

Textbook for Undergraduate and Graduate Students in Physics

SPECIAL RELATIVITY

A Geometric Approach

Course with Exercises and Answers

followed by the conference
Interstellar travel and antimatter

Mathieu ROUAUD

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Foreword

Special relativity, presented in the article published by Albert Einstein in June 1905, has deeply changed our physical concepts. The well-established theories of the time, Newton's old mechanics and Maxwell's brand new theory of electromagnetism, were fundamentally incompatible. In the first, there is an addition law for velocities, while in the second, an invariant speed is required: the speed of light in vacuum. In Newton's theory, in line with the relativity of motion introduced by Galileo, the speed of an object depends on the observational reference frame, so how could the speed of light in vacuum be a fixed fundamental constant? For inertial frames, special relativity reconciles mechanics and electromagnetism, at the cost of calling into question the absolute nature of space and time. Space and time are now relative and form a new absolute: the space-time. The theories of matter and light are thus unified in their natural spatiotemporal framework. Albert Einstein's historical approach is based on the constancy of the speed of light in a vacuum. The modern approach, which made it possible to build the Standard Model, is based on another logic: symmetries. This new approach is deeper and breaks free from the historical bias of the early 20th century. The structure of space-time imposes a speed limit. This maximum speed is

specific to space-time and is not linked to a material object. This new constant is specific to the container, the space-time, and not to the content, for example, light rays. This new vision is conceptually very different and sheds light on the true nature of physical laws. In this book, we focus on visual and graphical methods that help develop understanding without the systematic use of equations. This geometrical approach will be highlighted and will allow the reader to make sense of the equations that will follow. The path followed is not academic, but pragmatic and utilitarian. From the first pages you will master the tools that will allow you to apply special relativity independently. We are not studying general relativity here. We specify this because confusion is frequent between the two theories. That said, for those who want to understand general relativity, you must first have understood the special. General relativity deals with gravitation and is based on its own principles. Small notable exception, we will sometimes make analogies with the black hole to help delimit the two theories.

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TIME DILATION AND LENGTH CONTRACTION

In this chapter, we introduce special relativity and we present the first geometrical tool: the triangle of times

∞ UNITS OF TIME AND DISTANCE

These two physical quantities, time and distance, are of different natures. Impossible, for example, to say if a second is greater or less than a meter, that makes no sense.

We can use a speed to link a distance to a time, but the speed depends on the observer; this link would therefore be perfectly arbitrary. It is always true in classical mechanics, but in special relativity we have a novelty, we have an invariant speed: the maximum speed. This fundamental constant makes it possible to unambiguously associate a distance with a time. This distance is called *light-time*.

For example, the light-year corresponds to the distance traveled in vacuum by light during a year.

The speed of light in vacuum is about a billion km/h, it is named c and is precisely fixed at:

$$c = 299\,792\,458 \text{ m/s}$$

It is the speed of any electromagnetic wave in vacuum, whether it be radio, infrared, visible, ultraviolet, X-rays or gamma rays.

We specify well, in vacuum, because in a transparent material, such as air, water or glass, the speed is lower and depends on the wavelength.

A light-year, denoted l.y., is worth about 10,000 billion km. The star closest to our Sun, Proxima Centauri, is located about 4 ly. Our Sun is 8 light-minutes from Earth, the Moon is one light-second, and an adult human measures between 5 and 6 light-nanoseconds:

$$1 \text{ l.ns.} \approx 33 \text{ cm}$$

We can now freely compare distances and times, expressing the distances in units of light-time.

∞ FRAMES

Any measurement of a physical quantity is carried out in a given frame of reference.

The quantity can be a time, a distance, a velocity, an acceleration, a force, etc.

The reference frame, as in Newtonian mechanics, is defined by a reference solid considered fixed.

For example, a train can be taken as a reference. More precisely, a wagon of this train makes it possible to locate any object. We consider, arbitrarily, a point of the wagon as the origin. Then, from this point, we count how many times we have to move, end to end, a rigid ruler of one meter in the direction front-back, right-left and up-down to reach this object. We get a set of three numbers that uniquely defines the position of the object. If the object is fixed this will be sufficient, but if it moves, it will also be necessary to define a date. We then have a set of four numbers called event:

$$E(x, y, z, t).$$

For the date, we must proceed more precisely than in classical mechanics. Time is no longer absolute, and instead of a single clock we must have a set of synchronized clocks over the whole space.

Depending on the case, we can use the terrestrial reference frame, the heliocentric reference frame, the galactic reference frame, etc.

These frames of reference are in motion with respect to each other and for the same event we will have different sets of coordinates.

∞ EINSTEIN'S POSTULATES

Albert Einstein postulates in his article of June 1905¹ that the laws of physics are the same in all inertial frames of reference (1st postulate), and that in these same frames the speed of light in vacuum is invariant (2nd postulate) .

In Newtonian mechanics, for the statement of Newton's three laws, we did not speak of inertial frames but of Galilean frames, which amounts to the same thing. For example, in classical mechanics, in a frame rotating with respect to a Galilean frame, Newton's second law is no longer verified and new forces, called inertial, appear. A rotating frame with respect to an inertial frame is therefore not inertial.

How to define an inertial frame? A frame is inertial if the postulates are verified. The simplest is to have a inertial frame of reference, then all the frames in uniform rectilinear translation with respect to this first frame of reference are also of inertia.

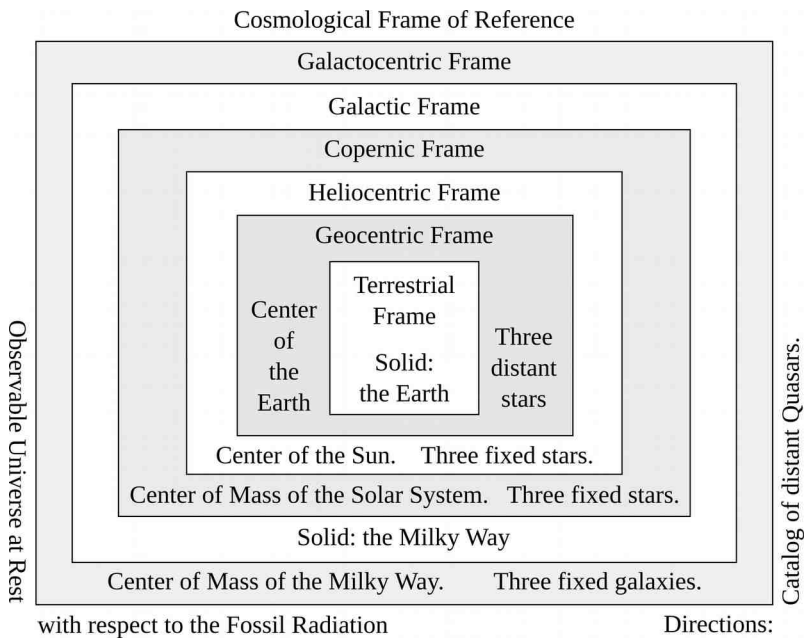
1 “*On the Electrodynamics of Moving Bodies*”, June 30 1905, English Translation.

The farther away we aim at an object, such as a distant star, the more its motion can be neglected. For example, extremely massive and very distant quasars, several billion light-years away, are taken as fixed points and make it possible to define the cosmological reference frame. Fossil radiation, emitted 380,000 years after the Big Bang, 13.8 billion years ago, is homogeneous and isotropic in this frame of reference.

To come back to our train, if it runs in a straight line and at constant velocity in the terrestrial frame of reference, the reference frame of the train can be considered as inertial for an experiment of a few minutes. This duration is small compared to that of the rotation of the Earth on itself. This is a good approximation, and the terrestrial frame can be considered here as inertial. The more precise the measurements, the shorter the duration of the experiment for the approximation to remain valid.

For a satellite, the terrestrial frame of reference is no longer inertial. A low-Earth-orbiting satellite goes around in 1 hour 30 minutes, a not insignificant time compared to the Earth's rotation which lasts about 24 hours. We then consider the geocentric frame of reference, with the origin at the center of the Earth, and in which the Earth is in rotation around its own axis relative to distant stars assumed to be fixed.

**In an inertial frame of reference,
an object keep moving in a
straight line at a constant speed
when no forces act upon it.**



Unable to position and date an event without landmarks. If you hide a treasure you will indicate its position relative to a point of origin: for example, "from the hundred-year-old oak tree, 22 steps west, 47 steps south and dig at three feet." If I say that I was born in 1992, it is in reference to an origin date, placed arbitrarily as a common reference point.

A reference frame is associated with a solid to which a chronology is added. A minimum of four fixed objects relative to each other is required. For chronology, in special relativity a single clock is no longer sufficient: one can imagine a solid made up of rigid bars of unit length, all placed perpendicular to each other in order to form a three-dimensional network, and at each node of this network we place a clock; all the clocks are synchronized, and the whole forms what is called a crystal of clocks.

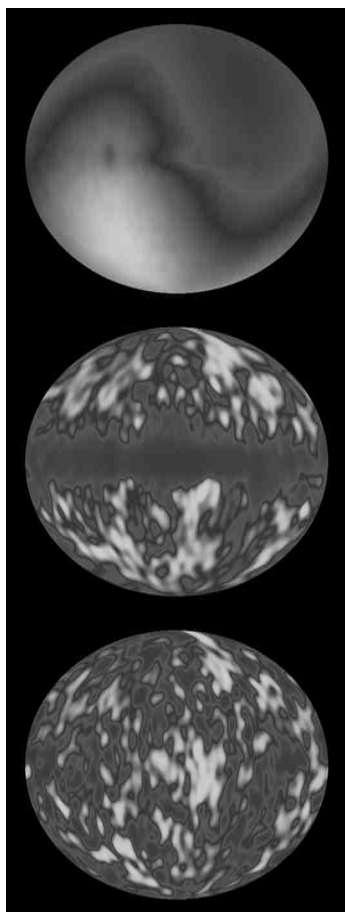
The largest object in the Universe is the Universe itself. Let's use it as a reference solid. In cosmology, the Universe can be seen as a fluid of galaxies which extends everywhere: any point of the Universe can be considered as the center. But, two remarks: first of all the Universe cannot be observed as a whole, because the further one looks far, the more one goes back in time. The oldest visible object is fossil radiation emitted 13.4 billion years ago when the Universe became transparent. Secondly, if we take a point where this fossil radiation is uniform, everything leads us to think that this point is motionless in the Universe.

Image opposite, the data collected by the COBE satellite on the cosmic diffuse background.

On the first image we visualize the anisotropy due to the displacement of the Earth in relation to the cosmological frame of reference, this is due to the Doppler effect and we thus evaluate a speed of 350 km/s.

In the second image, we have stray light from our own galaxy.

Finally, at the very bottom, we get an image of the Universe at its beginnings: it is homogeneous in the cosmological frame of reference and we can use quasars for the directions.



Thus the frames of reference nest one in the other: for the *Voyager* probe we consider the Copernic frame of reference, which has for origin the center of mass of the solar system and the directions of distant stars. For an interstellar journey to Proxima Centauri we will consider the galactic frame of reference. Indeed, over a journey of a few years or decades, the Milky Way and its stars can be assumed to be fixed; for example, our galaxy turns on itself in some 250 million years, much longer than our journey to the stars².

∞ THE TRIANGLE OF TIMES

There is not an absolute, unique and universal time. Times are multiple and relative. Each observer, or object, lives his own time. Times are plural, each time follows its course, and, when we compare them, we see that they evolve at different rates. These rhythms will be all the more different the greater the relative velocity between two inertial frames of reference. For each inertial frame we can define a unique time for a set of objects which are motionless with respect to each other.

Let us name R such an inertial frame of reference. Consider a fixed point $M_1(x_1, y_1, z_1)$ in R . At this point,

2 Continuation of the reflection on inertial frames of reference in the conclusion of the course on four-vectors.

two events occur at the date t_1 and t_2 :

$$E_1(x_1, y_1, z_1, t_1) \quad \text{and} \quad E_2(x_2=x_1, y_2=y_1, z_2=z_1, t_2).$$

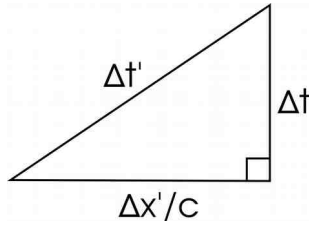
For example, a lamp that turns on and off. Second example, in the case of an interstellar journey, let us take for R the reference frame of a rocket, t_1 corresponds to the date of departure from the solar system, and t_2 indicates the date of arrival near Proxima Centauri. Dates measured on a clock fixed relative to the rocket.

