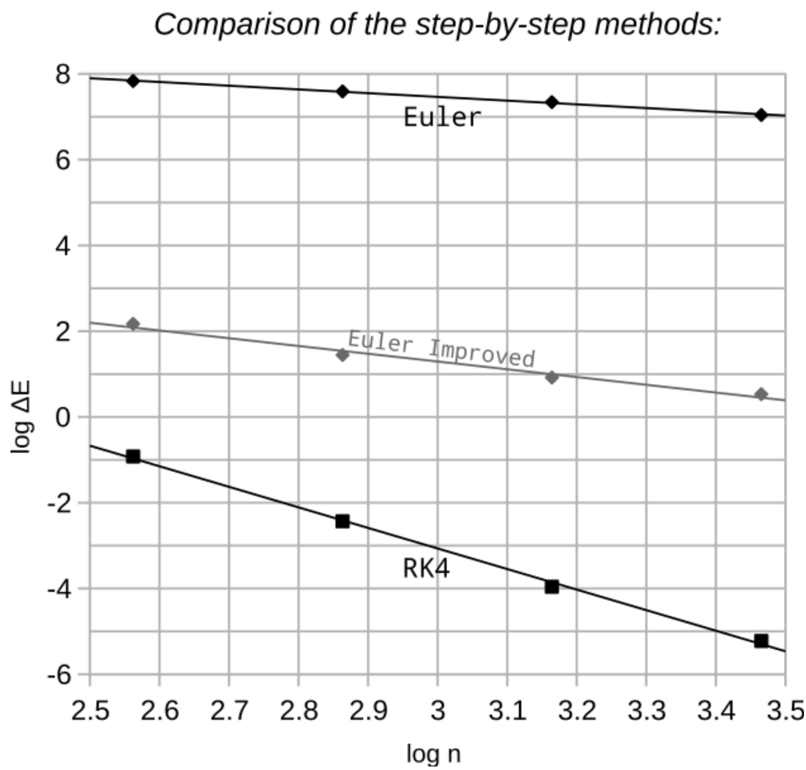
b- For h=1 day, h/2 and h/8: $r_{max}=151.998\times10^6$km, v_{min}=29,310.6 m/s and the year is 365.01 days. In accordance with the data entered.

Runge-Kutta 4 Method			
Step	$r=\sqrt{x^2+y^2}$	$\%e_r$ / turn	ΔE /kg /yr
h=1day	$1.4709807807\times10^{11}$m	27×10^{-9}	-120 mJ
h/2	$1.4709807603\times10^{11}$m	14×10^{-9}	-3.7 mJ
h/4	$1.4709807499\times10^{11}$m	6×10^{-9}	-0.11 mJ
h/8	$1.4709807447\times10^{11}$m	3×10^{-9}	-0.006 mJ

This method largely outperforms the previous ones.

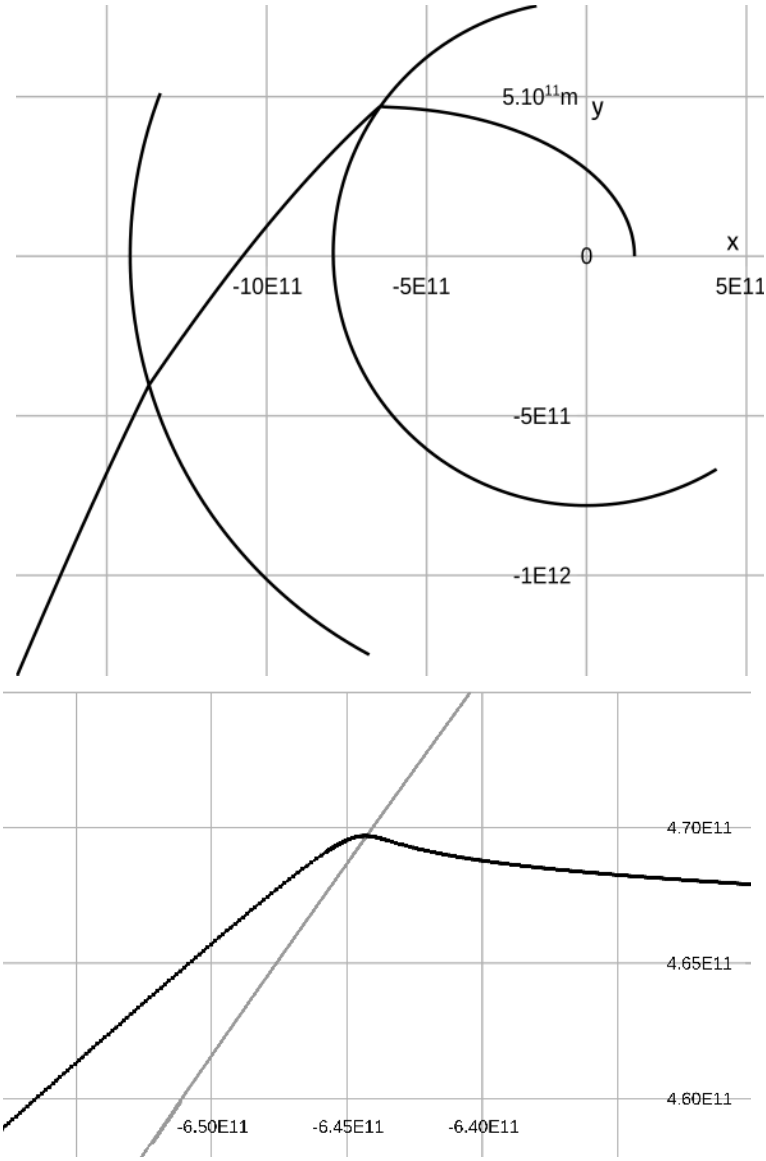
File: .../docs/Terre-Soleil-RK4.ods

We were interested in the variation of distance over one revolution and the variation of mechanical energy over one year. For a mathematical study of the error, we perform an experiment of fixed duration *T*, then we increase the numerical work *n* on this interval [0, T]. And to calculate the global error, we must compare the numerical value with the theoretical one at *t=T*. However, we have not determined the expression of *r(t)*, but only *r(θ)*. For the mechanical energy, it is much simpler to compare to the theoretical value, because the theory imposes a constant energy. So we compare the initial value of the energy to that at an arbitrary *T*. Here we have chosen *T= 365* days on the graph that follows.

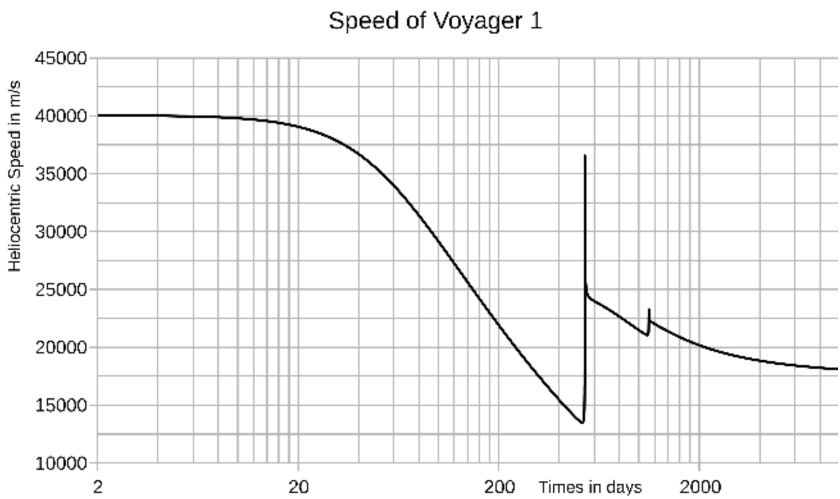


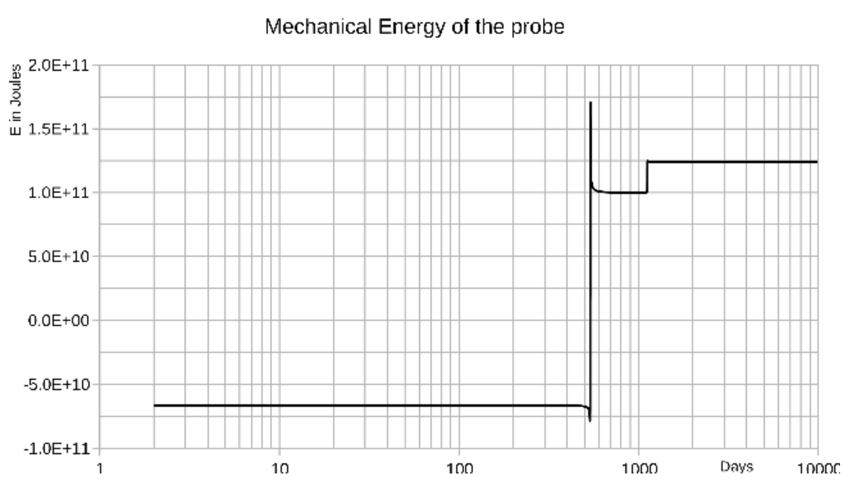
We plot the decimal logarithm of the error versus the decimal logarithm of the numerical work. On the left, the values for a step h of one day and a duration of 365 days, i.e. approximately one year (n=365). Then the points for h/2, h/4 and h/8 (n=730, 1095 then 1460) still over one year. The more the numerical work n increases, the smaller the error, and the faster the method is of higher order. For the Euler method the error decreases linearly with the step, here, over a decade, the error decreases by a factor 10. We see that the improved Euler method is indeed a method of order 2, over a decade, the error is divided by 100. For Runge-Kutta of order 4 we have a factor 10^4.

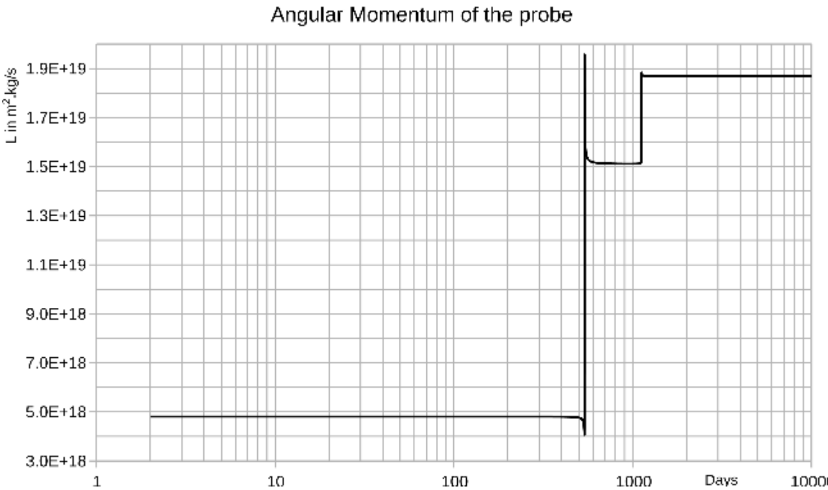
3 - *Voyager 1* :



The trajectory of the Voyager 1 probe seen in the heliocentric reference frame. On top, the departure at *the Earth level, followed by the slingshot at the level of Jupiter, then Saturn, to join then the interstellar medium. At the bottom, the slingshot effect appears clearly at the level of Jupiter. Unlike the trajectory in the Galilean reference frame centered on Jupiter, the heliocentric trajectory is not hyperbolic.*







We find the characteristics of the motion of *Voyager 1* in agreement with our results with Kepler's formulas and the historical data provided by NASA. For Jupiter, we have the values of velocities at the beginning of the slingshot, at the peak, and at the end of the slingshot which correspond.

Between two slingshots, there is conservation of the mechanical energy and angular momentum with respect to the Sun, the values are in adequacy. Between two planets, the probe can be considered as isolated, hence the two conserved quantities. On the other hand, at the time of the deviations, there is a transfer of energy between the probe and the planet. For example, the probe receives kinetic energy from Jupiter, so Jupiter slows down, but, given the mass of the planet compared to the probe, it is undetectable. At the moment of the interaction between the probe and the planet the mechanical energy and the angular momentum of

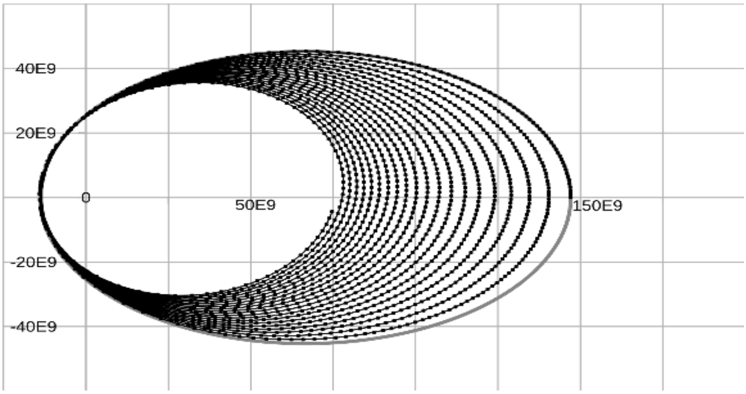
the two bodies are conserved in the heliocentric reference frame.
To perfectly chain the two slings, without trajectory correction, a very fine adjustment of the initial conditions is required.
Concerning the step, it would be very expensive, in computing time and quantity of data to be memorized, if we maintained it constant. We have chosen a step of two days in the interplanetary space, of two hours at the approach, and of one minute at the slings level. The step is automatically adapted according to the distance to the planet and the speed of the probe. It is an adaptation operated "by hand", the method is not general but adapted to this particular problem. There are adaptive step by step methods. The most classical one consists in estimating the local error at each step. Also, we could adapt the step according to the radius of curvature and the speed of the probe. Indeed, any trajectory is locally, in the vicinity of a point, contained in a plane, called the osculating plane. And the particle moves locally according to an osculating circle of radius *R*, called radius of curvature. We could consider that, at each step, in order to follow the curvature, the particle should not cover a too large portion of the circle. An adaptive method could impose an angular step *Δθ* rather than a temporal step *h*:

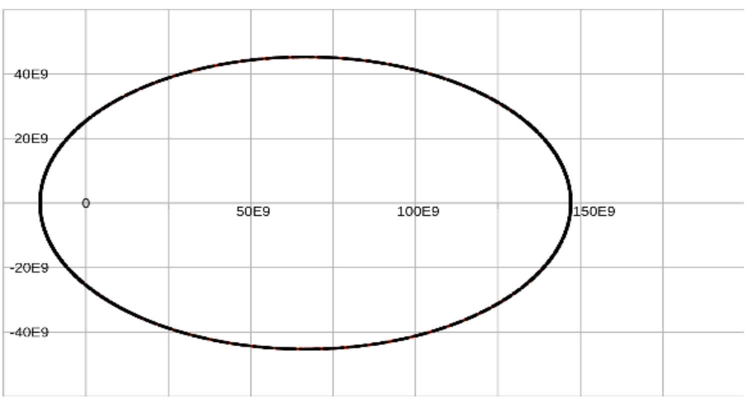
$$\vec{a}=\frac{d\,v}{dt}\vec{u}_t+\frac{v^2}{R}\vec{u}_n \;\; => \;\; R=\frac{v^2}{\sqrt{a^2-\left(\frac{dv}{dt}\right)^2}} \;\; \text{with} \;\; \frac{d\,v}{dt}=\frac{\vec{a}\cdot\vec{v}}{v}$$

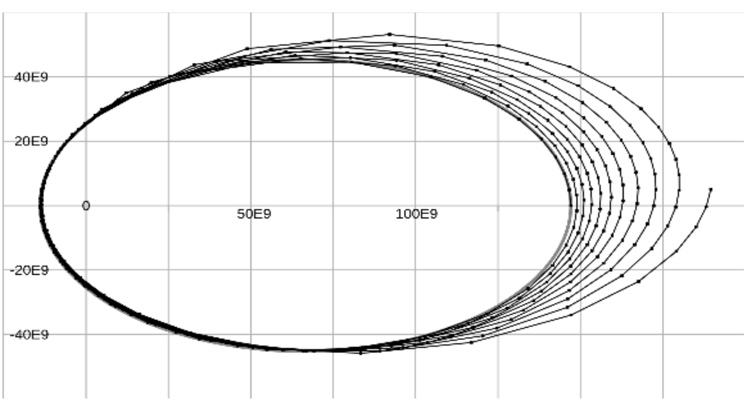
and finally $h=\frac{R\Delta\theta}{v}$.

We did not implement this method, however, using these formulas, we have calculated $\Delta\theta$ at each step to control good tracking and we have angles at most 1.5 degrees. First of all, this adaptive method would not be suitable for a 3D motion where the osculating plane can change permanently (as for the helical motion of a charged particle in a magnetic field), moreover, the step from n to $n+1$ is evaluated from the situation at t_n. But many unexpected things can happen between t_n and t_{n+1}. For example, during a step of two days, a planet or an asteroid can appear from "nowhere". We did not do it, but as here we know the position of the planets, we could anticipate, at each step, the next step. This would make the calculation lighter.

Comparison of the constant step RK4 with the variable step RK4: For our resistance test, we took the Earth-Sun system with a perigee starting velocity of only 12.5 km/s. For all three numerical experiments T=1825 days ≃ 5 yrs.
Top: h=1 day, n=1825. Unstable.
Middle: variable h, Δθ=3°, n=1533. Stable with lower n.
Bottom: variable h, Δθ=11°, n=353. Unstable.


40E9
20E9
0
50E9
150E9
-20E9
-40E9


40E9
20E9
0
50E9
100E9
150E9
-20E9
-40E9


40E9
20E9
0
50E9
100E9
-20E9
-40E9

The RK4 scheme for *Voyager 1* with its 12 degrees of freedom:

$$\frac{d x_V}{dt}=A(x_V, y_V, v_{xV}, v_{yV}, x_J, y_J, v_{xJ}, v_{yJ}, x_S, y_S, v_{xS}, v_{yS})=v_{xV}$$

$$\ldots$$

$$\frac{d v_{xV}}{dt}=C(DOFs)=-GM\frac{x_V}{(x_V^2+y_V^2)^{3/2}}-GM_J\frac{x_V-x_J}{((x_V-x_J)^2+(y_V-y_J)^2)^{3/2}} \quad \ldots$$

$$\ldots \ldots\ldots$$

$$A_1=A(x_{Vn},\ldots,v_{ySn}) \quad \ldots \quad L_1=L(x_{Vn},\ldots,v_{ySn})$$

$$A_2=A\left(x_{Vn}+\frac{h}{2}A_1,\ldots\right) \quad \cdots \quad L_4=L(\ldots,v_{ySn}+hL_3)$$

$$x_{Vn+1}=x_{Vn}+\frac{h}{6}(A_1+2A_2+2A_3+A_4)$$

$$\ldots \quad \ldots$$

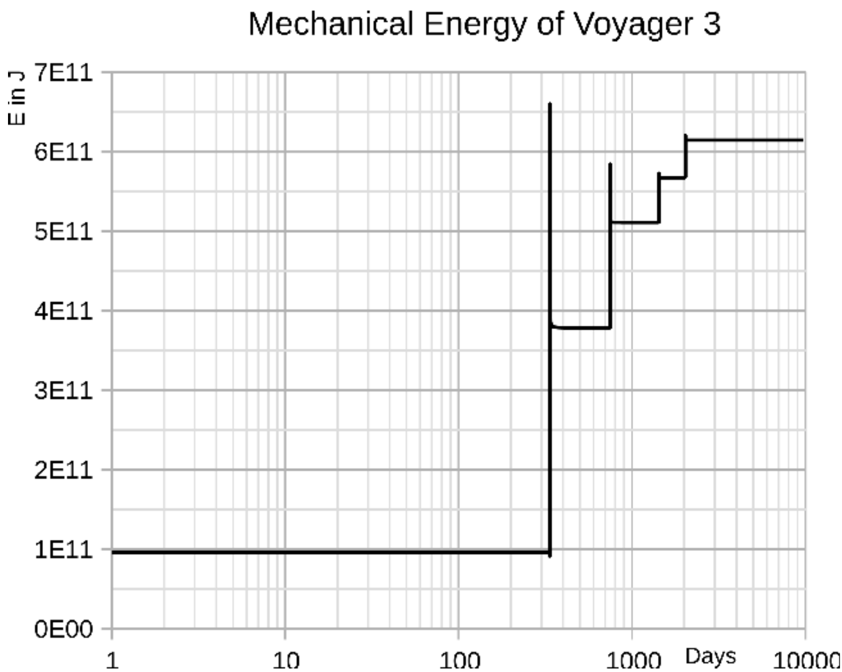
$$v_{ySn+1}=v_{ySn}+\frac{h}{6}(L_1+2L_2+2L_3+L_4)$$

The functions have been placed in a macro in Basic language.

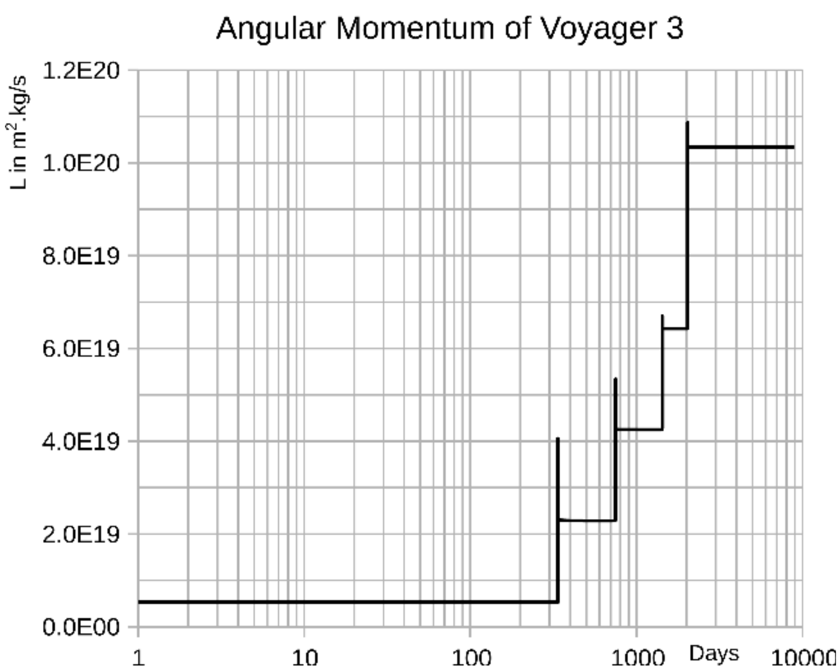
File: Voyager-1-RK4.ods

4 - *The Voyager 3 Project*:

The route of the probe and the speed curve have already been given during the conference page 288. We also note an excellent agreement with the file FrondesVoyager3.ods After 27 years, we have an interstellar speed of 39,300 m/s or 141,000 km/h.



Mechanical Energy of Voyager 3
E in J
7E11
6E11
5E11
4E11
3E11
2E11
1E11
0E00
1
10
100
1000
Days
10000



Angular Momentum of Voyager 3
L in m2.kg/s
1.2E20
1.0E20
8.0E19
6.0E19
4.0E19
2.0E19
0.0E00
1
10
100
1000
Days
10000

We made a trajectory correction of 331 m/s at the periastron of Uranus.

5. Calculation of propellant masses *Exercise p320.*

1 - The system {astronaut + wrench} is isolated. It follows from this that the conservation of momentum in a galilean frame of reference. Let's take the barycentric reference frame, initially the whole is motionless, and then each part goes in the opposite direction. We must throw the wrench, as hard as possible, in the opposite direction of the station:

$$\vec{0}=m\vec{V}+M\vec{v} \quad \text{and} \quad v=\frac{m}{M}V\simeq 0.36\ km/h$$

On the other hand, mechanical energy is not conserved. In this case the internal forces between the different parts of the system also intervene. The work of the internal forces is null for a solid where the distances between the different parts remain constant. The mechanical energy counts the macroscopic potential and kinetic forces. Here the kinetic energy is initially zero and then increases. The kinetic energies of the wrench and the astronaut have different values:

$$E_{c\ wren.}=\frac{1}{2}mV^2 \quad \& \quad E_{c\ astro}=\frac{1}{2}Mv^2=\frac{1}{2}\frac{m^2}{M}V^2=\frac{m}{M}E_{c\ wren.}$$

The momentum is proportional to the velocity, while the kinetic energy varies as the square.

2 - For a rocket the mass is ejected continuously and the mass of the rocket varies over time.

Nevertheless the principle is the same for each time interval: $0 = dm\, v_e + m\, dv$

and $\Delta v = \int dv = -\int \frac{dm}{m} v_e = -v_e \int \frac{dm}{m} = v_e \ln\left(\frac{m_i}{m_f}\right)$.

We well find the rocket equation.
Flow of propellants: $D = dm/dt = -ma/v_e$.
For the photonic rocket: $E = pc = mc^2$ then $p = mc$, where *m* is the mass of matter and antimatter which are annihilated. Finally we replace the ejection velocity v_e by *c*. After calculation for non-relativistic velocities:

$$m_M + m_{AM} = m_U \left(e^{\frac{\Delta v}{c}} - 1\right) \simeq m_U \beta = m_U \frac{d_{al}}{T_a}$$

In the first sum we have the masses of matter and antimatter that annihilate, and if we add the efficiency *r=0.1*:

$$m_{AM} \simeq \frac{m_U}{2r} \frac{d_{al}}{T_a}$$ (m_U : payload mass)

For 35,000 years we find 460 grams, but in fact we need twice less because the probe has already acquired half of the speed by slingshot effect.
If the dilation of time cannot be neglected it is necessary to consider the proper time because we are in the reference frame of the rocket. But even for a 50 years trip, time can still be considered, in a good approximation, as absolute.

3 - The voyage to Proxima at constant acceleration gives, for an artificial terrestrial gravity, a halfway speed of 95 % of *c* and a γ of 3. Clearly we cannot do without special relativity. We reason in the proper reference frame and during dτ the velocities remain classical. Proper acceleration and momentum conservation:

$$a=\frac{dv}{d\tau}=g, \quad 0=dm\,c+m\,g\,d\tau \quad \text{and} \quad D=\frac{dm}{d\tau}=-\frac{mg}{c}.$$

We integrate: $$\tau=\int d\tau=-\frac{c}{g}\int\frac{dm}{m}=\frac{c}{g}\ln\left(\frac{m_i}{m_f}\right).$$

Then: $$m_{AM}=m_U\frac{e^{\frac{g\tau}{c}}-1}{2}$$

The kinematic study gives $\tau\simeq$6.84 yrs for a round trip (chapter: *Accelerated motion*). For a photonic reactor with 100% efficiency:

One way: $m_{AM}\simeq 18\,m_U$.

Round trip: $m_{AM}\simeq 666\,m_U$.