The duration between the two events is $\Delta t = t_2 - t_1$.

If we now measure the four coordinates of these two events from a second inertial frame R' , in uniform rectilinear translational motion at the velocity \vec{v} with respect to R , we measure a second duration $\Delta t' = t'_2 - t'_1$.

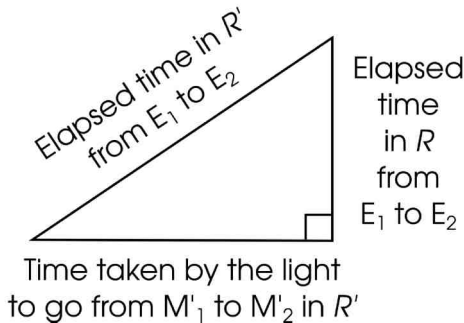
From the point of view of R' , the events E_1 and E_2 have space-time coordinates (x'_1, y'_1, z'_1, t'_1) and (x'_2, y'_2, z'_2, t'_2) , and now occur at two distinct points $M'_1(x'_1, y'_1, z'_1)$ and $M'_2(x'_2, y'_2, z'_2)$. The first duration Δt is called *proper time*, because the events are at rest in R ; the second duration $\Delta t'$ is called *relative time*, because the events are moving with respect to R' . The reference frame R' will have traveled, with respect to R , the distance $\Delta x' = x'_2 - x'_1$ during $\Delta t'$ (case where the x-axes are oriented along \vec{v}).

We then have *the triangle of times* which allows us to answer many of our questions:



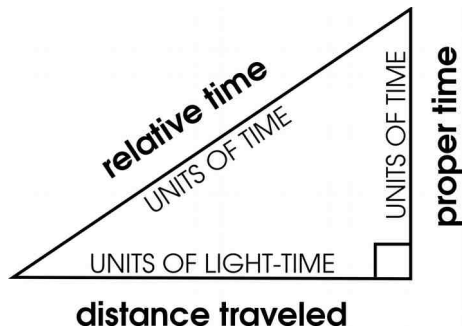
We use this triangle as a starting point to build special relativity. Later we can demonstrate its validity using Einstein's postulates or symmetries.

Each side of the triangle corresponds to a distinct time:



We can memorize it in the following form:

*The
Triangle
of
Times*

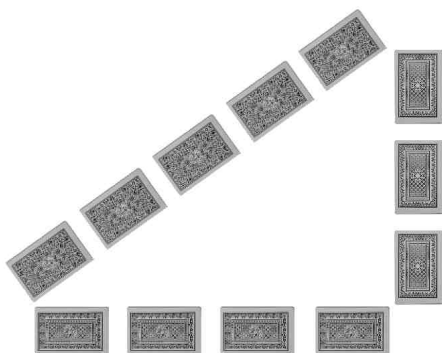


The *triangle of times* is easy to remember and apply. Take the case of an interstellar journey Sun - Proxima Centauri and use a card game to solve the problem.

The base of the right triangle is the distance in light-years. We place one card per light-year, so, here horizontally, four cards. Then we vertically place the number of cards that correspond to the travel time for the astronauts, one card per year.

We decide to complete the trip in three years, measured with a clock at rest in the frame of reference of the vessel.

How long will the journey measured from the galactic frame of reference last? It's simple we count the number of cards needed for the hypotenuse:



Relative time is 5 years and proper time 3 years. The *triangle of times* allows you to directly visualize the *time dilation*: $\gamma = \Delta t' / \Delta t$.

Here, the *gamma factor* is 5/3. The speed of the

vessel is in R' : $v = \Delta x' / \Delta t'$. Here, the speed is 4/5 of the limit speed so 80% of c . As the hypotenuse is the longest side, time can only expand, and the speed of light in vacuum cannot be exceeded.

The first two exercises on page 25 allow you to familiarize yourself with these concepts.

∞ LENGTH CONTRACTION

We previously envisioned a trip from the Sun to Proxima that lasts 3 years for astronauts. We could ask ourselves: «The ship takes three years while light takes four years, so we go faster than light!?» Question that comes up regularly among students at the time of the introduction to special relativity.

This is of course not the case. Rather, it should be reformulated as follows: if a terrestrial observer sends a light pulse with a laser, he will have to wait for his clock to indicate four years elapsed before the ray reaches Proxima Centauri; while an observer traveling at 80% c will have to wait for his clock to indicate three years elapsed before joining Proxima. And the terrestrial observer will observe well the vessel arriving after the ray, just as the astronaut leaving at the same time as the ray will never exceed it. To be logical, all reasoning must be carried out in a fixed frame of reference. If we change the frame of reference, we change our

point of view, and we have to rethink the situation.

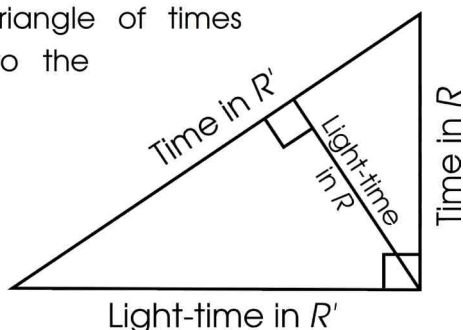
First of all, to measure a velocity in a given frame of reference, it is necessary to divide a distance by a time, taking care that the two quantities are measured in this same frame of reference.

In the question asked by the student, he divides a distance measured in the galactic frame of reference by a time measured in the frame of reference of the vessel. It does not make sense³, the quantity obtained does not correspond to the speed of an object.

The answer, to this apparent paradox, is that the Sun-Proxima distance measured from the frame of the vessel is not 4 ly, the length is contracted and is less than 3 ly.

The length contraction factor is equal to the time dilation factor.

Do we have the equivalent of the triangle of times? Not really, because, if we are trying to construct a triangle of lengths, one of the sides does not correspond to a physical quantity, directly measurable. On the other hand, we can add a fourth time in the triangle of times which corresponds to the time taken by the light measured in the reference frame R :



3 The discussion will be prolonged and deepened when studying four-vectors and four-velocity.

All the triangles are in the same proportions, and the light-time measured in R is the shortest.

The Sun-Proxima distance measured from the vessel is 2.4 ly.

∞ SPATIOTEMPORAL PERSPECTIVE EFFECT

Suppose the astronaut's heart beats at 60 beats per minute. If the time dilation is two, from the point of view of observers on Earth, his heart beats more slowly, once every two seconds. And if for another observer the gamma is equal to three, there will be a beat every three seconds according to the latter. But it goes without saying that for the astronaut, from his point of view, his heart beats quite normally, once a second. Its frame of reference is inertial as for the other two observers.

Also, by the relativity of the motion, the astronaut who observes the inhabitants of the Earth will have the impression of a symmetrical slowing down.

It should be noted that this slowing down of the clocks is the same whether one moves away or that one approaches. This phenomenon is different in nature from the Doppler effect, where, when a source approaches, the received signal is of higher frequency, and when the source moves away, it is lower.

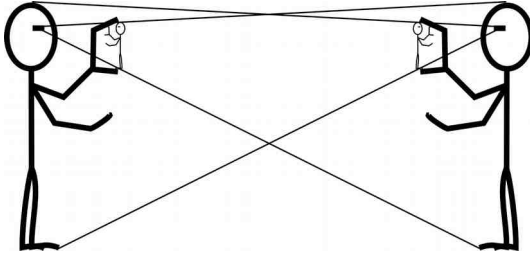
A classic confusion consists in confusing what we see with what is. When you look at a star, you see

the light that it emitted many years ago, possibly not there anymore, or even not existing. Yet spontaneously when we look at the starry sky we feel united to the cosmos, here and now. This illusion stems from our daily habits in a world where maximum speed is very high compared to our routine motions. We can assume the instantaneous propagation of light, we see what is. If the speed limit were 10 km/h, we would be used to these differences. Often we imagine ourselves watching, with the naked eye or with a terrestrial telescope, the astronaut in his vessel moving away and performing his motions in slow motion, but this thought experiment is false, it is not about that.

We do not «see», we measure with the crystal of clocks. The first time you apprehend it, the approach may seem somewhat conceptual, but with practice it becomes natural, and you stop saying that you see the clocks slowing down. It is necessary to have in mind the two reference frames of inertia such as meshes, one immobile, and the other in motion, and imagine the two successive events and the dates recorded locally by each of the synchronized clock crystals.

However, we can make analogies with spatial perspective effects. When you look at someone in the distance, he is very small, you can look at him from head to toe between two fingers. He can do the same, it's symmetrical. There is a contraction of the lengths, and nobody imagines the phenomenon

as real, the other is not small like a smurf.



The contraction takes place in all directions.

Another perspective effect that produces a contraction of the lengths: the rotation. When I show you a book from the front, then I turn it 90° on a vertical axis, you only see its edge, and the cover has reduced in size to zero during the rotation. The apparent contraction occurred horizontally only.

In special relativity, the two observers are in motion with respect to each other, and it is this motion that simultaneously creates the contraction of lengths and the dilation of time. The lengths are only contracted in the direction of the relative velocity. We recall that, unlike previous analogies, it is not what we see but what we measure.

Contrary to what we sometimes hear, it is not a spatiotemporal rotation. We will see the transformation to be performed between the coordinates (x, y, z, t) of R and (x', y', z', t') of R' in the chapter *Changing reference frame*, this is not a rotation.

∞ TWIN EXPERIMENT

This is a thought experiment proposed by Paul Langevin in 1911. We hope that one day we will have a space-time ship to make it happen! Although not performed with real twins, it has, for the moment, been performed with atomic clocks. We sometimes talk about the twin *paradox*, but it is a reality, not a paradox; this misleading name comes from misunderstandings. Langevin, the main defender of relativity in France at the beginning of the 20th century, did not speak, at the Bologna Philosophy Congress in 1911, of paradox, or of twins... but of a Jules Verne-style voyage by cannonball! It is the mathematician and physicist Hermann Weyl who speaks of twins in 1918. It is the philosopher Henri Bergson who devotes an entire book, published in 1922, on special relativity, which speaks of paradox and gives an erroneous interpretation of the experience.

Now let's explain this experiment. We take two twins as they celebrate their 20th birthday on Earth. Right after the birthday, they leave each other, one stays on Earth and the other leaves for Proxima at 80% c . According to the *triangle of times*, we have 5 years elapsed for the twin who remained on Earth and 3 years for the one who travels to Proxima. Then the traveling twin returns to Earth, which doubles the times. The twin on Earth is now 30 years old and the

one who has made the round trip 26 years. Our twins are no longer the same age.

The image is striking because the two twins can directly compare their two clocks with a difference of four years. It is less abstract than a measurement via a crystal of clocks. The postulates of special relativity consider inertial frames of reference. We can at some point have the clocks of two different frames coinciding, but then they just move away from each other at constant speed. Thus, the twin experiment cannot be understood on the basis of Einstein's first two postulates alone.

We see a cumulative effect of time dilation on the round trip, why not also a cumulative effect of contractions: a younger and flattened astronaut...!? Time and space do not have equivalent natures: a left-right motion can be compensated by a right-left motion, for time it is impossible, there is the principle of causality and one can only go from the past to the future, one can only move forward in time and the proper times are added.

Before concluding on the twins' experiment one last point. Doesn't it seem absurd to you that the traveler leaves just like that at 800 million km/h, implied instantaneously? It is of course impossible, a physicist is only interested in physically acceptable situations, it would require infinite energy and the force due to the acceleration exerted would also be infinite. In short, even if the acceleration phase lasted a few seconds, it is not conceivable that such a powerful reactor could exist, and the occupants would simply

be crushed... The spaceship actually sees its speed increase continuously, which can be modeled by a succession of inertial reference frames of increasing velocities.

A new postulate completes special relativity, it is the *clock hypothesis* which has been verified experimentally:

Two clocks of the same instantaneous speed v , one being accelerated and the other not, undergo the same time dilation factor γ .

The clock measures the *proper time* and we add the times of the traveler over the whole of his space-time round trip:

$$\tau = \int d\tau = \int \frac{dt'}{\gamma}$$

The *proper time* is the time measured by a clock at rest in relation to the phenomenon to be studied. We had called it Δt , but often to emphasize its peculiarity we use the Greek letter τ . On the other hand, measuring a relative time requires *two* different clocks previously synchronized.