The quantities of antimatter are important here. For the same travel time and the same payload, a manned trip at constant acceleration requires more energy than a trip at constant speed (as for the probe). Propellant used at the beginning produces an increase in speed which benefits the whole trip, whereas used a little before the halfway point it will hardly be used, because it will start, just after, the deceleration phase. To respect the tolerance of the human body to the g, while minimizing the quantity of propellants used, we can vary the average acceleration of the vessel:

Voyage with variable acceleration

1st period (speed up): average acceleration 2 *g*.

Activity: 12 hours at 1.2 *g*.

Sleep / Rest: 12 hours at 2.8 *g*.

2ème period (until turning over): av. acc. 0.3 *g*.

Activity: 8 hours at 0.9 *g*.

Sleep / Weightlessness: 16 hours at zero *g*.

$$\tau=\tau_1+\tau_2 \qquad D=D_1+D_2 \qquad D_1=\frac{c^2}{g_1}\left(ch\left(\frac{g_1\tau_1}{c}\right)-1\right)$$

$$\beta_{1max}=\sqrt{1-\frac{1}{\left(1+\frac{g_1 D_1}{c^2}\right)^2}} \qquad \tau_2=\frac{c}{2g_2}\ln\left(\frac{1+\beta_{max}}{1-\beta_{max}}\frac{1-\beta_{1max}}{1+\beta_{1max}}\right)$$

$$D_2=\frac{c^2}{g_2}\left(\frac{1}{\sqrt{1-\beta_{max}^2}}-\frac{1}{\sqrt{1-\beta_{1max}^2}}\right) \qquad \frac{m_i}{m_{1/2}}=e^{\frac{g_1\tau_1+g_2\tau_2}{c}}$$

If τ_1=0.5 yr then D_1=0.3 ly and $\beta_{1\,max}$=78%*c*. With β_{max}=88 %, D=2 ly and τ=1.6 yr. Significant fuel economy:

One way: $m_{AM}\simeq 8\,m_U$, Round trip: $m_{AM}\simeq 134\,m_U$.

The outward journey to Proxima lasts 3.2 years for the astronauts. The maximum speed is much lower: this allows to decrease the size of the front shield of the rocket which protects from the collisions with the particles of the interstellar medium. This medium is very diluted 10^{-21} kg/m^3 but at relativistic speeds the energetic contribution of the impacts is to be considered (Exo *Bouclier de protection* of the book of Semay).

4 - *Voyager 3 Project*: We will consider two cases, the one of the flyby, and the one of the orbiting.

Flyby: the speed of the probe must be increased by 4.8 km/s. Let's take 5 km/s to foresee also the corrections of trajectory:

$$\Delta v = v_e \ln\left(\frac{m_S + m_e}{m_S}\right) \quad \text{and} \quad m_e = m_S\left(e^{\frac{\Delta v}{v_e}} - 1\right)$$

NA: m_S=800 kg, v_e=4 km/s and $m_{propel.}$=2000 kg.

<u>Orbiting</u>: if the situation is symmetrical, we can double with Δv=10 km/s, half to accelerate and half to slow down.

We summarize the values in a table:

Δv	m_{probe}	$m_{propel.}$	m_{total}	Rockets
5 km/s	0.8 t	2 t	3 t	Ariane 6, Falcon 9, etc.
10 km/s	0.8 t	9 t	10 t	Ariane 6, Falcon Heavy, etc.
20 km/s	0.8 t	119 t	120 t	Saturn V, StarShip.
10 km/s	12 t	134 t	150 t	StarShip.

6. Planetary alignments

Exercise p321

1 - $\begin{cases} \theta_A(t) = \omega_A t + \theta_A(0) \\ \theta_B(t) = \omega_B t + \theta_B(0) \end{cases}$, origin of dates on an alignment

$\theta_A(0) = \theta_B(0)$, next alignment $t = T_{AB}$:

$$\theta_A(t) - \theta_B(t) = 2\pi = (\omega_A - \omega_B) T_{AB} = \left(\frac{2\pi}{T_A} - \frac{2\pi}{T_B}\right) T_{AB}$$

2 - $T_{TJ} = T_T T_J / (T_J - T_T) \simeq 1.092\ yrs \simeq 1\ yr\ 1\ m$.

After one revolution of the Earth, Jupiter will have rotated one twelfth of a revolution.

Next Sun-Earth-Jupiter alignments:

20/08/2021	27/09/2022	03/11/2023	07/12/2024	10/01/2026
10/02/2027	12/03/2028	12/04/2029	13/05/2030	15/06/2031

19/07/2032	25/08/2033	02/10/2034	08/11/2035	12/12/2036
14/01/2038	15/02/2039	16/03/2040	16/04/2041	17/05/2042
20/06/2043	24/07/2044	30/08/2045	07/10/2046	12/11/2047

3 - $T_{JS} = T_J T_S / (T_S - T_J) \simeq 19.86\ yrs \simeq 19\ yr\ 10\ m$

The two large gaseous planets are aligned every 20 years or so. The rotation of the Earth being much faster we will have, over the same period, a correct alignment with four bodies. For a quick search, we start by using an astronomy software (*Stellarium* / Situation : Sun / Ecliptic grid), and we refine with *Miriade*:

Sun-Earth-Jupiter-Saturn alignments

Date	18/07/2020	22/03/2040	20/11/2059	05/08/2080
Separat°	6°	11°	4°	4°

4 - $T_{UN} \simeq 171.47\ yrs$ and $T_{UN}/T_{JS} \simeq 8.6 \simeq 9$.

We could therefore have a suitable alignment every 171 years. In 2162, the Earth and the four giants are grouped on an angle of 60°. The previous alignment was in the years of the launch of the probes *Voyager 1* and *2*.

7. Motion of the stars

Exercise p322

1- $v_{t\alpha} = \mu_\alpha d_0$ $\quad v_{t\delta} = \mu_\delta d_0$ $\quad v_t^2 = v_{t\alpha}^2 + v_{t\delta}^2$

$$v_t = \mu d_0 \qquad \mu^2 = \mu_\alpha^2 + \mu_\delta^2$$

Particular units:

$$v_t(km/s) = \frac{10^{-3}}{3600} \times \frac{\pi}{180°} \times \frac{3.10^8}{10^3} \times \mu(mas/yr) \times d_0(ly)$$

$$v_t(km/s) = 1.454\times10^{-3}\ \mu(mas/yr)\ d_0(ly)$$

(same formulas along α and δ)

Speed of the star: $v=\sqrt{v_r^{\ 2}+v_t^{\ 2}}$

α Cen C : $\mu=3859\,mas/yr$ $\quad v_{t\alpha}=-23.33\,km/s$

$v_{t\delta}=4.75\,km/s$ $\quad v_t=23.81\,km/s$ and $v=32.56\,km/s$.

Over a century, we can consider the apparent motion constant and simply multiply:

$$\Delta\mu_\alpha\simeq\mu_\alpha\times\Delta t \quad \text{and} \quad \Delta\mu_\delta\simeq\mu_\delta\times\Delta t.$$

α Cen C : $\Delta\mu_\alpha\simeq-0h6'18''$ $\quad \Delta\mu_\delta\simeq0°1'17''$

$\alpha\simeq14h23'25''$ and $\delta\simeq-62°39'29''$

2- We note that the radial velocity, tangential velocities, and proper motions vary with time. At approach the radial velocity is negative, it cancels at the minimum distance, and becomes positive when the star moves away. Conversely, the tangential velocity is maximum at perihelion.

a- $\vec{v}=\dfrac{d\overrightarrow{SM}}{dt}=\overrightarrow{cst}$ then $\vec{d}=\overrightarrow{SM}=\vec{v}\,t+\overrightarrow{SM_0}$

$$d^2=d_0^{\ 2}+2\overrightarrow{SM_0}\cdot\vec{v}\,t+v^2t^2 \ \ \& \ \ d(t)=\sqrt{d_0^{\ 2}+2\,d_0\,v_{r0}t+v^2t^2}$$

b- Perihelion: $\dfrac{d\,d(t)}{dt}=0$ then $t_m=-\dfrac{d_0\,v_{r0}}{v^2}$ & $d_m=d_0\dfrac{v_{t0}}{v}$.

α Cen C : $d_m=3.10\,ly$ and $t_m=26{,}660\,yrs$.

c- We will go through the Cartesian coordinates to return to the spherical coordinates. For more coherence in the book, we use the spherical coordinate system used in physics. Few changes, nevertheless the notations are different from those of astronomy, and, the colatitude is preferred to the latitude (exercise page 169):

$$\theta=\frac{\pi}{2}-\delta \qquad \varphi=\alpha \qquad \vec{d}_0=\begin{pmatrix} d_0 \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} x_0=d_0\sin\theta_0\cos\varphi_0 \\ y_0=d_0\sin\theta_0\sin\varphi_0 \\ z_0=d_0\cos\theta_0 \end{pmatrix}$$

$$\vec{v}=\begin{pmatrix} v_{r0} \\ v_{\theta 0}=-v_{t\delta 0} \\ v_{\varphi 0}=v_{t\alpha 0} \end{pmatrix}=\begin{pmatrix} v_x=v_{r0}\sin\theta_0\cos\varphi_0+v_{\theta 0}\cos\theta_0\cos\varphi_0-v_{\varphi 0}\sin\varphi_0 \\ v_y=v_{r0}\sin\theta_0\sin\varphi_0+v_{\theta 0}\cos\theta_0\sin\varphi_0+v_{\varphi 0}\cos\varphi_0 \\ v_z=v_{r0}\cos\theta_0-v_{\theta 0}\sin\theta_0 \end{pmatrix}$$

$$\vec{d}=\begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix}=\begin{pmatrix} x=d\sin\theta\cos\varphi \\ y=d\sin\theta\sin\varphi \\ z=d\cos\theta \end{pmatrix}=\begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}t+\begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

$$\begin{cases} \tan\varphi(t)=\dfrac{v_y t+y_0}{v_x t+x_0}=\dfrac{y(t)}{x(t)} \\ \cos\theta(t)=\dfrac{v_z t+z_0}{d(t)}=\dfrac{x(t)}{d(t)} \end{cases}$$

$\underline{\alpha \text{ Cen C}}$ **:** $\theta_0=\frac{\pi}{2}-\delta_0=152.68°$ $\quad \varphi_0=\alpha_0=217.43°$

$$\vec{d}_0=\begin{pmatrix} x_0=-1.547\,ly \\ y_0=-1.184\,ly \\ z_0=-3.771\,ly \end{pmatrix} \quad \text{and} \quad \vec{v}=\begin{pmatrix} v_x=-9.44\,km/s \\ v_y=22.15\,km/s \\ v_z=21.90\,km/s \end{pmatrix}$$

$$\begin{cases} \tan\varphi_m=\dfrac{v_y t_m+y_0}{v_x t_m+x_0}\simeq-0.329 \\ \cos\theta_m=\dfrac{v_z t_m+z_0}{d_m}\simeq-0.589 \end{cases}$$

$$\begin{cases} \textit{Equatorial}: \\ \alpha_m\simeq 161°47' \\ \delta_m\simeq -35°59' \end{cases} \qquad \begin{cases} \textit{Ecliptic}: \\ \alpha_m\simeq 180°8' \\ \delta_m\simeq -39°46' \end{cases}$$

Positions of the stars at the arrival of the spacecraft

	Equatorial		d_0	$\mu_{\alpha 0}$	$\mu_{\delta 0}$	v_{r0}	t	d	v_{probe}	Equatorial		Ecliptic	
Star	$long._0$	$lat._0$	l.y.	mas/yr	mas/yr	km/s	yrs	l.y.	km/s	long.	lat.	long.	lat.
Alpha Centauri	219.9°	-60.8°	4.4	-3678	482	*-24.7*	27800	3.0	33	160.3°	-34.0°	177.5°	-38.6°
Ross 248	355.5°	44.2°	10.3	113	-1592	*-77.8*	36300	3.0	25	-0.1°	-28.7°	-12.4°	-26.1°
Gliese 445	176.9°	78.7°	17.1	748	481	*-111.7*	44300	3.3	23	-62.8°	17.1°	-56.6°	37.5°
Ross 128	176.9°	0.8°	11.0	608	-1223	*-31.0*	71100	6.3	27	208.7°	-46.4°	224.4°	-32.2°
Ross 154	282.5°	-23.8°	9.7	639	-194	*-10.5*	75000	7.5	30	-57.6°	-27.8°	-61.4°	-7.6°
Gliese 1061	54.0°	-44.5°	12.0	745	-374	*-20*	77000	7.8	30	95.9°	-50.2°	103.4°	-73.4°
Sirius	101.3°	-16.7°	8.6	-546	-1223	*-5.5*	84000	8.5	30	82.2°	-46.3°	74.6°	-69.4°
Gliese 682	255.6°	-44.3°	16.3	-706	-938	*-34.9*	85700	10.2	36	166.9°	-62.0°	209.0°	-58.5°
Wolf 1061	247.6°	-12.7°	14.0	-94	-1184	-21.6	86300	10.5	36	242.4°	-54.2°	251.2°	-32.5°
Kruger 60	337.0°	57.7°	13.0	-865	-462	*-33.9*	88700	6.3	21	-75.0°	10.7°	-72.2°	33.2°
Gliese 687	264.1°	68.3°	14.8	-321	-1270	*-28.8*	90000	10.5	35	252.3°	15.1°	248.4°	37.2°
Tau ceti	26.0°	-15.9°	11.7	-1730	855	*-16.6*	100000	12.6	38	-27.4°	13.6°	-20.1°	23.2°
Procyon	114.8°	5.2°	11.4	-717	-1035	*-3.2*	100000	12.5	37	94.7°	-22.4°	96.3°	-45.8°

Coordinates of 13 nearby stars for a slow interstellar travel. Six input data allow to compute the position of the stars over the time (rectilinear motion hypothesis). The radial velocity is the initial condition which can have the most uncertainties.
In italics the data prior to the one collected by the Gaia satellite.