It is thus possible, without ambiguity, to calculate the actual time taken by the traveler for the round trip. Calculation made from the galactic inertial frame R' . Note that if we do the calculation from another inertial frame of reference R'' we would find the same proper duration τ .

On the other hand, a direct calculation is impossible from the reference frame of the vessel because this one is not of inertia⁴.



Joseph Hafele and Richard Keating, in 1971, experimentally verify the «clock hypothesis», the third postulate of special relativity. With few resources and a lot of perseverance, they went around the world twice, one to the east and the other to the west. They were in commercial planes with atomic clocks and many passengers. On the way back, they compare with a clock that has remained on the ground⁵.

Photo: Time Magazine, October 18, 1971.

Training: exercises 3, 4 & 5 on page 26.

4 The calculation can be done from the point of view of the accelerated reference frame using the non-Minkonskian metric given on page 143.

5 *L'expérience cruciale de Hafele et Keating* by Pierre Spagnou, pdf, 27 pages, March 2018.

∞ USE OF EQUATIONS

The triangle of times, page 10, gives by application of the Pythagorean theorem :

$$(\Delta t')^2 = (\Delta t)^2 + \left(\frac{\Delta x'}{c}\right)^2 \quad \text{besides} \quad \gamma = \frac{\Delta t}{\Delta t'} \quad \text{and} \quad v = \frac{\Delta x'}{\Delta t'}$$

$$\text{then} \quad (\gamma \Delta t)^2 = (\Delta t)^2 + \left(\frac{v}{c} \gamma \Delta t\right)^2 \quad \text{and} \quad \gamma = \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

we also note beta : $\beta = \frac{v}{c}$ which expresses the speed with respect to c ,

So, we have the following relation for gamma:

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}$$

Knowing this expression of the gamma factor by heart makes it possible to do without the triangle of times.

Training: exercises 6 to 9 on page 28.

◦ Transformation of volumes and angles

• **Volumes** : Only the lengths along the direction of the relative velocity between the two frames of reference are contracted. Let us take the case of a rectangular parallelepiped along the axes ($Oxyz$) at rest in R , then if $\vec{v} = v\vec{i}$: $\Delta x' = \Delta x/\gamma$, $\Delta y' = \Delta y$ and $\Delta z' = \Delta z$,

$$\text{from where : } \boxed{V' = \frac{V}{\gamma}} .$$

True relationship whatever the shape of the object. Indeed, any object can be decomposed into infinitesimal parallelepipedal volumes each contracted by the same factor γ , the integral, sum of infinitesimals, is therefore also.

A cube in R flattens in R' while keeping the same section perpendicular to \vec{v} . A sphere in R flattens in the direction of \vec{v} in R' .



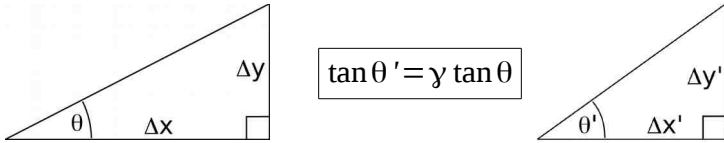
The distance measurement protocol ensures that each position of the object is measured at the same time t' in R' .

This is of course only a perspective effect, nothing physical here, if for example the cube is a box which contains a gas, this one is not compressed and no risk that this one liquefies!

Concerning what is seen by an observer, there is a new deformation due to the propagation of light

rays to the point of observation. The distance from a point of the object to the observation point varies and the object photographed on a sensor consists of light points which correspond to different instants t' at the level of the object, the measurements are not then simultaneous in R' . This more subtle aspect is discussed in the chapter *The Appearance of Things*.

• **Angles** : Consider a right triangle. A side of length Δx along \vec{v} , and a second perpendicular along y and of length Δy . We measure the angle θ between the side of length Δx and the hypotenuse. The triangle is at rest in R and $\tan \theta = \Delta y / \Delta x$. In R' : $\tan \theta' = \Delta y' / \Delta x'$. $\Delta x' = \Delta x / \gamma$ and $\Delta y' = \Delta y$ then:



When you see a star in the sky, you measure its position using angles. These angles are modified by the motion of the Earth in its orbit in the galactic frame of reference. The apparent angle θ_a under which we see a star is not simply the angle θ' because we must also take into account the propagation of light rays to our telescope. The color of a star is also modified, see the chapter *The Appearance of Things* for more details.

Exercises

Methods of resolution:

- ♥ (card game)
- ✂ (ruler, triangle, protractor and compasses)
- √ (equations)

Difficulty : ▲▲△ (simple) / ▲▲△ / ▲▲▲ (complex)

Data :

Speed of light (vacuum) $\simeq 300\,000$ km/s

Distance Sun-Proxima $\simeq 4$ light-years

Distance Sun-Barnard $\simeq 6$ light-years

Distance Sun-Sirius $\simeq 9$ light-years

Radius of the Earth $\simeq 6\,400$ km

1. ♥ ▲▲△ The Crystals of the Pop exomoon

In the galactic year 2110, you undertake the Sun-Barnard voyage to study the crystals of the Pop exomoon. After eight years in your rocket, you land on Pop. In what galactic year are we then, and what was the speed of your rocket?

Answers page 337.

2. ♥ ▲△△ One-way ticket for Sirius

It is decided, in 2154, for your 30 years, you will leave for Sirius with the antique ship $\beta 6$ of your friend Zu. Too eager to change air and make a new start. The ship is not very fast, but spacious and comfortable. At what age will you arrive, and will you be able to attend the festival of the two suns of 2168, or will you have to wait for the one of 2178?

*Dream Series $\beta 6$: model 2110-2124 / Speed 60% of c .
Answers page 337.*

3. ♥ ▲△△ Parcel delivery

Your job? The delivery of parcels throughout the galaxy. And you are the first on the market because you have the fastest SpaceTruck!

"... to trade between the Sun and Proxima, I only need 4 years of travel time for the round trip. And a profit of 5 million Blings, imagine how much money I make !!"

How long does it take to deliver, what is the speed of the ship and the time dilation?

Answers p338.

4. ♥ ▲▲△ Twin on his way to Sirius

Twins are 20 years old in 2132, the most intrepid leaves for Sirius and returns in 2156.

How old are the twins then?

What was the speed of the rocket?

Answers page 338.

5. \ ▲▲△ Cruel dilemma?

We are in 3021. Denys lives in the galactic center. He has just received terrible news: during his stay in the spiral arm of Perseus, he caught a virus, he will die in exactly 32 years, and there is no cure ...

In addition to that, he has just received a very precise mission order: to defuse a gamma ray bomb located at 26 ly before it destroys the whole galaxy, explosion planned in 3052.

And most important of all, to be there, at the center of the galaxy, for the great secular galactic celebration of 3082!

Denys has a ship with a gamma equal to two.

What can you offer Denys?

Answers p339.

✓ The use of equations is the most complete and general method. Nevertheless, we believe that its systematic use, from the very beginning of learning, makes it difficult to understand phenomena intuitively. Moreover, the mathematical language to be mastered unnecessarily blocks many people who are passionate about physics.

The equations are very practical in the two cases where the triangle of times is very stretched: for slow motions where the speed is very low in front of that of light, or, on the contrary, for fast motions where the speed is very close to the maximum speed (ultra-relativistic cases).

6. ✓ ▲▲△ Muons

Cosmic rays are made up of high-energy particles. Many of those that hit the Earth's atmosphere are protons. They come from the Sun, our galaxy and beyond. Fortunately for life on Earth, many of these particles are destroyed in the upper atmosphere and create showers of other, less energetic particles. We are interested here in the case of muons created in this way. When you are by the sea, an average of 170 muons reach the ground per square meter per second. Every second that you take to read the statement of this exercise dozens of muons pass through you.

Muons have a half-life $t_{1/2}$ of 1.5 microseconds. This means that if you take a large number of muons at rest, only half of them will remain after 1.5 μs , and since they do not age, only a quarter will remain after 3 μs , and so on.

Let's take the example of a muon created at an altitude of 10 km and which moves vertically towards the ground with a speed of 99.9% c .

What do you think about the probability of this muon reaching the ground (sea level)?

Answers page 340.

7. ✓ ▲▲▲ High-speed train journey

In 2012, the longest high-speed train line is in China and connects Beijing to Guangzhou. Its length is 2300 km and the journey time is eight hours.

You have two atomic clocks. You synchronize them, then, you leave one of them at the station in Beijing, and, the other one accompanies you for your round trip Beijing-Guangzhou.

On the return trip, what will be the time difference between the two clocks?

- Accuracy of on-board atomic clocks: 10^{-14} s/s.
- The trip is considered at constant speed, which will give a good approximation.
- A necessary mathematical tool here, a series expansions: if epsilon is very small compared to one, $\epsilon \ll 1$, then $(1+\epsilon)^\alpha \simeq 1+\alpha\epsilon$. Here $1/\sqrt{1-\beta^2} = (1-\beta^2)^{-1/2} \simeq 1 + \frac{1}{2}\beta^2$.

Answers page 341.

8. ✓ ▲▲▲ Satellite

Let's consider a low altitude satellite, such as, the International Space Station. The satellite is placed at an altitude of 500 km and travels at 27,000 km/h in the geocentric reference frame. This frame of reference is considered to be inertial in this exercise.

One clock is placed in the International Space Station and a second is kept motionless in the

geocentric frame of reference. Synchronization and time comparison protocols are perfectly respected. What is the time difference after a revolution?

- *The satellite's frame of reference is not inertial and we apply the clock hypothesis.*
- *Unlike Hafele and Keating's experience, the clocks remain at a constant altitude, so we don't have to take into account the effects of gravity.*

Answers page 342.

9. ✓ ▲▲▲▲ Hafele-Keating experiment

Here we will try to find the results of Hafele and Keating established in 1971.

For a round-the-world trip to the east, they found that the onboard clock aged less than about 60 ns compared to the clock on the ground, on the other hand, for a round-the-world trip to the west, the onboard clock aged more by about 300 ns.

We simplify the problem, only one plane is enough to go around the world. The flight is equatorial at an altitude of 10 km. The plane has a speed of 1000 km/h from the ground. At the equator, the ground is moving at 1674 km/h relative to the geocentric reference frame, here considered Galilean. The takeoff and landing phases are considered fast enough to be neglected.

Concerning gravitation, time slows down when gravitation increases:

$$\Delta t' = \left(1 + \frac{gh}{c^2}\right) \Delta t, \quad h : \text{altitude}, \quad g = 9.81 \text{ m/s}^2$$

$\Delta t'$ is the time spent in altitude, Δt on the ground.
(general relativity in the weak-field limit)

You can imagine three clocks, the first stationary in the geocentric reference frame, the second at rest in the plane and the third on the ground.

Are your results in agreement with those of the experiment carried out in 1971?

Answers page 342.

SPACETIME DIAGRAM

After the *triangle of times*, we present here a second geometrical tool, a diagram, which broadens our vision of space-time, gives a synthetic representation of situations and makes it possible to answer a very large set of questions.

∞ WORLDLINES

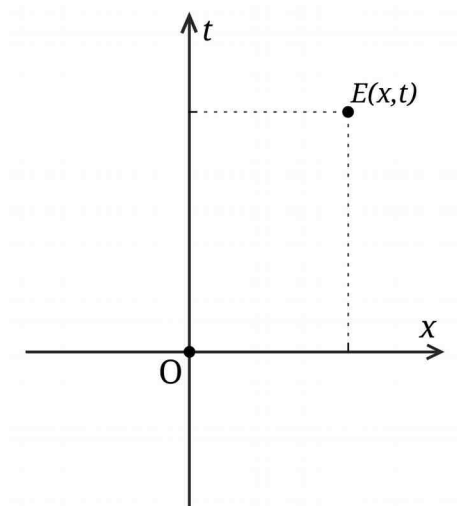
The triangle of times is enough to study the motion of a single moving object with constant velocity. When the velocity of the object varies, or we have several moving objects, we prefer space-time diagrams. For example, for the twin experiment, the traveling twin's direction of velocity changes between the outward and return journey.

The world-line of an object contains all of its physical information: all of its positions through time, and therefore the evolution of its velocity, acceleration and force exerted on the particle.

A worldline represents the set of events experienced by an object.