Graphs on the following pages

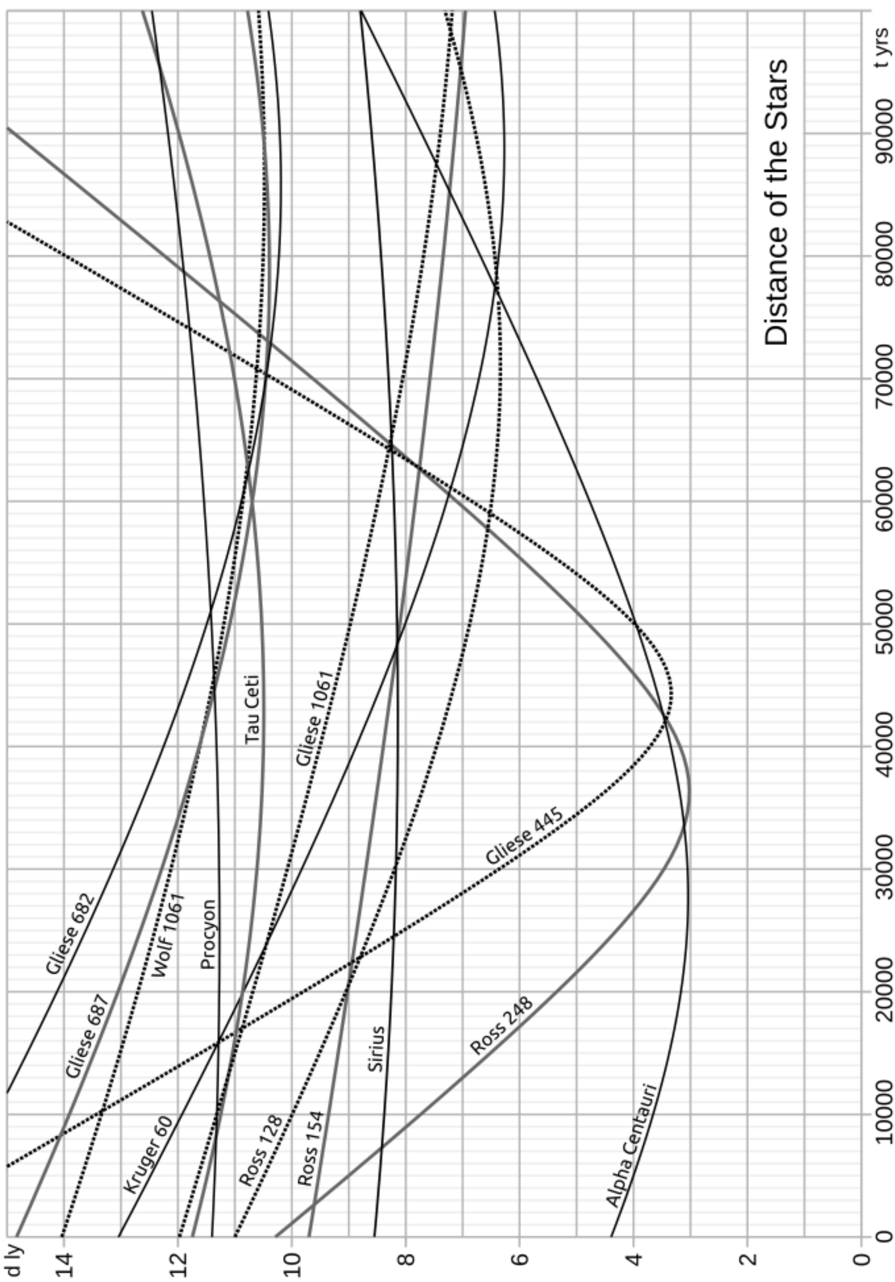
Distance of the stars:

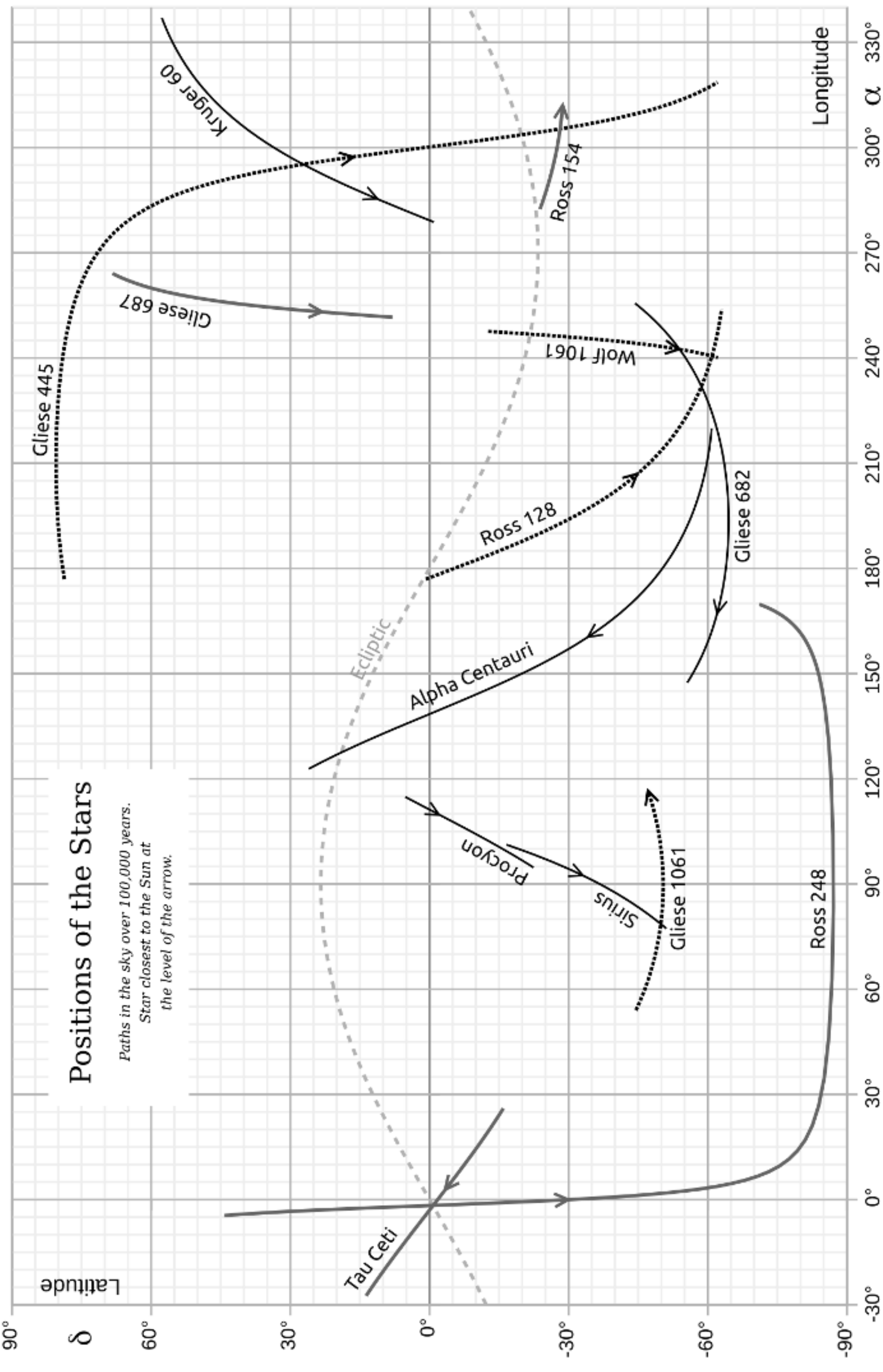
The graph represents the evolution over 100,000 years of the distance of thirteen stars accessible with a ship moving at a speed of less than 40 km/s.

The stars of Barnard and Teegarden do not appear. It is true that Barnard is only 6 ly away, and moreover, it is approaching us at high speed, and will be close in 10,000 years at only 3.8 ly. Nevertheless, the probe would have to go at 115 km/s to reach it at its perihelion, and then the star will move away at too high a speed. As for Teegarden, it is located at 12.6 ly and is moving away from us at high speed (positive radial velocity of 68 km/s).

Position of the stars :

Evolution from today to 100,000 years from now of the equatorial coordinates of nearby stars in the sky. The arrow indicates the position where the star is closest. The gray line represents the current position of the ecliptic. The celestial sphere is projected on the plane of the leaf of paper in Mercator projection, so near the poles, the trajectories appear stretched.


Distance of the Stars
d ly
t yrs
0
2
4
6
8
10
12
14
0
10000
20000
30000
40000
50000
60000
70000
80000
90000
Gliese 682
Gliese 687
Wolf 1061
Procyon
Kruger 60
Ross 128
Ross 154
Sirius
Tau Ceti
Gliese 1061
Gliese 445
Ross 248
Alpha Centauri



Positions of the Stars
Paths in the sky over 100,000 years.
Star closest to the Sun at
the level of the arrow.
Latitude
δ
Longitude
α
90°
60°
30°
0°
-30°
-60°
-90°
-30°
30°
60°
90°
120°
150°
180°
210°
240°
270°
300°
330°
Ecliptic
Tau Ceti
Ross 248
Gliese 1061
Sirius
Procyon
Alpha Centauri
Ross 128
Gliese 682
Gliese 445
Wolf 1061
Gliese 687
Ross 154
Kruger 60

8. Can a pair of primordial black holes be used as a stargate?

Exercise p329.

1 - For the planet Mars, the slingshot effect is weak because the mass is small and the minimum approach distance is limited by the radius of the planet. With a pair of PBHs, we have, at the same time, a high orbital speed and an approach distance that can approach zero. Let's express the characteristics of the system with two bodies of equal masses in circular orbits. We take the results from the end of the exercise *Sling effect*:

$$\mu=\frac{M_1 M_2}{M_1+M_2}=\frac{M}{2} \quad \text{and} \quad \frac{a^3}{T^2}=\frac{GM}{2\pi^2}$$

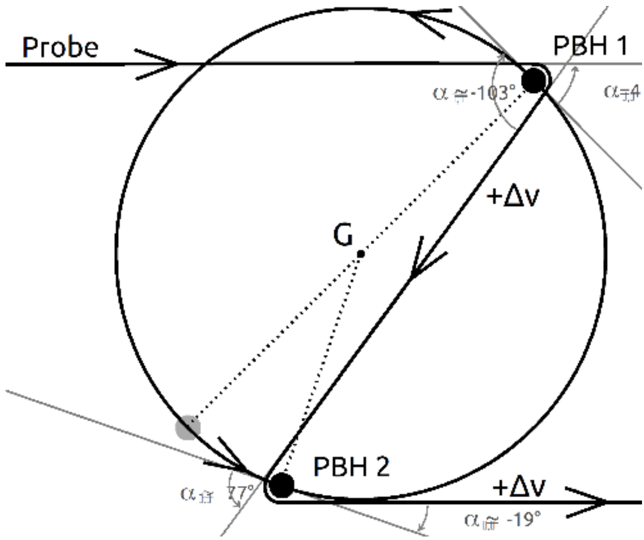
Distance between the black holes: $d=a=GM$

$$T=\sqrt{\frac{2\pi^2 d^3}{GM}} \quad \text{and} \quad v_{fi}=\frac{2\pi a}{T}=\sqrt{\frac{2GM}{d}}$$

v_{fi} is the speed of the fictitious particle and $v=v_{fi}/2$.

d	380 000 km	10 000 km	1000 km	1 km	10 m
T	8.5 days	53 min	1min40s	3 ms	3 µs
v	1.6 km/s	36 000 km/h	113 000 km/h	0.3 % of c	**3 % of c**

<u>d=10 000 km</u>: We arrive on the first PBH with a speed of 100,000 km/h (*Voyager 3* without propellant boost). Let us take the case where the probe arrives at 45° with the trajectory of the first PBH:



A pair of transneptunian Primordial Black Holes to accelerate a probe.

The probe first goes backwards, it is deviated of 120°, and, has a speed of 138,000 km/h. It arrives at the second PBH in less than 4 minutes. During this time this one turned of 26°. The second slingshot propels the probe forward with a speed of 188,000 km/h. Proxima is reached in 18,000 years. The minimum approach distance for the second PBH is 66 km. This is much larger than the Schwarzschild radius but nevertheless the tidal forces are already important. We will estimate in the next question a minimum approach distance for our PBHs of more than 60 km so that a human can support the tidal forces.

For a probe, we can approach black holes a little faster. Even if a probe can resist much more severe constraints than a human being, we will still be limited by the tidal forces that could destroy the probe.

d=10 m: Here we reach speeds close to *c*. We could exceed 10% of the speed of light and reach Proxima in less than 50 years. But a manned mission is impossible because of the tidal forces, and even for a probe it is not possible, the distance between the PBHs is of the order of the size of the probe...

In conclusion: if such pairs exist they would be good accelerators for missions, but not stargates.

2 -

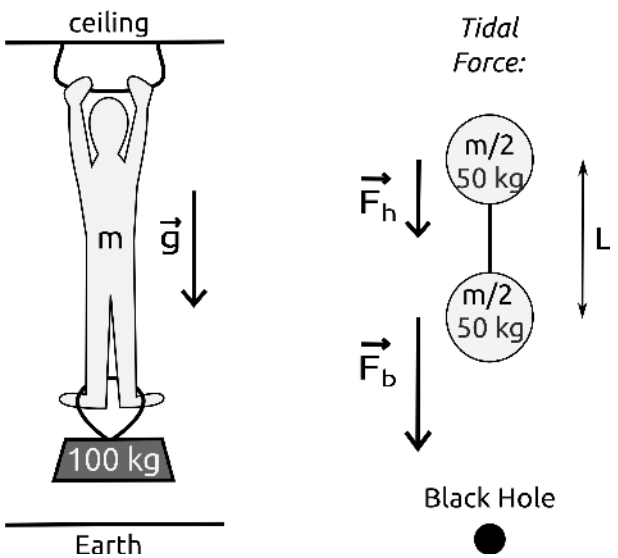


Illustration of tidal forces: Imagine yourself hanging from the ceiling with your own mass suspended at your feet. You would only hold on for a short time before letting go! In a spaceship, when going around a black hole, there should be no forces on you, because you are in free fall. As in the international space station where the tidal forces are not perceptible by the occupants in weightlessness. But when you are too close to a massive star, the difference in gravitational force between your feet, *directed towards the attracting star, and your head becomes non-negligible: the gravitational field can no longer be considered as uniform. For example, for Saturn, a gas giant, too close to it, the natural satellites can no longer exist and are crushed into rings. We have modeled the astronaut by two masses one meter apart. The parts of his body closer to the star undergo a higher gravitational force.*

Calculation of the differential tidal force:

$$F_b - F_h = F_{tidal} = mg = \frac{G M_{PBH} m/2}{r_{min}^2} - \frac{G M_{PBH} m/2}{(r_{min}+L)^2}$$

L is very small in front of r_{min} then:

$$F_{tidal} = \frac{G M_{PBH} m}{2 r_{min}^2}\left[1-\left(1+\frac{L}{r_{min}}\right)^{-2}\right] \simeq \frac{G M_{PBH} m}{2 r_{min}^2}\left[1-\left(1-2\frac{L}{r_{min}}\right)\right]$$

and $$r_{min} = \sqrt[3]{\frac{G M_{PBH} L}{g}} \simeq 60\,km.$$

9. Antiproton-proton collision

Exercise p330.

1 - In the reference frame of the center of inertia, the total impulse is zero and the two protons arrive one in front of the other with the same opposite velocities. After collision, at the threshold for pair creation, we have four particles at rest: the two initial protons and the proton/antiproton pair.

As the total energy is conserved:

$$2E = 4E_0,\quad 2m\gamma c^2 = 4mc^2,\quad \gamma = 2 \ \&\ \beta = \frac{\sqrt{3}}{2} \simeq 87\%$$

Back in the laboratory reference frame, we calculate the velocity of the proton on its target:

$$\beta_{lab} = \frac{\beta+\beta}{1+\beta^2} = \frac{4\sqrt{3}}{7} \simeq 99\% \quad \text{and} \quad \gamma_{lab} = 7.$$

$$E = \gamma E_0 = E_k + E_0 \quad \text{and} \quad E_{k\ min} = 6E_0 \simeq 5.63\ GeV$$

2 - To begin with, a very energetic antiproton is difficult to trap, hence the interest in first slowing it down. If it first encounters a target and creates a pair, the antiproton thus created will be much less energetic. Its energy at the threshold in the laboratory :

$$E = \gamma E_0, \quad \gamma = 2 \quad \text{and} \quad E_k = E_0 \simeq 1\,GeV < 6E_0.$$

We obtain antiprotons of kinetic energy six times less. So we could use the p's of kinetic energy above 6 GeV to create $\bar{\text{p}}$'s. We could create them in large quantities, because protons have a flux 10,000 times higher than $\bar{\text{p}}$'s.

10. Helical motion

Exercise p330.

1 - $\begin{cases} \rho(t) = r \\ \theta(t) = \omega t \\ z(t) = v_z t \end{cases} \qquad z = \frac{v_z}{\omega}\theta$

2 - $\begin{cases} \dot{x}=-\omega r\sin\omega t \\ \dot{y}=\omega r\cos\omega t \\ \dot{z}=v_z \end{cases}$ $\begin{cases} \ddot{x}=-\omega^2 r\cos\omega t \\ \ddot{y}=-\omega^2 r\sin\omega t \\ \ddot{z}=0 \end{cases}$ $\vec{a}\cdot\vec{v}=0 \quad \vec{a}\propto\overrightarrow{MH}$

3-4 - $v=\sqrt{\omega^2 r^2+v_z{}^2}=cst \qquad \frac{dv}{dt}=0 \qquad a=\omega^2 r=cst=\frac{v^2}{R}$

$$R=r+\frac{v_z{}^2}{\omega^2 r} \qquad p=\frac{2\pi v_z}{\omega} \qquad R=r+\frac{p^2}{4\pi^2 r}$$

5 - $l=\int_{\theta=0}^{\theta=2\pi}\sqrt{r^2 d\theta^2+dz^2}=2\pi\sqrt{r^2+\frac{v_z^2}{\omega^2}}=2\pi r\sqrt{1+\frac{p^2}{4\pi^2 r^2}}$

$$v_\parallel=v_z \qquad v_\perp=\omega r \qquad v^2=v_\perp^2+v_\parallel^2 \qquad r=\frac{v_\perp^2}{v^2}R=\cos^2\alpha R$$

$$l=2\pi\frac{v_\perp^2}{v^2}R\sqrt{1+\frac{v_\parallel^2}{v_\perp^2}}=2\pi R\frac{v_\perp}{v}=2\pi r\frac{v}{v_\perp}$$

$$l=2\pi R\cos\alpha=\frac{2\pi r}{\cos\alpha}$$

These last two expressions depend only on the geometric quantities r and α, they depend only on the shape of the trajectory and not on the speed at which it is traveled. These formulas are therefore also true for non-uniform helical motion.

11. The magnetosphere

Exercise p331.

1 - Magnetic force: $\vec{f}=q\vec{v}\wedge\vec{B}$.

Force power: $P=\vec{f}\cdot\vec{v}$ then $P=0$.

Kinetic power theorem: $P=\frac{dT}{dt}$

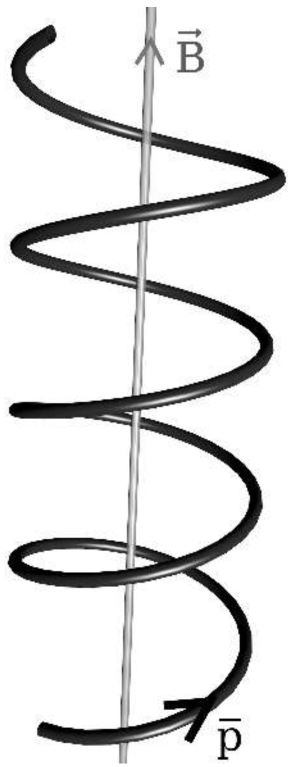
(*E* or *T* because the energy of mass is constant).

So $T=(\gamma-1)mc^2=cst$ and $v=cst$.

2 - Helical trajectory

Solving the system of differential equations:

$$m\gamma\frac{d\vec{v}}{dt}=q\vec{v}\wedge\vec{B} \qquad \vec{v}=\vec{v}_{\perp}+\vec{v}_{\parallel} \qquad (\gamma \text{ is here constant})$$



Motion, projected in a plane perpendicular to $\vec{B}$, circular of radius $R=\frac{m\gamma v_{\perp}}{|q|B}$.

A negative particle rotates in the trigonometric direction for a forward field $\vec{B}$.

Cyclotron angular frequency:

$$\omega=\frac{|q|B}{m\gamma}.$$

Motion, projected along the direction of $\vec{B}$, uniform.

Constant pitch helix.

Charged particles wrap around the magnetic field lines.

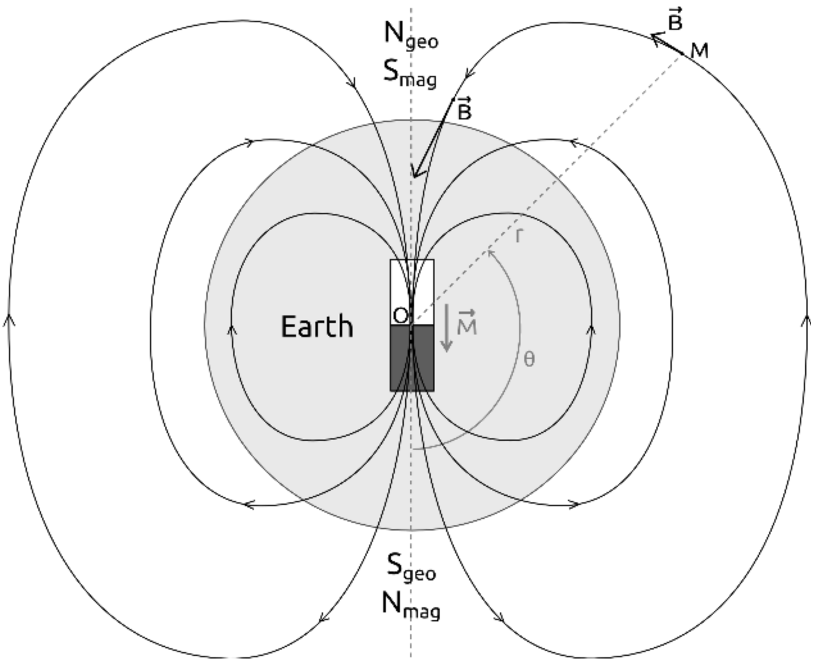
3 - Magnetic field lines flow from North to South (magnetic poles). The North of a compass is attracted to the Earth's magnetic South located not far from the geographic North (historical convention of a magnetic "North").

The magnetic flux is conserved: $\Phi=\oiint \vec{B}\cdot\vec{dS}=0$.

Consequence: by conservation of the flux on a field tube, the magnetic field is more intense when the field lines tighten.

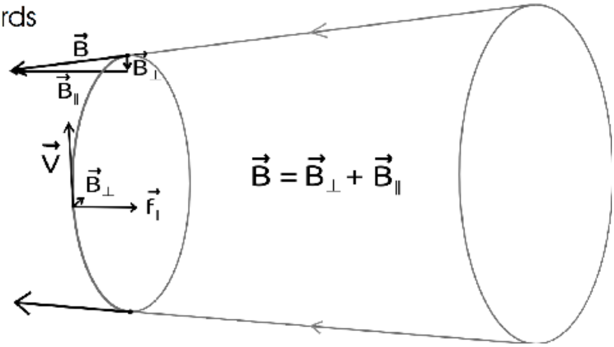
Components of $\vec{B}$ in spherical coordinates:

$$B_r = \frac{\mu_0}{4\pi} M \frac{2\cos\theta}{r^3}, \quad B_\theta = \frac{\mu_0}{4\pi} M \frac{\sin\theta}{r^3} \quad \text{and} \quad B_\varphi = 0.$$



$$\mu_0 = 4\pi \times 10^{-7}\, m.T/A \quad \text{and} \quad M = 7.7 \times 10^{22}\, A.m^2$$

4 - When the lines of a tube tighten, the field is no longer homogeneous and a force appears which pushes the particle towards the zone of weak field:

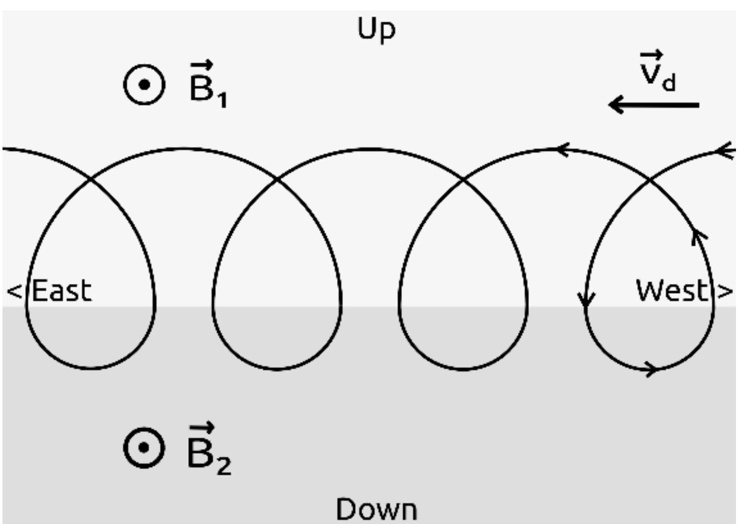


$$\vec{f} = q\,\vec{v} \wedge (\vec{B}_\perp + \vec{B}_\parallel) = \vec{f}_\perp + \vec{f}_\parallel \quad \text{and} \quad \vec{f}_\parallel = q\,\vec{v} \wedge \vec{B}_\perp.$$

The charged particles, trapped in the magnetosphere, go back and forth between the poles (to give some orders of magnitude, an electron can make a round trip in one second, for protons and antiprotons it is longer: a few tens of seconds. It depends on the energy of the particle and its distance from the Earth).