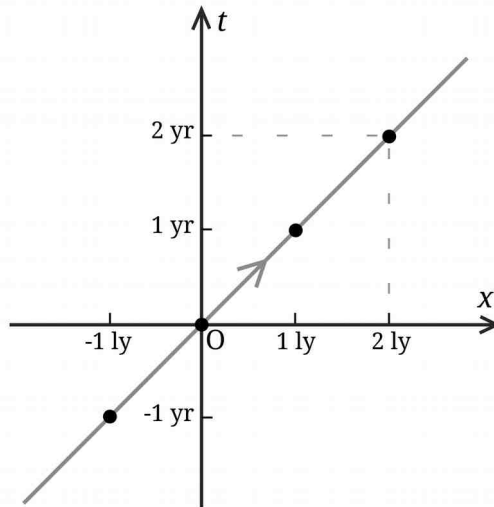
∞ MINKOWSKI DIAGRAM

The spacetime diagram is often called a Minkowski diagram in the context of special relativity. In the case of a rectilinear motion, a spatial axis is sufficient and the diagram will be represented in a plane. The horizontal axis represents the x -coordinate of the object and the vertical axis the time t . Each point in the diagram corresponds to an event. Point O corresponds to the origin event — both temporal and spatial.



Let's start by considering the motion of a photon which "passes" by O and which goes to the right. The successive events "experienced" by the photon create its worldline. We graduate the axes in natural

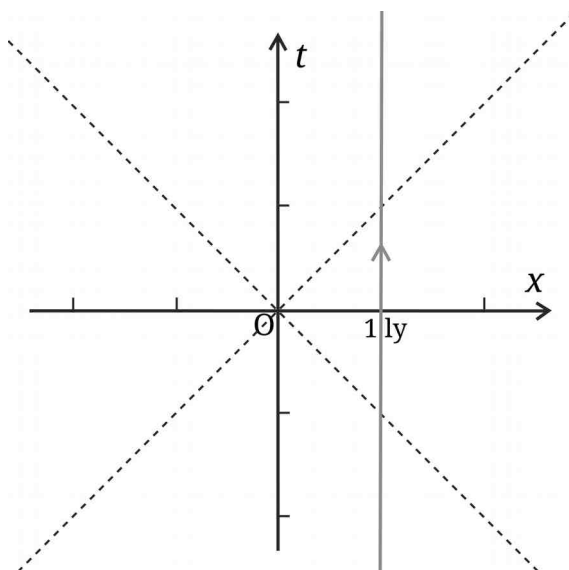
space-time units and we choose the year as the unit of time.



A year ago the photon was located one light-year to the left, it is now here, it will be one light-year to the right in a year, etc.

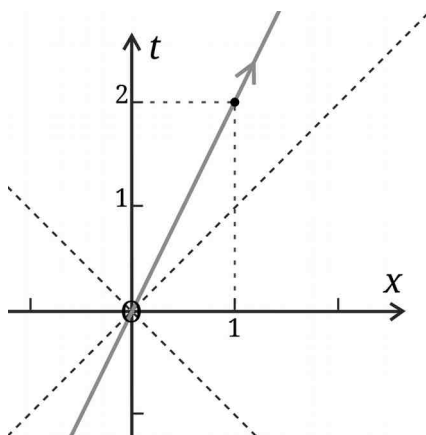
In addition, we consider a second photon, which also passes through the origin, but which moves in the other direction, from right to left. The two photon worldlines are shown in dotted lines and are often present to aid the reading of Minkowski diagrams. In the case of an immobile particle in the observational frame of reference, the worldline is a vertical line oriented upwards.

On the following diagram we have the world line of an object at rest in the observational frame of reference and located one light year to the right of the origin of the frame.

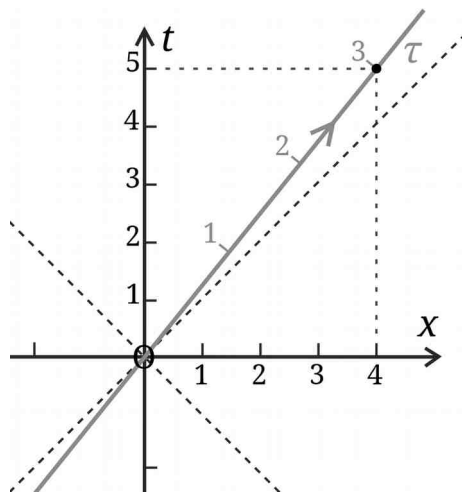


We now consider the general case of a particle which passes through O and moves to the right with a constant velocity v . As a particle cannot go faster than light, the worldline is represented by a straight line of inclination intermediate between the vertical line (time axis) and the dotted line of the corresponding light ray.

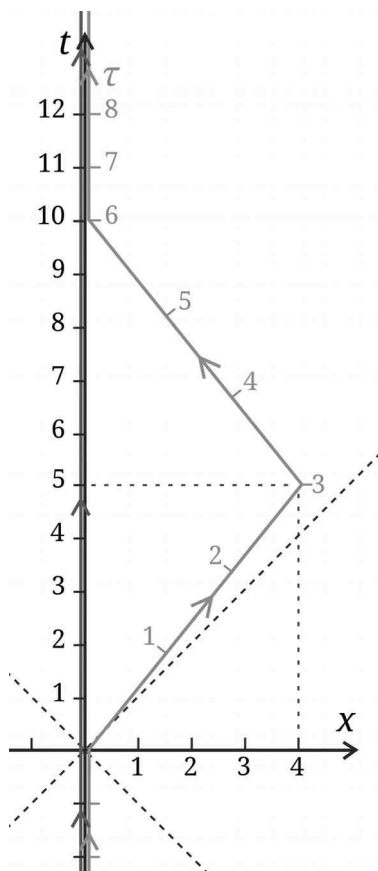
On this example, the object moves at 50% of c , it travels one light-year in two years.



We now know that there is dilation, the time for a moving object is not the same as for an object at rest. We take the example of a trip at 80% c . With the triangle of times we obtain the proper time τ which we add on the worldline of the moving object. The dilation of time appears clearly.



For the twin experiment we visualize the two world-lines of each on the same diagram:



The worldlines are represented in the frame of reference of the twin who remained on Earth, more precisely the galactic frame of reference which is an excellent inertial frame of reference. We cannot directly reason from the reference frame of the traveler, the latter is not inertial because his velocity varies.

∞ USE OF EQUATIONS

Equation of worldlines

These straight line equations are used to determine dates and positions, appointments and reception of spatial messages.

Ship passing through O and heading to the right at speed v :

$$v = \frac{x}{t} \quad \text{then} \quad t = \frac{1}{\beta} \frac{x}{c} \quad \text{with} \quad \beta = \frac{v}{c}$$

Ship passing through A and heading left at speed v' :

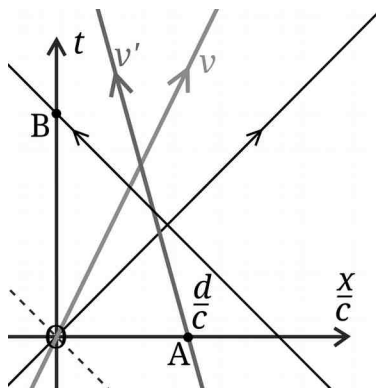
$$t = -\frac{1}{\beta'} \frac{(x-d)}{c} \quad \text{with} \quad \beta' = \frac{v'}{c}$$

Photon passing through O and heading to the right:

$$t = \frac{x}{c}$$

Photon which passes by B and goes to the left:

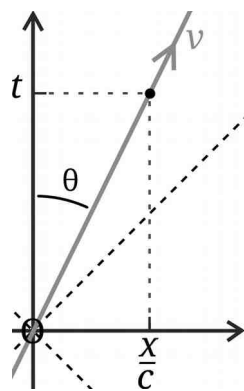
$$t = -\frac{x}{c} + t_B$$



Angles

The more the speed increases, the more the world-line of the spaceship, initially vertical, inclines at an angle θ which tends towards 45° when the speed approaches the maximum speed c .

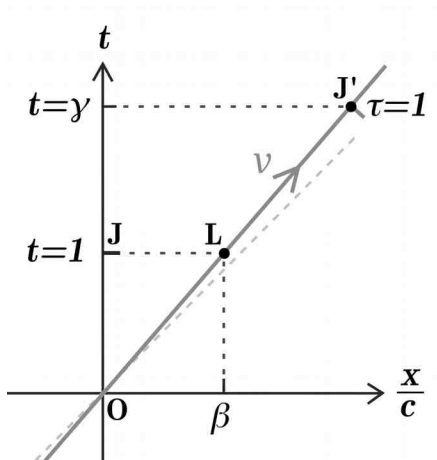
$$\tan \theta = \frac{x/c}{t} = \beta$$



β	0	0.1	0.25	0.5	0.6	0.8	0.9	0.94	1
θ	0°	6°	14°	27°	31°	39°	42°	43.3°	45°
γ	1	1.005	1.03	1.15	1.25	1.67	2.3	3	$+\infty$

Scale factor

On the worldline of the ship, the proper time axis, time passes more slowly and the graduations are more spaced.



$$OJ=1$$

$$OL=\sqrt{1+\beta^2} OJ$$

$$OJ'=\gamma OL$$

$$OJ'=\gamma \sqrt{1+\beta^2} OJ$$

$$\frac{OJ'}{OJ}=\sqrt{\frac{1+\beta^2}{1-\beta^2}}=\sqrt{2\gamma^2-1}$$

v % of c	50	60	75	80	87	95	99	99.5
γ	1.15	1.25	1.51	5/3	2	3.2	7	10
OJ ($t=1$)	1	1	1	1	1	1	1	1
OJ' ($\tau=1$)	1.29	1.46	1.89	2.13	2.6	4.4	10	14

Exercises

† : resolution by Minkowski diagrams.

1.† ▲△△ Draw the Minkowski diagrams of chapter 1 exercises 1 to 5.

Answers p344.

2.†√ ▲△△ **Interstellar communications**

In the Twins Experiment page 17, when the traveling twin lands on the planet Proxima b, it takes a photo and sends it to Earth as an electromagnetic wave.

When will the twin on Earth receive the photo?

Throughout the journey, the twin on Earth follows his brother's journey using a very powerful telescope.

When will he see his brother land on the planet in his telescope?

If the twin on Earth looks through his telescope the moment his brother lands on Proxima b, 5 years after his departure, what does he see?

To send a birthday message to his brother when he lands on the exoplanet, when should he send it?

Make a Minkowski diagram that represents the worldlines of the twins and those of the photons that transmit the photo, the telescope images and the message.

Answers p347.

3.√ ▲▲△ **Call for help**

A cruise ship with more than 10,000 people on board undertakes the Proxima - Earth crossing at the speed of 50% of light.

Halfway through the journey, the ship calls for help. An emergency rescue shuttle leaves Earth at 90% c as soon as the electromagnetic distress message is received.

How long will the passengers have to wait before the arrival of the help ?

Answers p349.

4. ▲△△ **Tim, Tam, Tom**

We are in a slow universe where the maximum speed inherent to space-time is 20 km/h.

Tom, Tim and Tam are in the living room, the clock indicates 10 o'clock. They decide to meet there at 11 o'clock. Tom stays there. Tim leaves to run at 10 km/h. Tam goes to work at his office 10 light-minutes away with a bicycle that travels at 15 km/h

Tim has to be back by what time indicated on his watch?

How much work time will Tam have at his office? What time will his watch show when he returns?

Answers p350.

CHANGING REFERENCE FRAME

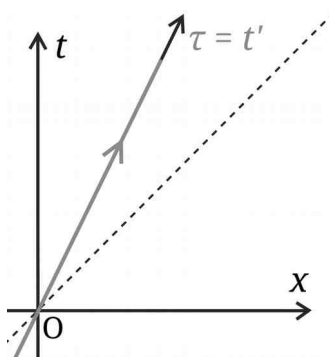
We will consider a second inertial reference frame. The first observational reference frame was the reference frame R of axes (x, y, z, t) , a frame often associated with the galactic frame of reference in the context of interstellar journeys.

The second frame of reference R' is in motion with respect to R , moved at a constant velocity. We say that R' is in uniform rectilinear translational motion with respect to R . For R' the origin is denoted O' and the axes (x', y, z', t') .