5 - The field lines weaken with altitude. The field is therefore more intense downwards with a smaller radius of curvature. Our two-zone model, combining half circles of different radii, gives an average eastward drift velocity for a negatively charged particle:

$$v_d = \frac{D_1 - D_2}{\frac{T_1}{2} - \frac{T_2}{2}} = 2v\frac{B_2 - B_1}{B_1 + B_2}$$



Drift of a charged particle in a non-uniform field. Electrons, protons, and antiprotons drift around the Earth.

6 - <u>Trap</u>:

a- $\vec{B}=B_r\vec{u}_r+B_\theta\vec{u}_\theta+B_\varphi\vec{u}_\varphi=B_x\vec{i}+B_y\vec{j}+B_z\vec{k}$

Without going into technical details:

$$\vec{u}_r=\sin\theta\cos\varphi\,\vec{i}+\sin\theta\sin\varphi\,\vec{j}+\cos\theta\,\vec{k}$$
$$\vec{u}_\theta=\cos\theta\cos\varphi\,\vec{i}+\cos\theta\sin\varphi\,\vec{j}-\sin\theta\,\vec{k}$$
$$\vec{u}_\varphi=-\sin\varphi\,\vec{i}+\cos\varphi\,\vec{j}$$

Then: $\vec{B}(x,y,z)=\dfrac{\mu_0 M}{4\pi r^5}\left[3xz\,\vec{i}+3yz\,\vec{j}+(2z^2-x^2-y^2)\vec{k}\right]$

with $r=\sqrt{x^2+y^2+z^2}$

b- $$\frac{d\vec{v}}{dt}=\frac{q}{m}\sqrt{1-\frac{v^2}{c^2}}\,\vec{v}\wedge\vec{B}=\frac{q}{m\gamma}\begin{pmatrix}v_x\\v_y\\v_z\end{pmatrix}\wedge\begin{pmatrix}B_x\\B_y\\B_z\end{pmatrix}$$

$$\begin{cases}\dot{v}_x=\dfrac{q}{mc}(v_yB_z-v_zB_y)\sqrt{c^2-v_x^2-v_y^2-v_z^2}\\[2ex]\dot{v}_y=\dfrac{q}{mc}(v_zB_x-v_xB_z)\sqrt{c^2-v_x^2-v_y^2-v_z^2}\\[2ex]\dot{v}_z=\dfrac{q}{mc}(v_xB_y-v_yB_x)\sqrt{c^2-v_x^2-v_y^2-v_z^2}\end{cases}$$

$$B_x(x,y,z)\qquad B_y(x,y,z)\qquad B_z(x,y,z)$$

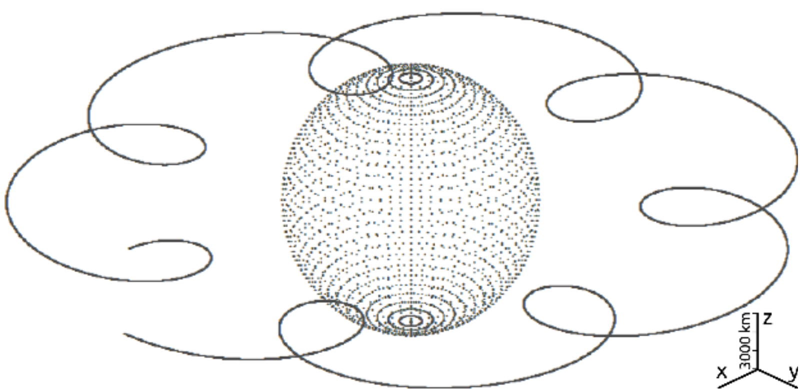
c- $$\begin{cases}\dot{x}=v_x=A(v_x) & A_1\ \ A_2\ \ A_3\ \ A_4\\ \dot{y}=v_y=B(v_y) & B_1\ \ B_2\ \ B_3\ \ B_4\\ \dot{z}=v_z=C(v_z) & C_1\ \ C_2\ \ C_3\ \ C_4\\ \dot{v}_x=D(x,y,z,v_x,v_y,v_z) & D_1\ \ D_2\ \ D_3\ \ D_4\\ \dot{v}_y=E(x,y,z,v_x,v_y,v_z) & E_1\ \ E_2\ \ E_3\ \ E_4\\ \dot{v}_z=F(x,y,z,v_x,v_y,v_z) & F_1\ \ F_2\ \ F_3\ \ F_4\end{cases}$$

d-

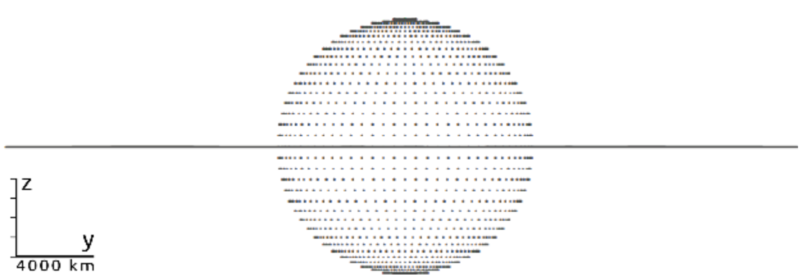
• ***Case of an antiproton of 2 GeV trapped in the equatorial plane:*** We will practice with this particular case where the motion is plane. Initially the antiproton is placed at 20,000 km from the center of the Earth and has a speed of 95% of *c* directed towards the East.

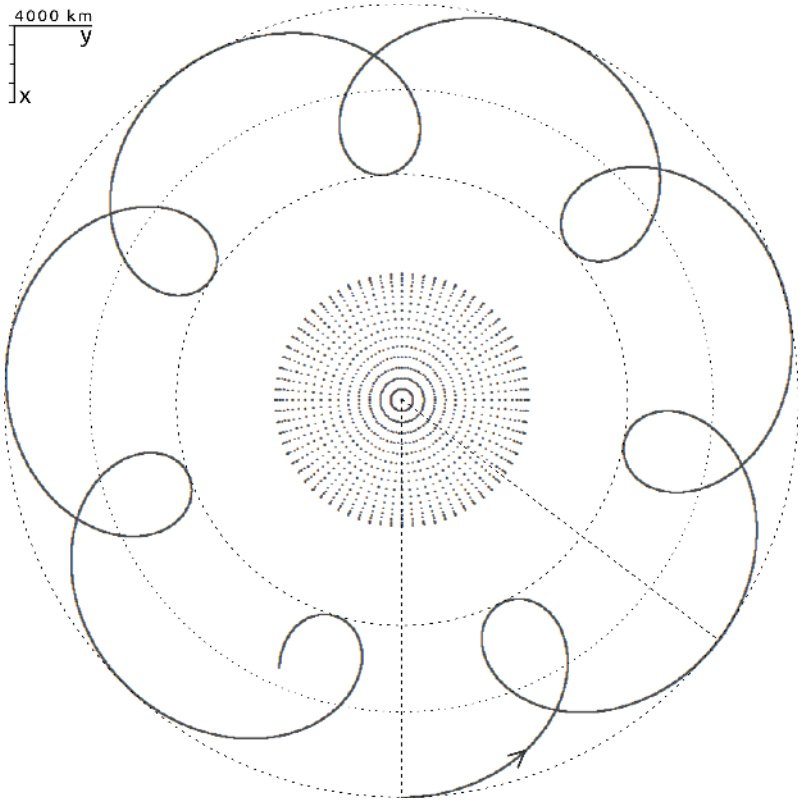
-Curves : **during** T=0.8 s with n=500,000 iterations:

Perspective:



Side view:





-Characteristics**:** The particle executes its cyclotron rotation while drifting towards the East. The antiproton goes around the Earth in a little more than 0.8 seconds and performs, in the same time, about 7 cyclotron rotations. On a cyclotron rotation the angle varies of 53° and it takes 0.12s:

$$T_{cyclo} \simeq 0.12 \text{ s} \quad \text{and} \quad T_{drift} \simeq 0.82 \text{ s}.$$

Let us find the order of magnitude of the cyclotron period with our formulas. First, the expression of the dipole magnetic field is simplified in the equatorial plane:

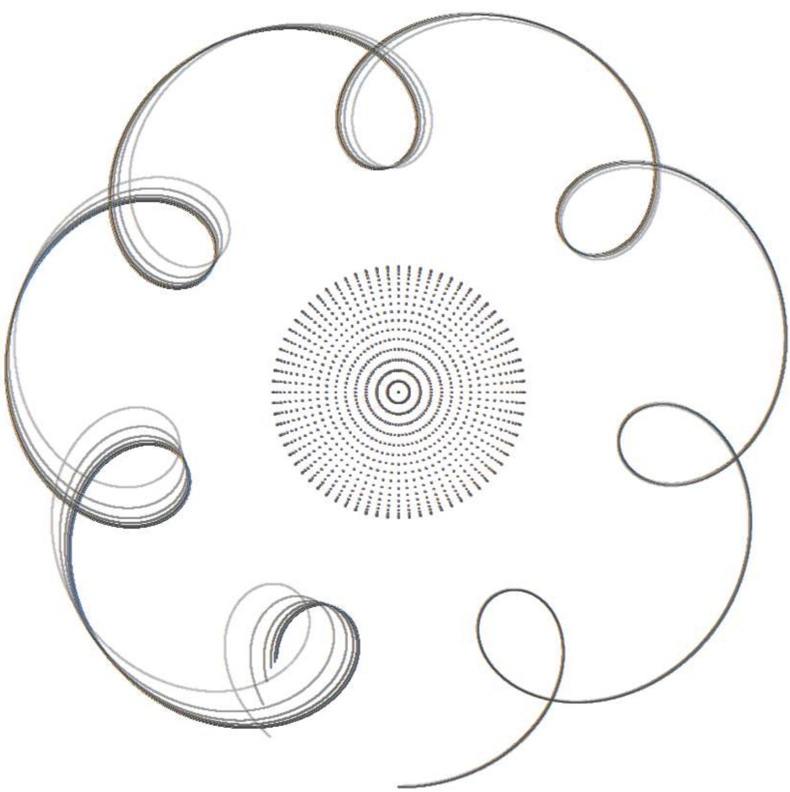
$$\vec{B}(x, y, 0) = -\frac{\mu_0 M}{4\pi r^3}\vec{k} \quad \text{with} \quad \frac{\mu_0 M}{4\pi} = 7.7 \times 10^{15} T . m^3$$

The trajectory is between $r_{min} \simeq 11{,}400$ km & $r_{max} \simeq 20{,}000$ km with $r_{avg} \simeq 15{,}700$ km. Hence $B_{max} \simeq 5.2 \times 10^{-6}$ T, $B_{min} \simeq 0.96 \times 10^{-6}$ T and $B_{avg} = 2.0 \times 10^{-6}$ T:

$$T_{cyclo} \simeq 2\pi \frac{m\gamma}{e B_{avg}} \simeq 0.10\, s$$

which is correct considering the great inhomogeneity of the magnetic field. The cyclotron frequency variations are greater than a factor of 5 between perigee and apogee. If the particle went around the Earth with a perigee of 20 000 km, it would take 0.44 s.

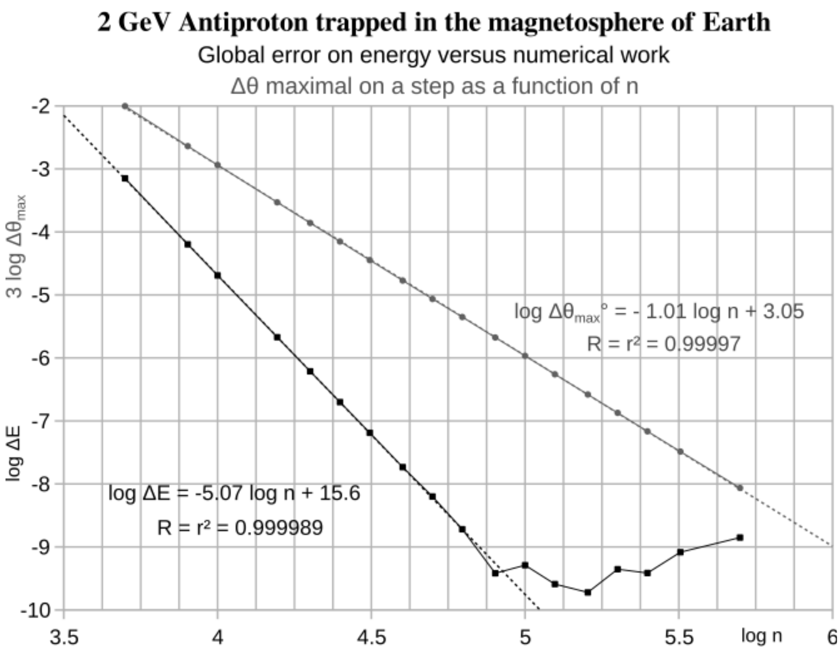
-Trajectory stability: We compute trajectories for increasing *n* and observe if they tend towards a stable trajectory.



The simulations converge: in black n=500,000, and, with each halving of the numerical work, in lighter gray each time, n=250,000, n=125,000, n=62,500, n=31,250, and n=15,625.

-Energy error:

The energy of a particle is constant in a magnetic field. We plot the error on the energy as a function of the number of iterations.