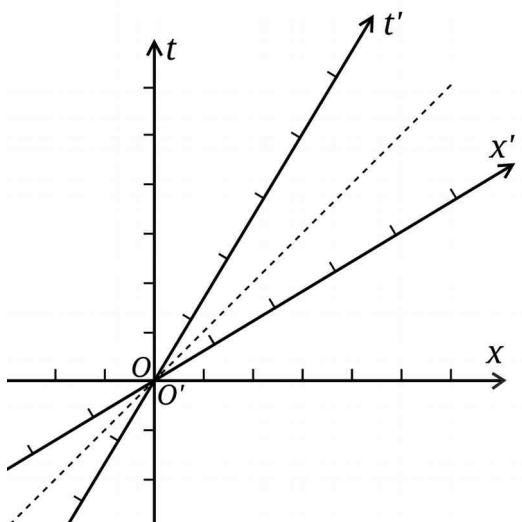
R' is then also a reference frame of inertia, where the principles of special relativity apply. This frame of reference R' will often be associated with the spaceship.

∞ SPACETIME DIAGRAM

We will build step by step the axes of R' in the Minkowski diagram of R . The proper time τ on board the space-time vessel corresponds to the time t' . The axis $O't'$ is thus identified with the ship's worldline.



The speed limit is the same in R and R' . This invariant shows that the axis $O'x'$ is necessarily symmetrical with respect to the worldline of a light flash that moves to the right and passes through O . We thus have the reference frame R' seen from the reference frame R :

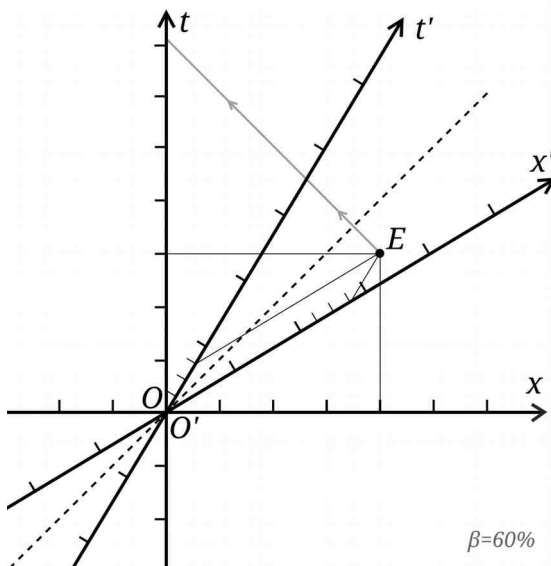


Let's show on an example how the coordinates are read. From Earth, we record, 3 years after the departure of the spacecraft, a huge stellar eruption produced by the star Proxima Centauri located 4 light-years away. The spacecraft is moving at 60% of c . In the reference frame of the spaceship where and when does the eruption occur?

In the galactic frame of reference R the event E has coordinates $(x=4, t=3)$.

In the vessel frame of reference R' we read on the

Minkowski diagram that the event E has coordinates $(x'=2.75, t'=0.75)$. The occupants of the ship will determine that the eruption occurred 9 months after their departure at a distance of 2.75 light-years.

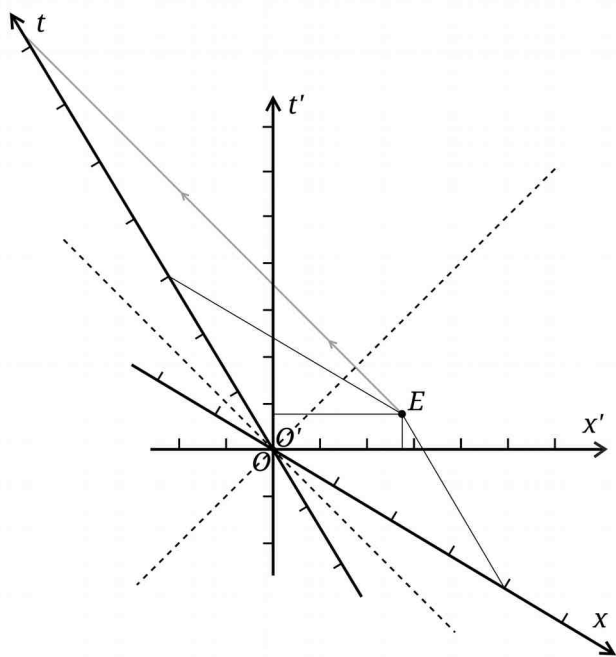


Nevertheless, the astronauts will see the flare in their telescope well after 9 months. Indeed, following the eruption, it is also necessary to allow time for the light to propagate to the telescope and to the eye of the observers. To complete this we have drawn in gray the worldline of a light beam emitted by the flare. It will first be observed in the spaceship after about 3 and a half years of travel, and it will then be observed on Earth 7 years after departure.

In Minkowski's diagrams, the coordinates indicated for an event are taken from local recordings made

using the reference solid and the associated clock crystal. Propagation times are not included.

All inertial frames of reference are equivalent in special relativity and we can also represent R from R' :



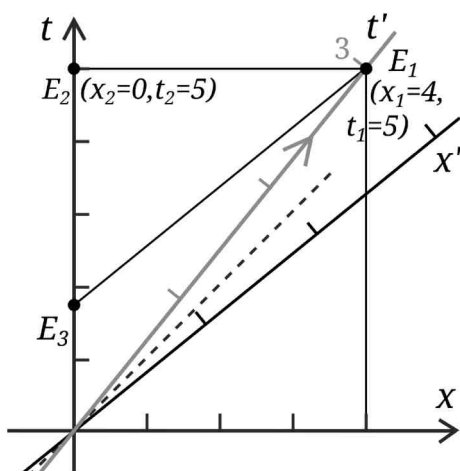
∞ RELATIVITY OF SIMULTANEITY

In the case of the ship heading towards Proxima at 80% c we had a 3-4-5 triangle of times. When the ship is at the level of Proxima 4 light-years away, before reducing its speed, 3 years have elapsed in

the starship and 5 years on Earth.

Consider these two events, E_1 , the ship is at the level of Proxima at 80% c , and, E_2 , the clock on Earth indicates 5 years since departure. An earthling can say to himself: "there, now that 5 years have elapsed for me, the ship is at this very moment at the level of Proxima. If I cannot see it directly with a telescope it is for technique reasons of finite speed information propagation time", he can then have the emotion of the moment shared with the astronauts. If we look at these two events in a space-time diagram they are effectively at the same time t , they are simultaneous events in R .

For this to be true, this emotion of a common moment must be shared, but for the observers on board the ship it is not. For them in R' the simultaneous event on Earth is E_3 , a much earlier event, less than two years after departure.



Simultaneity is relative to the frame of reference considered. In Newtonian mechanics, simultaneous events remained so in all frames of reference, in special relativity simultaneity is not an absolute

notion. In R , E_1 and E_2 are simultaneous, in R' , E_1 is earlier than E_2 .

∞ CAUSALITY

We can only go from the past to the future. It is pure logic, the cause produces an effect and not the opposite! The world is *One*, and this is only an obvious principle of consistency. If you could go back in time and travel in the past, you would destroy the present...

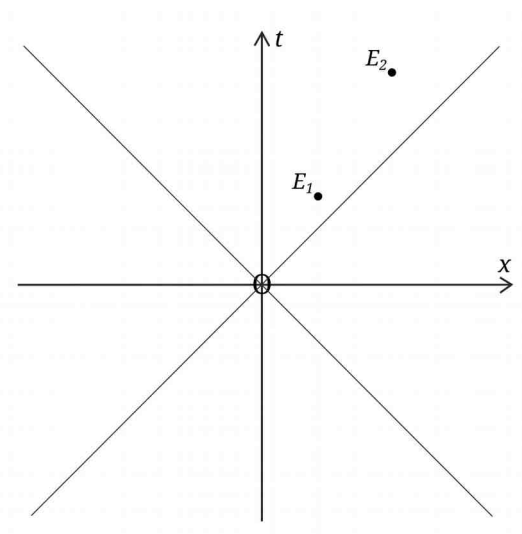
For example, you go 50 years in the past and during this time travel you die in a car accident, or just your actions do that your parents don't actually meet, etc. If you want to travel to the past at all costs, then you would need several presents and suppose parallel worlds which realize all possibilities.

In physics, we choose the simplest theory to explain the facts, there is only one world, *One* reality, the past cannot be changed, the future does not pre-exist, we cannot go back and the *arrow of time* is constantly advancing from the present to the future.

Special relativity of course respects the principle of causality. Not as simply as in the old mechanics, but just as rigorously. The fact that there are several times, the possibility of traveling in the future, a relative simultaneity, can create a confusion that we will clarify immediately.

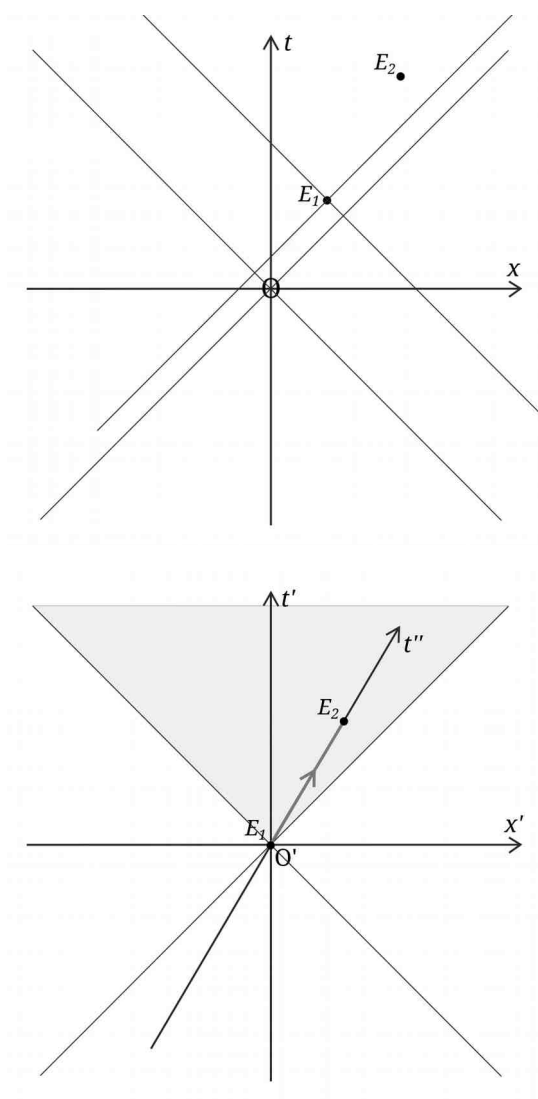
Let us take any two events E_1 and E_2 . If there is a causal link between them we can determine which event is prior, and this temporal ordering must be independent of the observation frame of reference. Two different cases can occur, let us represent the events on a diagram in an arbitrary observation frame R .

First case: there is a possible causal link between E_1 and E_2 . The two events have a constant temporal order whatever the observational reference frame.



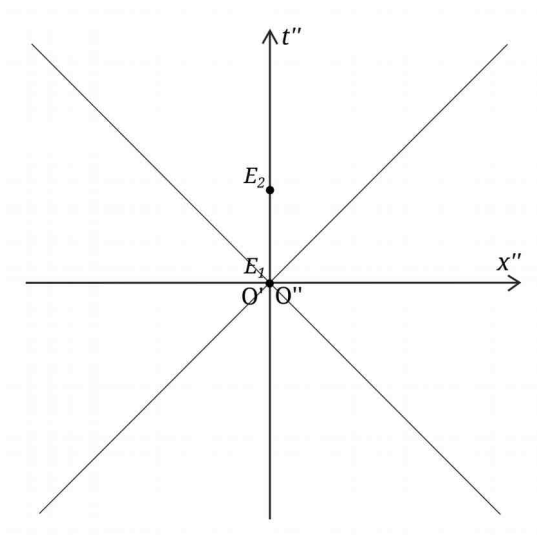
In R , E_2 is subsequent to E_1 because $t_2 > t_1$.

We then consider R' , a frame that is immobile with respect to R but with a new origin $O' = E_1$.



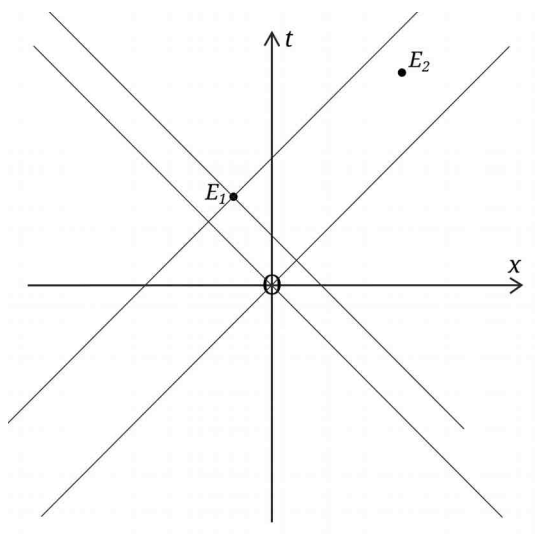
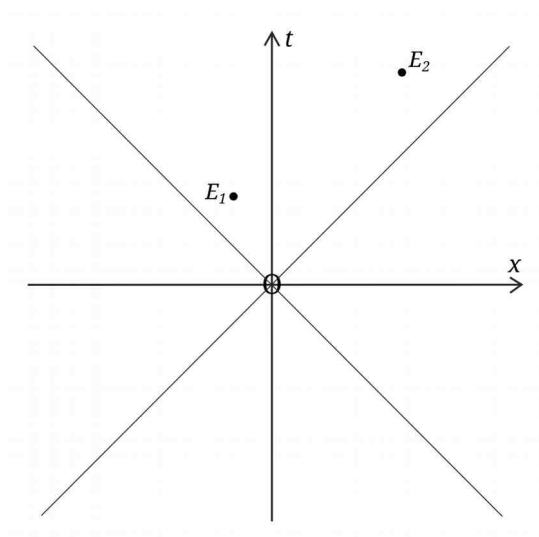
We note a possible causal link between the two events, for example a ship can connect the two points (its speed would not have to exceed the maximum speed), or a succession of events which propagate step by step like in a line of dominoes that fall and establish a causal chain.

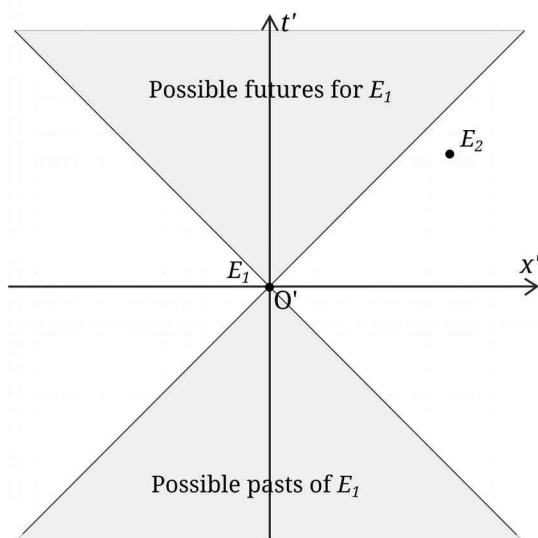
We can then place ourselves in the ship's proper frame R'' , the chronological order is not changed and we always have E_2 later than E_1 and $t_2'' > t_1''$.



Events E_1 and E_2 occur at the same place in R'' . It is in this proper frame of reference that the time interval between events is minimal: $t_2'' - t_1'' = \Delta t'' = \tau < \Delta t' = \Delta t$.

Second case: there is no possible causal link between E_1 and E_2 . The temporal order is not defined, E_1 is prior to E_2 in one frame of reference, the reverse in another, and the events are simultaneous in a third. This does not call into question the principle of causality, because there is no possible cause and effect link between these two events.





No material object or luminous object passing through E_1 can join E_2 , and vice versa. No object can go faster than light. These two events are independent and cannot interact. Looking for a timeline between them does not make sense. There is no proper frame where these two events are at rest.

∞ COMPOSITION OF VELOCITIES

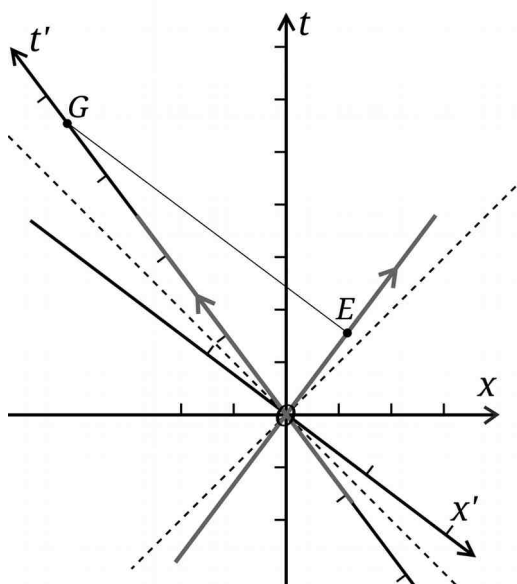
Two ships hurtle towards each other at 75% of maximum speed. If you get into one of the ships, how fast will you see the other ship coming towards you?

If we had the additivity of the speeds as in classical mechanics we would find 150% of c , speed above

the limit, which is, in fact, impossible.

We are going to represent on a diagram the worldlines of the two vessels in the galactic reference frame R . The two vessels approach, cross in O , then move away.

From the frame of reference R' of one of the two vessels, we measure the coordinates of the second and we will simply have its speed in R' .



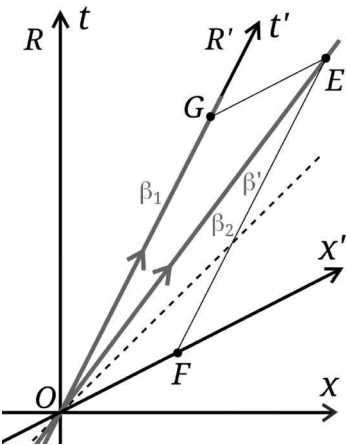
The distance OG corresponds to t' and measures 4.8 cm on the drawing. The distance EG corresponds to x' and measures 4.6 cm on the drawing. By dividing EG by OG we get the relative speed of the vessels:

$$v' = 96\% \text{ of } c$$

Let us now take a second case where the vessels move in the same direction, one at 50% and the other at 75% of c :

We divide EG by EF and we find a relative speed between the two vessels of 40% c .

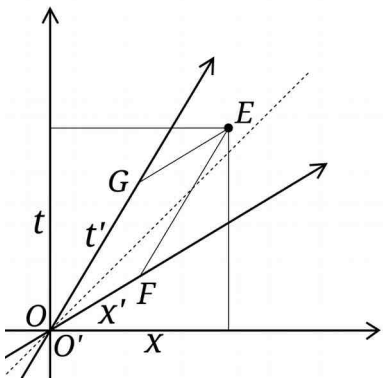
Clearly, for relativistic speeds, the speeds do not add or subtract. The law of composition of the speeds is different in special relativity and ensures that the speeds of the objects are indeed subluminal.



∞ USE OF EQUATIONS

◦ Lorentz transformation

For an event E , we want to express its coordinates (x',t') in R' in relation to that (x,t) in R .



For t' :
$$t' = \frac{OG}{\gamma \sqrt{1+\beta^2}}$$

We have applied the scaling factor to go from R to R' , a factor established in the previous chapter.

The coordinates of point G are given by the intersection of the two following lines:

$$t = \frac{1}{\beta} \frac{x}{c} \quad \text{and} \quad t - t_E = \beta \left(\frac{x}{c} - \frac{x_E}{c} \right)$$

(t' axis and straight line parallel to the x' axis which passes through E with a slope inverse to that of the t' axis)

After solving this system of equations :

$$t_G = \gamma^2 \left(t_E - \beta \frac{x_E}{c} \right) \quad \text{and} \quad \frac{x_G}{c} = \beta t_G$$

$$\text{So : } OG = \sqrt{t_G^2 + \left(\frac{x_G}{c} \right)^2} = \gamma^2 \sqrt{1+\beta^2} \left(t_E - \beta \frac{x_E}{c} \right)$$

$$\text{And finally : } t' = \gamma \left(t_E - \beta \frac{x_E}{c} \right)$$

Proceeding in a similar way for x' , we find :

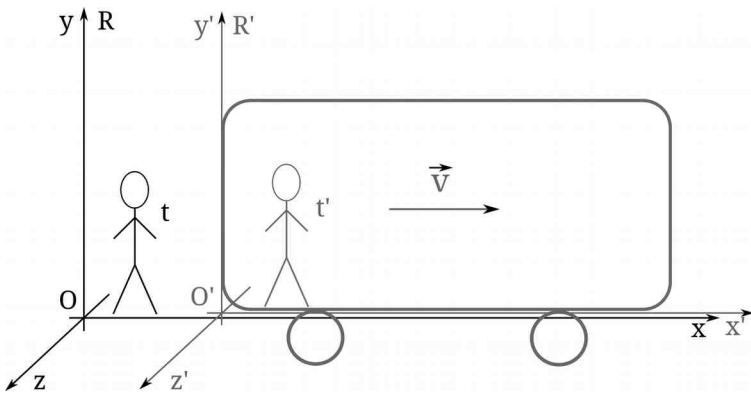
$$\frac{x'}{c} = \gamma \left(\frac{x_E}{c} - \beta t_E \right)$$

We obtain what is called the Lorentz transformation of the coordinates of an event. For a motion of R' with respect to Ox and the setting to zero of clocks

and spatial coordinates when they coincide in $O=O'$, we can, without losing generality, write :

**Lorentz
transformation**

$$\left\{ \begin{array}{l} \frac{x'}{c} = \gamma \left(\frac{x}{c} - \beta t \right) \\ y' = y \\ z' = z \\ t' = \gamma \left(t - \beta \frac{x}{c} \right) \end{array} \right.$$



At $t=0$ and $t'=0$, O and O' coincide, then O' moves away to the right as time passes. On a Minkowski diagram, in full agreement with the one above, O and O' are no longer points but worldlines, the axis of t and the axis of t' . The origins O and O' indicated are the spatio-temporal positions at $t=t'=0$.

To get the coordinates in R from those in R' , simply change the sign of the speed and thus β :

$$\left\{ \begin{array}{l} \frac{x}{c} = \gamma \left(\frac{x'}{c} + \beta t' \right) \\ y = y' \\ z = z' \\ t = \gamma \left(t' + \beta \frac{x'}{c} \right) \end{array} \right.$$

Within the limits of low speeds we find the classical transformation of coordinates. Spatial and time coordinates are then disconnected to let space and time both absolute:

Galilean
transformation

$$\left\{ \begin{array}{l} x' = x - vt \\ y' = y \\ z' = z \\ t' = t \end{array} \right.$$

In this book we made the pedagogical choice to start from the triangle of times to construct the special relativity. We could also start from the Lorentz transformation. In what follows we find the time dilation, the length contraction and the existence of a relativistic invariant using this transformation.

• **Time dilation:** we have events that occur at the same location in R , so $x_2=x_1$ and $\Delta x=x_2-x_1=0$, separated by a time interval $\Delta t=t_2-t_1$. What happens to this time interval in R' ? $\Delta t'=(\gamma\Delta t-\beta\Delta x/c)$ then $\Delta t'=\gamma\Delta t$. QED

• **Length contraction:** we can imagine a ruler at rest in R placed on the x -axis, $L=\Delta x=x_2-x_1$. The protocol for measuring a length in a given frame of reference requires to determine the positions of the ends at the same time in this frame. Measurement of the relative length L' in R' : $\Delta x=\gamma(\Delta x'+\beta c\Delta t')$ and $t'_2=t'_1$ then $L=\gamma L'$, and $L'=L/\gamma$. QED

• **Lorentz invariant:** In classical mechanics we had two invariant quantities: length $L=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ and duration Δt . Whatever the observational frame of reference, we had the Euclidean distance and the duration unchanged. This is no longer the case in special relativity. But we have another quantity that verifies this property: $\Delta s^2=c^2\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2$. Δs is the *spacetime interval* between any two events, it corresponds to a sort of Minkowskian distance. Its property of invariance is verified by carrying out the calculation of Δs in a second inertial reference frame R' :

$$\Delta s'^2=c^2\Delta t'^2-\Delta x'^2-\Delta y'^2-\Delta z'^2$$

$$\Delta s'^2=\gamma^2(c\Delta t-\beta\Delta x)^2-\gamma^2(\Delta x-\beta c\Delta t)^2-\Delta y^2-\Delta z^2=\Delta s^2$$

We can write Δs^2 as a function of the speed v of an object which joins the two events along a rectilinear

and uniform trajectory:

$$\Delta s^2 = c^2 \Delta t^2 \left(1 - \frac{v^2}{c^2} \right)$$

Δs^2 can be of different signs, if there is a possible causal link between the events, $v \leq c$, Δs^2 is positive or null and the interval is said to be timelike or lightlike (null vector), if it is negative, $v > c$, Δs^2 is spacelike. When Δs^2 is not spacelike, we can link the interval Δs to the proper time τ :

$$\tau = \frac{\Delta s}{c} = \Delta t \sqrt{1 - \frac{v^2}{c^2}}$$

Proper time is the fundamental notion on which special and general relativity is built. This measure of the aging of a particle is invariant and absolute, unlike the space-time coordinates (ct, x, y, z) which are relative and have no physical meaning in themselves.