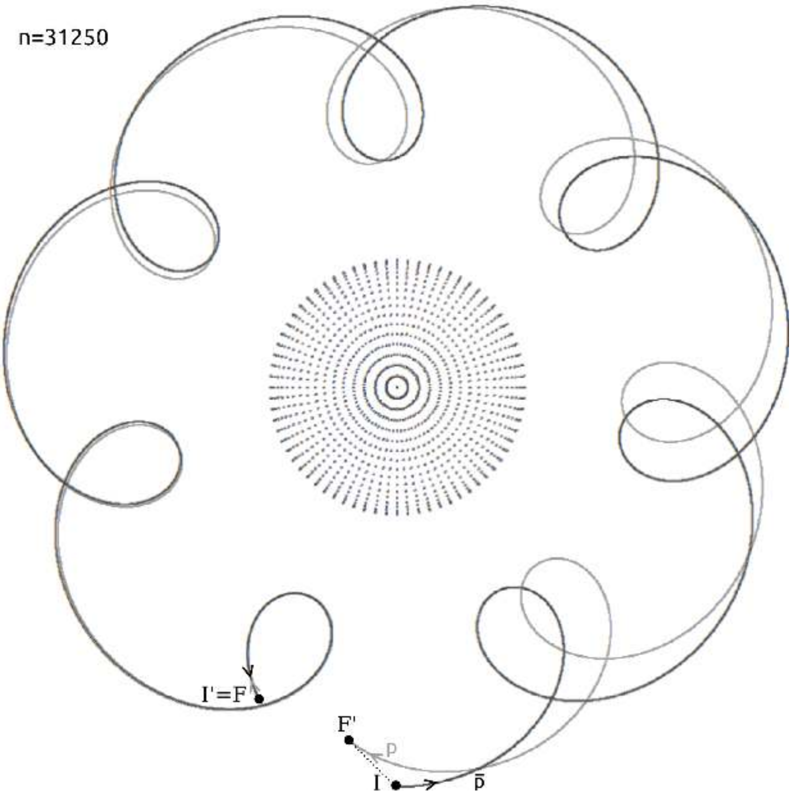


The behavior is excellent, the energy variations decrease exponentially and rapidly with work. We have a very regular decrease (correlation coefficient close to 1) and without accidents; we are reassured about the convergence. The hazardous variations in high work are normal, they are due to rounding errors. Indeed, in the program we have numbers with a precision of 14 significant digits, but for a precision of 10^{-10} MeV and an energy of 2000 MeV, the maximum precision is precisely *reached. In gray, the maximum angle variation over one step. We note this value reached at the end of the calculation on the last turn where the curvature is important. Here again the result is very good.*

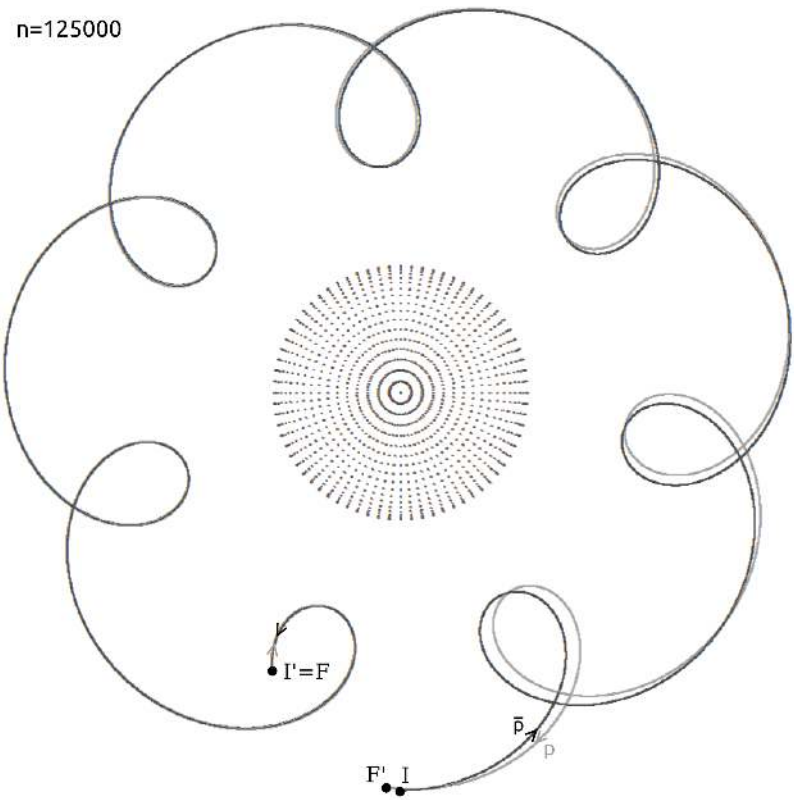
-The antiparticle test: Our trajectory starts from an initial point I to reach a final point F. We now consider another particle of the same mass and opposite charge which starts with the final conditions of the first one I'=F to reach its final point F'. The particle of opposite charge leaves from the same position with a speed in the opposite direction. Logically the antiparticle must go back and perfectly resume the path in the other direction to end up in F'=I. The force is the same and the change of sign of the velocity is like going back in time. The trajectory not being exact we can estimate the difference with the ideal trajectory.

Evolution of the distance between I and F':

T=0.8 s	step	n	$d_{F'I}$
RK4	h= 25.6 µs	31,250	3,240 km
RK4	h= 6.4 µs	125,000	688 km
RK4	h= 1.6 µs	500,000	167 km
RK4 A	Δθ= 0° 0' 5.65''	A 31,250 / R 31,542	2,354 km



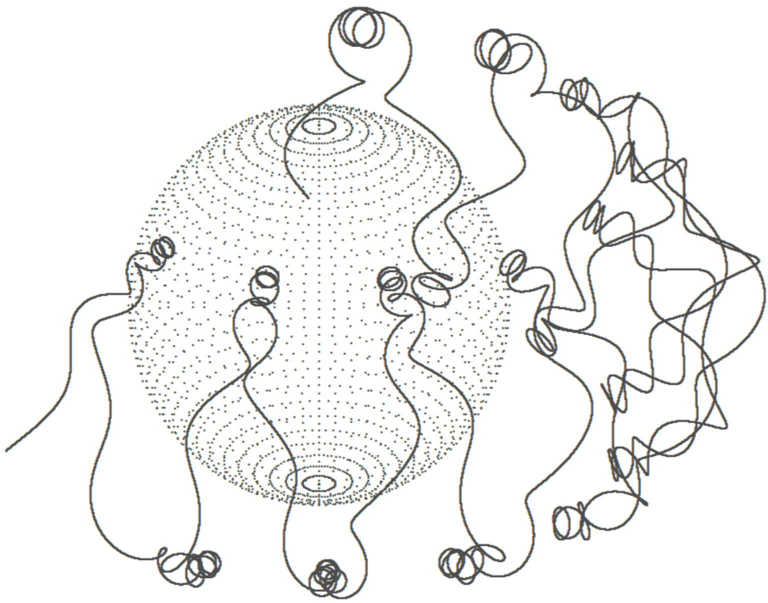
Principle of path reversal: *Just as a photon takes the same path if it goes back in the other direction, in the same way that the* Voyager *probe would take the slings in reverse order to slow down to the Earth, a proton goes back through the path of the antiproton to return to its initial position. A simulation that does not approximately verify this property is not valid.*

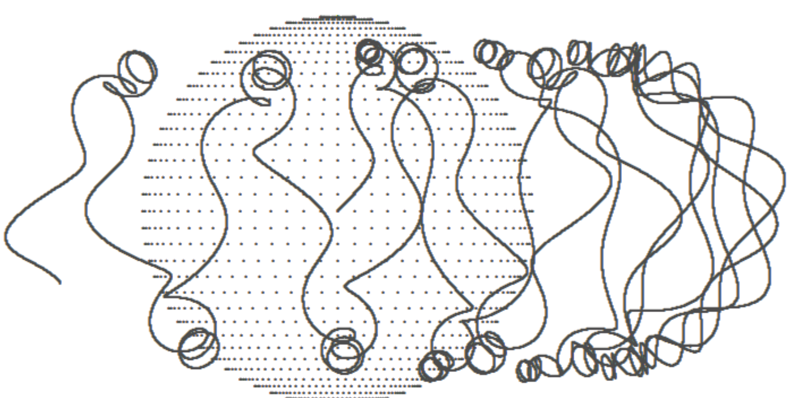


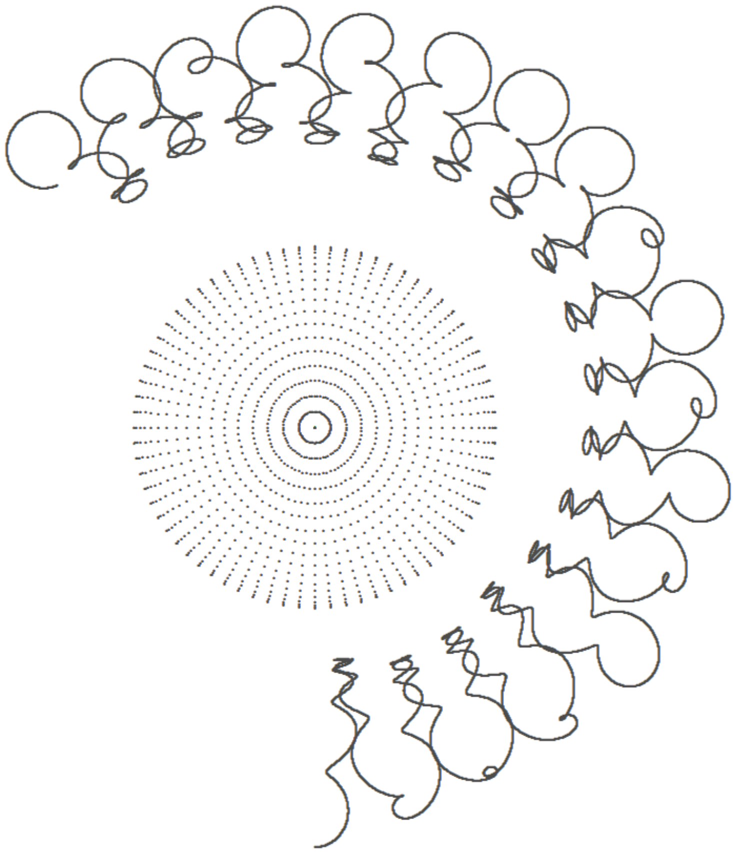
• ***Case of an antiproton of 500 MeV:***

Initially the antiproton is placed in the equatorial plane at 14,800 km from the center of the Earth with a velocity of 76% of *c* directed towards the Northeast.

-Curves : during T=2 s with n=500,000:







-Characteristics: The antiproton goes around the Earth in about 3 seconds. In the same time there are about 13 go and return between the poles. And on each go and return between the mirrors we count about twenty cyclotron rotations:

$T_{drift} \simeq 3.2$ s , $T_{poles} \simeq 0.24$ s and $T_{cyclo} \simeq 8.9$ ms to 42 ms.

The cyclotron period is inversely proportional to the magnetic field strength. The magnetic field varies from 2.4×10^{-6}T at the equator to 11×10^{-6}T at the poles. Approaching the poles the radius of curvature becomes small around 300 km. It is equalized at the mirror points with the cyclotron radius, the pitch of the helix becomes zero and the motion is momentarily plane with a velocity, parallel to the magnetic field, equal to zero. At the beginning, at the equator, the radius of curvature R_{curv} is 2,167 km, the cyclotron radius R_{cyclo} 1,083 km and the helix pitch *p* 6,806 km.

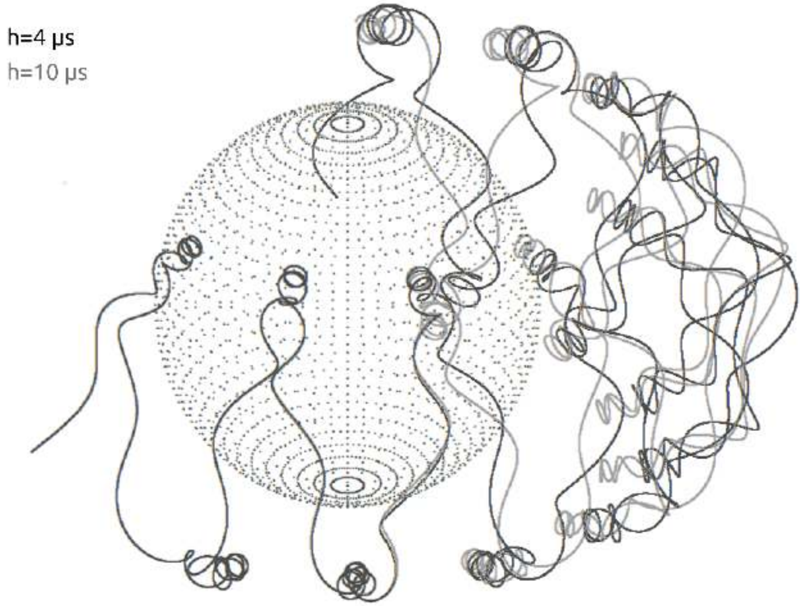
Radius of curvature calculation: $\frac{dv}{dt}=0$ => $R_{curv}=\frac{v^2}{a}$.

Helix pitch calculation: $v_{\parallel}=\frac{\vec{v}\cdot\vec{B}}{B}$ & $p=v_{\parallel}T_{cyclo}$.

Cyclotron radius: $v_{\perp}=\sqrt{v^2-v_{\parallel}^2}$ & $R_{cyclo}=\frac{v_{\perp}}{\omega_{cyclo}}$.