◦ Composition of velocities

We use the notations in the figure on page 61. β_1 and β_2 are the speeds in R of starships 1 and 2 expressed as a percentage of c . β' is the speed of vessel 2 in R' .

The first equation is for the world line of ship 1 in R , the

$$\left\{ \begin{array}{l} t = \frac{1}{\beta_1} \frac{x}{c} \\ t - t_E = \beta_1 \left(\frac{x}{c} - \frac{x_E}{c} \right) \\ t_E = \frac{1}{\beta_2} \frac{x_E}{c} \end{array} \right.$$

second for the line (EG) and the third the relationship between the coordinates of a point E on the worldline of ship 2 .

The first equation applied to point G gives:

$$OG = \sqrt{t_G^2 + \left(\frac{x_G}{c}\right)^2} = \sqrt{1 + \beta_1^2} t_G$$

Besides :

$$t_G - t_E = \beta_1(\beta_1 t_G - \beta_2 t_E) \quad \text{and} \quad t_G(1 - \beta_1^2) = t_E(1 - \beta_1 \beta_2)$$

After some calculus, we have EG as a function of β_1 , β_2 and t_G . We calculate the relative speed:

$$\beta' = EG/OG.$$

Then :

$$\beta' = \frac{\beta_2 - \beta_1}{1 - \beta_1 \beta_2} \quad (\text{vessels in the same direction}),$$

$$\beta' = \frac{\beta_1 + \beta_2}{1 + \beta_1 \beta_2} \quad (\text{vessels in opposite directions}).$$

We find the good results for the two examples of the course:

$$\beta' = \frac{0.75 + 0.75}{1 + 0.75 \times 0.75} = 0.96 \quad \text{and} \quad \beta' = \frac{0.75 - 0.5}{1 - 0.5 \times 0.75} = 0.4$$

In terms of speeds we have:

$$v' = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

If the speeds are small compared to c , the denominator tends towards 1 and $v' = v_1 + v_2$, we find the classical additivity of the velocities again.

Second method : We reasoned with objects which move at constant velocities. We can do a more general calculation using the Lorentz transformation. Definition of the instantaneous velocities with respect to (x,t) and (x',t') in R and R' :

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt} \quad \text{and} \quad v' = \frac{dx'}{dt'}$$

these quantities should be noted v_x and $v_{x'}$, we will write v and v' for ease of reading.

From Lorentz's transformation:

$$x' = \gamma(x - \beta ct) \quad \text{and} \quad ct' = \gamma(ct - \beta x) \quad \text{with} \quad \beta = u/c$$

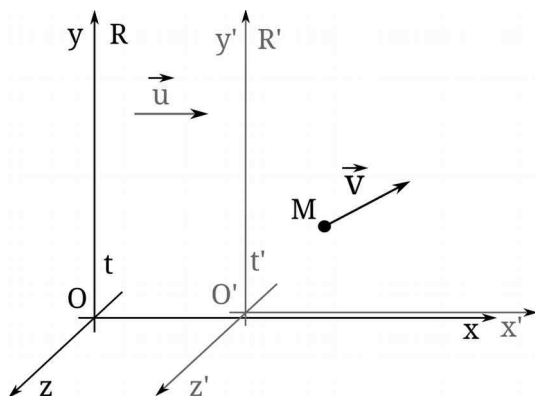
hence for infinitesimal variations:

$$dx' = \gamma(dx - \beta c dt) \quad \text{and} \quad c dt' = \gamma(c dt - \beta dx)$$

And by dividing the two equations:

$$\frac{dx'}{c dt'} = \frac{dx - \beta c dt}{c dt - \beta dx} \quad , \quad \frac{v'}{c} = \frac{\frac{v}{c} - \beta}{1 - \beta \frac{v}{c}} \quad \text{and} \quad v' = \frac{v - u}{1 - \frac{uv}{c^2}}$$

u is the speed of R' compared to R .



We can easily obtain the two other components of the velocity for y and z ⁶, but we limit ourselves here to the rectilinear motion.

◦ Transformation of accelerations

With respect to x and x' : $a_x = \frac{dv_x}{dt}$ and $a_{x'}' = \frac{dv_{x'}'}{dt'}$

Simply noted a and a' thereafter.

$$a' = \frac{dv'}{dt} \frac{dt}{dt'} = \frac{a(1 - \frac{uv}{c^2}) + (v-u) \frac{u}{c^2} a}{(1 - \frac{uv}{c^2})^2} \frac{1}{\gamma(1 - \frac{uv}{c^2})}$$

(quotient rule)

$$\text{Then : } a' = \frac{1}{(1 - \frac{uv}{c^2})^3 \gamma^3} a$$

In the case where M is initially at rest in R' the initial

velocity v is zero and $a' = \frac{a}{\gamma^3}$.

6 Done in exercise on page 101 (composition of velocities and accelerations in 3D).

Exercises

1. ▲▲△ Composition of velocities

a - Two vessels are heading towards each other at 50% of c . What is their relative speed?

b - Two vessels are moving in the same direction, one at 80% of c and the other at 50% of c . What is their relative speed?

Answers p351.

2. ▲▲△ Two vessels

Two spaceships A and B produce the following events in the galactic frame R :

$$E_{A,1}(x_A=0, y_A=0, z_A=0, t_1=0)$$

$$E_{B,1}(x_B=2, y_B=2, z_B=2, t_1=0)$$

$$E_{A,2}(2, 0, 0, t_2=4) \quad E_{B,2}(4, 4, 4, t_2=4)$$

$$E_{A,3}(4, 0, 0, 8) \quad E_{B,3}(5, 5, 5, 8)$$

Distances and times in light-years and years.

R considered of inertia.

a - What are the average velocities of the ships between $t=0$ and $t=4$?

Same question between $t=4$ and $t=8$.

b - What are the average accelerations of the vessels between $t=0$ and $t=8$?

c - Vessel A has a translational, rectilinear and uniform motion. We call R' the reference frame of vessel A . Is the frame of reference R' inertial?

Determine the coordinates of the events of vessel B as seen from vessel A .

Can the trajectory of vessel B in R be rectilinear? Is it the same for the trajectory of B seen from R' ?

d - In R' , determine the average velocity of vessel B between t'_1 and t'_2 , then between t'_2 and t'_3 .

e - In R' , determine the average acceleration of vessel B .

f - Could you determine the average acceleration felt by the passengers of vessel B ?

g - Accelerations are calculated here in ly/yr^2 , how to convert them into m/s^2 ?

Deduce the acceleration to which the astronauts are subjected as a percentage of the Earth's gravity field at sea level: $g=9.81 \text{ m/s}^2$.

Answers p351.

3.√ ▲▲△ **Low speeds limit**

Two cars drive face to face at 90 km/h . What is their relative speed? Determine the difference with the classical limit.

Answers p356.

THE APPEARANCE OF THINGS

Sometimes we naively forget to take into account the duration of the propagation of the signals to our eye, as if we were spontaneously seeing spacetime as a whole.

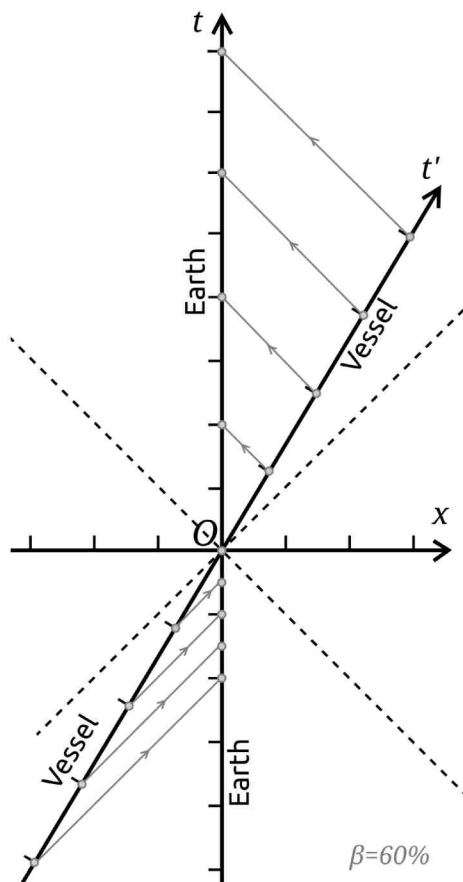
We will begin by studying the Doppler effect where, due to relative motion, the color of objects is modified. The color of light depends on the period of the light wave. This quantity is a time, and we could think that it is sufficient to take into account the time dilation. The perceived period would simply be multiplied by γ as the travel time in the twin experiment. Except that the twins end up in the same place and there is no delay due to the propagation of a signal at finite speed. For the Doppler effect the frequency will not simply be divided by γ , and moreover, it will differ depending on whether the vessel is moving closer or further away.

After studying the Doppler effect, we will take pictures of a relativistic ruler, followed by a contemplation of the starry sky in a starship each time faster.

∞ DOPPLER EFFECT

The Doppler effect can be experimented with all kinds of waves: sound waves, electromagnetic waves, waves on water, etc... In all cases, we have a wave propagating at a finite speed, and a source and a receiver in relative motion. For example, for a sound wave that propagates in the air, if you get closer to the source the frequency is heard higher, and if the source moves away the sound is, on the contrary, perceived deeper.

Here we will focus on an electromagnetic wave that propagates in vacuum, or more precisely in spacetime. In this case, in addition to the delays or advances due to the propagation of the wave towards a moving object, the effect of spacetime perspective is added. Let's take the example of a yellow



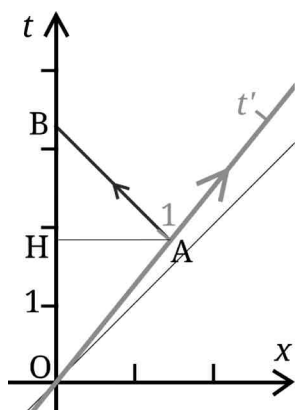
light, with a wavelength of 600 nm, emitted by a vessel moving at 60% c. To simplify, let's imagine that the vessel emits regular flashes of light at the frequency of the wave. We have drawn the world-lines of these flashes on a Minkowski diagram. We see on Earth the flashes closer when the ship is approaching and further apart when the ship is moving away. The time between the reception of two flashes corresponds to the period of the signal, we measure on the diagram, when the ship is approaching the Earth:

$T = T'/2$ so $f = 2f'$ and $\lambda = \lambda'/2$ then $\lambda = 300$ nm,
the light received is in the ultra-violet.

When the ship moves away:

$T = 2T'$ so $f = f'/2$ and $\lambda = 2\lambda'$ then $\lambda = 1200$ nm,
the light received is in the infrared.

◦ Use of equations



A periodic signal is emitted in R' with a period T' , and, is received in R with a period T . We call r the ratio between these two periods : $r = T/T'$.

In the event that the source and receiver move away:

$$r_+ = OB_R/OA_{R'}$$

For $OA = 1$ on the axis of t' we

have $OH = \gamma$ on the axis of t .

In (x, t) : $OA = \gamma \sqrt{1 + \beta^2}$ (scale factor).

The triangle BHA is right isosceles in H : $AH = BH$.