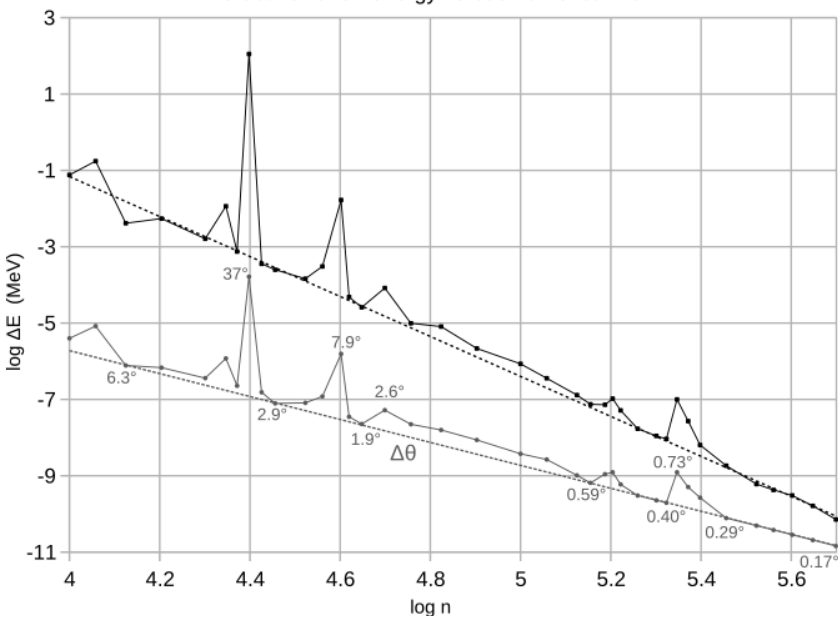
-Stability of the trajectory: In black, we recognize the trajectory for *n=500,000*. In gray, the one for *n=200,000*. Until the fourth mirror there is a good correspondence, then the curves diverge and become significantly different. On the gray curve the curvatures become stronger and the spins are more dived.

-Energy error: Here the error curves are more uneven than for the 2 GeV antiproton in the equatorial plane. We regularly have peaks above a baseline. The peaks on the energy variation and the angle variation are correlated. As we could suspect, it is on the most bent parts of the trajectory, that the energy has difficulty to be conserved.



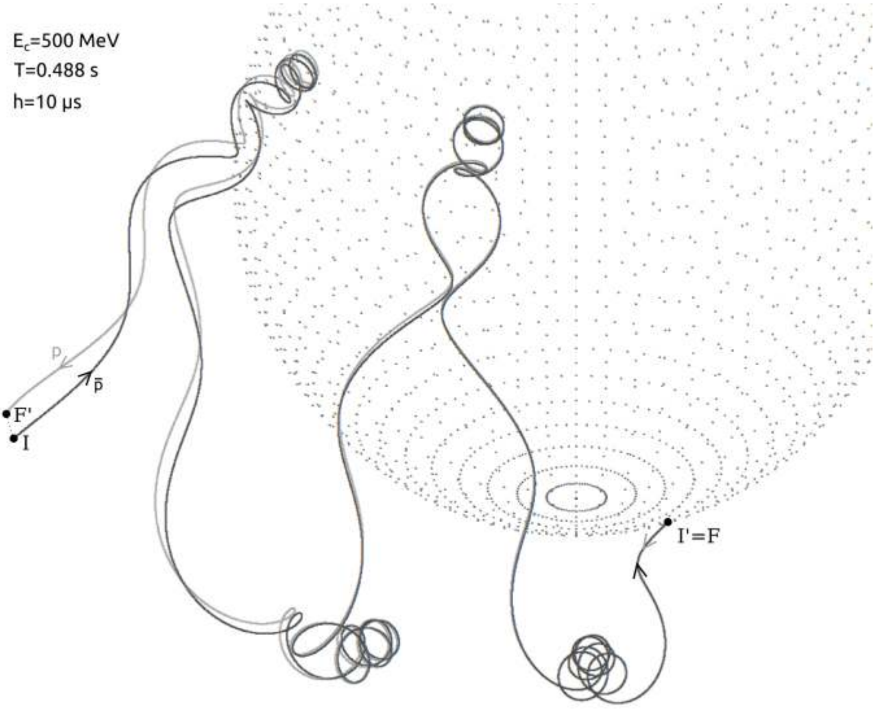
Antiproton of 500 MeV trapped in the Earth's magnetosphere

Global error on energy versus numerical work



-Antiparticle test: We perform the test on two go and return trips between the poles.

<u>T=0.488 s</u>	temporal step	n	$d_{F'I}$
RK4	h= 10 μs	48,800	463 km
RK4	h= 4 μs	122,000	188 km
RK4	h= 1 μs	488,000	39 km



We succeed in simulating an antiproton belt in the Earth's magnetosphere. For accurate results a high computing power is needed.

File: www.voyagepourproxima.fr/magnetique.php
source code: .../docs/magnetique.txt

12. Penning trap

Exercise p333.

1 - This vector field derives from a potential if the cross derivatives are equal:

If V exists: $\vec{E}=-\vec{\nabla}V=-\frac{\partial V}{\partial x}\vec{i}-\frac{\partial V}{\partial y}\vec{j}-\frac{\partial V}{\partial z}\vec{k}$

For example,

$$\frac{\partial V}{\partial x\partial y}=\frac{\partial V}{\partial y\partial x} \quad \text{if and only if} \quad \frac{\partial E_x}{\partial y}=\frac{\partial E_y}{\partial x}.$$

Now all the cross derivatives are zero, so the condition is verified.

Integration: $V=\frac{U_0}{r_0^2}\left(\frac{x^2+y^2}{2}-z^2\right)+cst$, $r_0=\sqrt{2}\,z_0$.

2 - $\vec{F}=q\vec{E}$ then $\vec{F}(O)=\vec{0}$: position of equilibrium.

Stability: $E_p=qV$ (potential energy).

$\frac{\partial^2 E_p}{\partial z^2}=+\frac{2eU_0}{r_0^2}>0$, stable along (Oz).

$\frac{\partial^2 E_p}{\partial x^2}=-\frac{2eU_0}{r_0^2}<0$ and $\frac{\partial^2 E_p}{\partial y^2}=-\frac{2eU_0}{r_0^2}<0$,

unstable in the plane (Oxy).

3-a- Magnetic force:

$$\vec{F}=-e\vec{v}\wedge\vec{B}=-e\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix}\wedge\begin{pmatrix}0\\ 0\\ B_0\end{pmatrix}=-e\begin{pmatrix}B_0\dot{y}\\ -B_0\dot{x}\\ 0\end{pmatrix}$$

The force along z is zero: the equilibrium and stability along this direction are not modified.

3-b- Fundamental principle of dynamics:

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix}=-e\begin{pmatrix}-\frac{U_0}{r_0^2}x+B_0\dot{y}\\ -\frac{U_0}{r_0^2}y-B_0\dot{x}\\ \frac{2U_0}{r_0^2}z\end{pmatrix}\Rightarrow\begin{cases}(1)\quad \ddot{x}=\frac{eU_0}{mr_0^2}x-\omega_c\dot{y}\\ (2)\quad \ddot{y}=\frac{eU_0}{mr_0^2}y+\omega_c\dot{x}\\ \qquad \ddot{z}=-\frac{2eU_0}{mr_0^2}z\end{cases}$$

(1)+j(2) $\Rightarrow\quad \ddot{\rho}-j\omega_c\dot{\rho}-\frac{eU_0}{mr_0^2}\rho=0$

Characteristic equation with $\rho=Ae^{rt}$:

$$r^2-j\omega_c r-\frac{eU_0}{mr_0^2}=0 \quad \text{and} \quad \Delta=-\omega_c^{\,2}+\frac{4eU_0}{mr_0^2}.$$

The solutions are harmonic if the exponential has complex argument:

a negative discriminant gives $\omega_c>\sqrt{\frac{4eU_0}{mr_0^2}}$.

Then: $B_0>\sqrt{\frac{4mU_0}{er_0^2}}=B_c$.

Oscillations along z: $\ddot{z}=-\frac{eU_0}{mR^2}z$ and $\ddot{z}+\omega_z^{\,2}z=0$,

solutions of the form: $z(t)=z_M\cos(\omega_z t+\varphi)$,

=> $\omega_z=\sqrt{\frac{2eU_0}{mr_0^2}}$.

3-c- $r=j\omega_c\frac{1}{2}\left(1\pm\sqrt{1-\frac{4mU_0}{er_0^2B_0^2}}\right)$

solutions: $\rho=Ae^{j\omega_c' t}+Be^{j\omega_m t}$, $(A,B)\in\mathbb{C}^2$,

$$\omega_c'=\omega_c\frac{1}{2}\left(1+\sqrt{1-\frac{4mU_0}{er_0^2B_0^2}}\right)$$

$$\omega_m=\omega_c\frac{1}{2}\left(1-\sqrt{1-\frac{4mU_0}{er_0^2B_0^2}}\right)$$

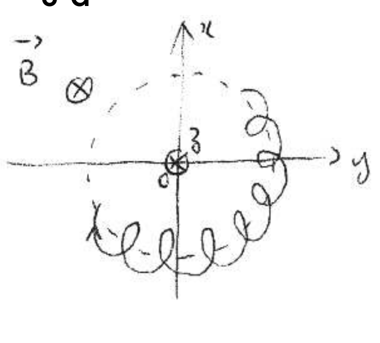
N.A.: $B_c\simeq 2.14\,mT\ll B_0$,

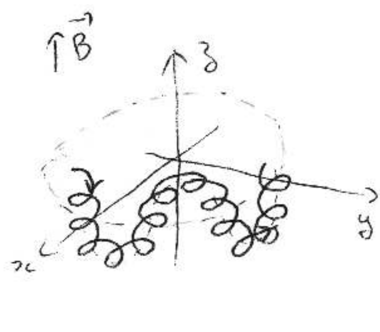
$\omega_z\simeq 145\,rad/s$, $f_z\simeq 58\,kHz$,

$\omega_m\simeq 199\,rad/s\simeq 32\,turns/s$,

$\omega_c\simeq\omega_c'\simeq 52.7\times 10^6\,rad/s$ and $f_c\simeq f_c'\simeq 8.4\,MHz$.

3-d-



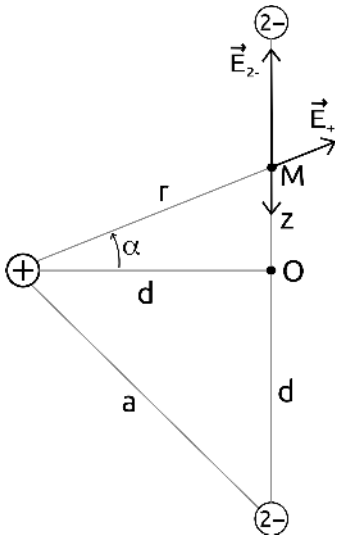


4 - Microscopic cage:

a - It is not an electric monopole, because the total charge is zero. It is not a dipole, because the barycenters of the positive and negative charges are identical. This charge distribution can therefore correspond to that of a quadrupole.

b - $B_0 = 2B_r\left(r=\frac{a}{\sqrt{2}}, \theta=0\right) = 2\frac{\mu_0}{4\pi}\mu_B\frac{2\cos\theta}{r^3}$

and $B_0 = \frac{1}{\pi}\frac{\mu_0\mu_B}{d^3} \simeq 10.5\,T$ with $d = \frac{a}{\sqrt{2}}$.



c - The total electric field is the sum of the electric fields generated by the six point charges.

We place us on the vertical axis (Oz) oriented along to the two magnetic moments.

At point M(z, 0, 0):

$$E = \frac{1}{4\pi\epsilon_0}\left[-\frac{(-2e)}{(d-z)^2} + \frac{(-2e)}{(d+z)^2} + 4\times\sin\alpha\times\frac{e}{d^2+z^2}\right]$$

With z small in front of d: $\sin\alpha \simeq \tan\alpha = z/d$

$$E \simeq \frac{e}{2\pi\epsilon_0 d^2}\left[\left(1-\frac{z}{d}\right)^{-2} - \left(1+\frac{z}{d}\right)^{-2} + 2\frac{z}{d}\left(1+\frac{z^2}{d^2}\right)^{-1}\right]$$

=> $E \simeq \frac{3e}{\pi\epsilon_0 d^3}z \sim 2\frac{U_0}{r_0^2}z$

d - $B_c = \sqrt{\frac{4 m U_0}{e r_0^2}} \sim \sqrt{\frac{6 m}{\pi \epsilon_0 d^3}} \simeq 32 \times 10^6 T \gg B_0$

The magnetic field is totally insufficient with a factor of about one million to trap antiprotons. Perhaps within a ferromagnetic material the B_0 field could reach such values.

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Articles and docs: www.voyagepourproxima.fr/docs/

Software: All the software used is free and open source.

Word processing and spreadsheet: *LibreOffice.*
Graphics: *Gimp*, *Inkskape* and *Blender*.
Operating system: *Ubuntu*.
Programming in *php* on *Linux* server.

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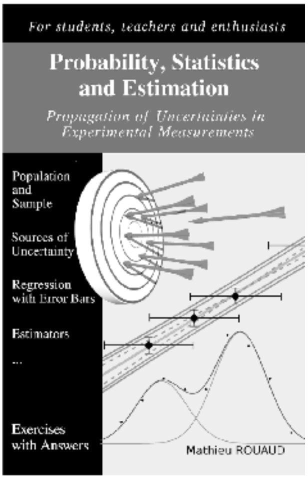

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