Pythagorean theorem in OHA :

$$r_+ = OB = OH + HB = \gamma + \sqrt{\gamma^2(1+\beta^2) - \gamma^2}$$

$$\text{When moving away : } r_+ = \gamma(1+\beta) = \sqrt{\frac{1+\beta}{1-\beta}}$$

$$\text{When getting closer : } r_- = \gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}$$

In terms of frequencies, $f = 1/T$:

$$f' = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} f \quad \text{and} \quad \boxed{T = \sqrt{\frac{1 \pm \beta}{1 \mp \beta}} T'}$$

Interval : $0 < r < 2\gamma$

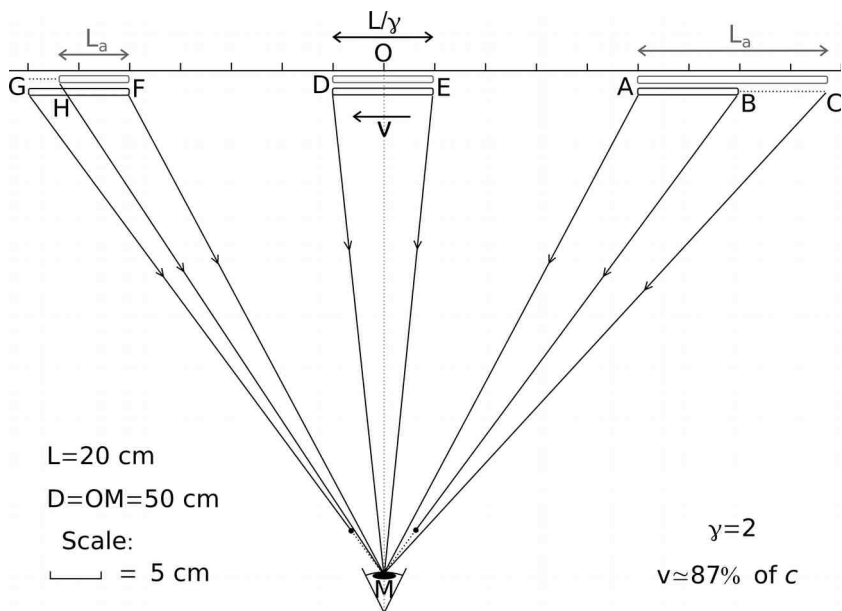
In the example of the course, $\beta = 0.6$, $\gamma = 1.25$ and the numerical application gives correctly $f = 2f'$ when transmitter and receiver are approaching, and $f = f'/2$ when they are moving away.

The Doppler effect shows that the color of a photon is not an absolute quantity. A photon is neither red, blue, nor yellow, it all depends from where you look at it. Its wavelength depends on the observational inertial frame of reference and there is no privileged observer.

A photon has other characteristics, such as chirality, which is intrinsic. A photon is either left or right and, unlike its color, it does not depend on the point of view.

∞ PHOTOGRAPH OF A MOVING RULER

A ruler moves at the velocity \vec{v} in the observational reference frame R . A graduated optical bench, fixed in R , makes it possible to locate the position of the two ends of the ruler. The proper length of the ruler, in the frame of reference R' where it is at rest, is denoted L . The length in the laboratory is the contracted length L/γ . We take different pictures of the ruler as it passes. On each photograph, we note the apparent length L_a , the difference between the abscissas of the two ends marked on the bench.



The contracted length corresponds to measurements at the same instant t of the position of the ends, while the image of the ruler which appears on

the photographic plate is formed by photons which reach the sensor at the same instant and which, due to time of different routes, were not emitted at the same time at the object level.

We do not yet know how to make a camera with such sensitivity and such a short shutter time, but it is not out of reach given current advances in optoelectronics. Second challenge, to animate a macroscopic object at a relativistic speed. The thinking exercise is excellent anyway, as it allows us to deepen our understanding of the theory.

Let us think in the laboratory reference frame R . The ruler of length L/γ comes from the right. The light rays emitted at the same time from the A and B ends will not reach the eye at the same time and will therefore not be in the same image. The ray emitted by B will arrive late.

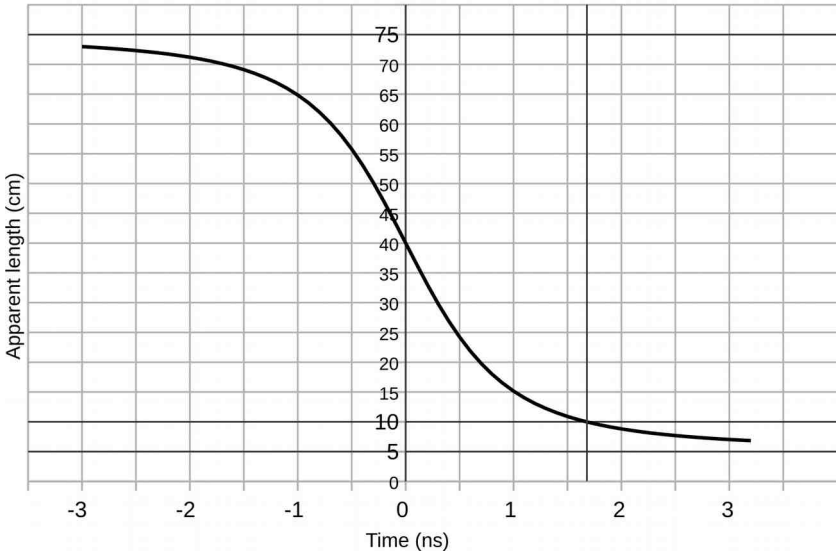
There is an earlier moment when the ray emitted by this end compensates this delay, it is the case of point C on the diagram. The apparent length is then greater than the contracted length.

When at $t=0$, the ruler is centered on O , the rays are emitted symmetrically and the apparent length is equal to the contracted length. This occurs for a photo taken at $t \simeq 1.7$ ns (light travel time from D , or E , to M).

For t positive, when the ruler moves away, the apparent length is instead smaller than the contracted length.

Below we have the curve of the apparent length versus time t :

Apparent Length of a Mobile Ruler



We can easily find the extreme values. When the ruler is still far away, the delay of the light beam from C is about AC/c . Moreover the ruler moves at the speed v , so, to catch up, BC is worth v times the delay.

$$\text{Then : } L_a = AB + BC = \frac{L}{\gamma} + v \frac{L_a}{c}$$

$$\text{So : } L_a = \frac{L}{\gamma(1-\beta)} = L \sqrt{\frac{1+\beta}{1-\beta}} \simeq 75 \text{ cm}$$

On the contrary, when the ruler moves away, to the limit of $t \rightarrow +\infty$,

$$\frac{HF}{c} = \frac{L_a}{c} = \frac{L/\gamma - L_a}{\beta c}, \quad \text{and} \quad L_a = L \sqrt{\frac{1-\beta}{1+\beta}} \simeq 5 \text{ cm}.$$

We finally find the same kind of formula as for the Doppler effect with inverted effects :

$$L_{a, t \rightarrow \pm\infty} = L \sqrt{\frac{1 \mp \beta}{1 \pm \beta}}$$

When an object gets closer, the perceived period is shorter and its length seen, in the direction of motion, is greater, on the contrary, when it moves away the perceived period is greater and its length seen smaller.

We also had an inversion of behavior between time and space with time dilation and length contraction.

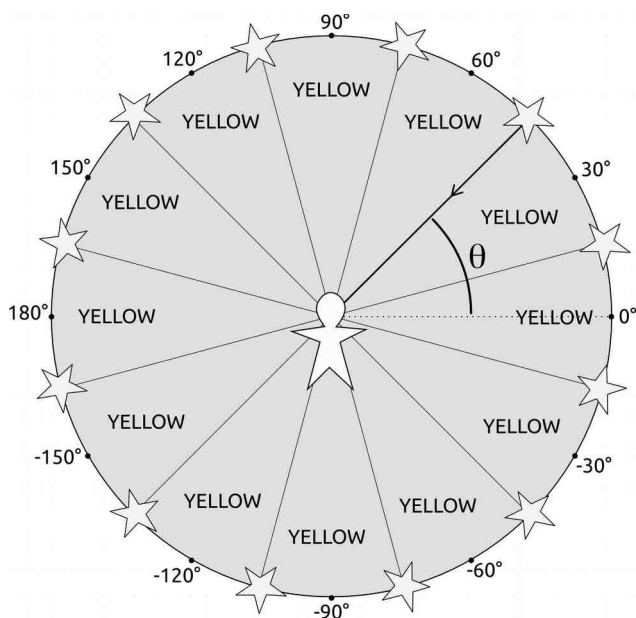
We just did the long-distance calculations. To find the complete curve of the length on the photograph as a function of time, we consider a three-dimensional Minkowski diagram (x, y, ct) .

The camera is represented by a vertical world line $(0, D, ct)$. The optical bench by the world plane $y=0$. The ruler by an inclined world strip. The resolution of the problem is in principle simple: find the intersection between the past light cone from the eye at time t with the world strip of the mobile ruler.

The intersection gives the position of the two ends in R : $E_1(x_1, 0, ct_1)$ and $E_2(x_2, 0, ct_2)$. We then have the apparent length $L_a = x_2 - x_1$. Except at O , we have well $t_1 \neq t_2$.

∞ THE STARRY SKY SEEN FROM THE SHIP

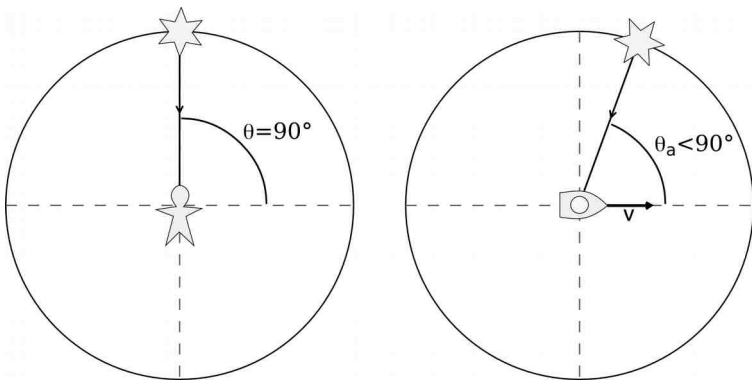
Let's determine the change in the perception of the starry sky as a function of the speed of the ship. In addition to the change in the perceived color of the stars by Doppler effect, their position in the sky is modified, this is called the aberration of light. When we are at rest in the galactic frame of reference, the stars are, as a whole, motionless. To simplify, we will consider yellow and homogeneously distributed stars.



Let's take the case of a star seen at rest in the galactic frame of reference perpendicular to the direction of motion of the spacecraft. Under which angle θ_a is this same star seen from the ship's frame

of reference?

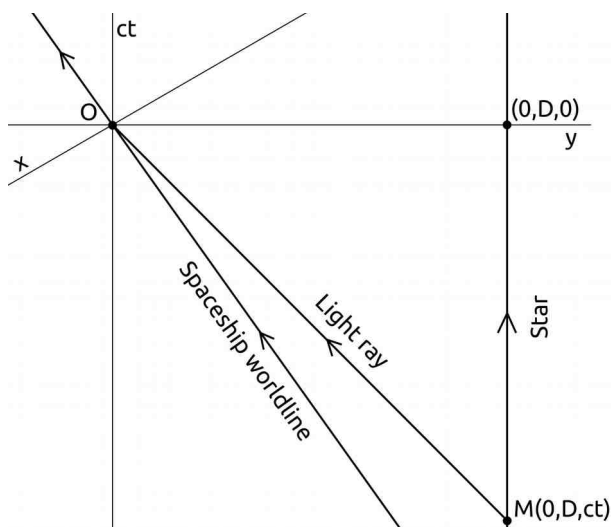
We can make an analogy with the rain that falls, seen through the windshield of a car it looks like the rain comes from the front, even if from the road it falls vertically. The demonstration in classic mechanics is quite simple, just apply the addition of the velocities. You can imagine that here the result will be, at least quantitatively, different.



We have to think again in a three-dimensional Minkowski diagram (x, y, ct) . As soon as we measure an angle, there are at least two dimensions of space. However, there is no need to use the third dimension of space, because there is invariance by rotation according to the direction of the vessel, otherwise, in addition to colatitude θ , we would have had to use longitude φ and we would have had to work in a four-dimensional Minkowski diagram (x, y, z, ct) .

We consider the galactic frame of reference and we start by studying the case $\theta=90^\circ$. The ship is moving in the direction of increasing x and the star is

located on the y axis at a distance D . We have three world lines, one for the spacecraft in the (x, ct) plane, one for the star, vertical, and one for the light ray in the (y, ct) plane.



We define a straight line by the intersection of two planes defined in Cartesian coordinates.

$$\text{Light ray worldline : } \begin{cases} x=0 \\ y+ct=0 \end{cases}$$

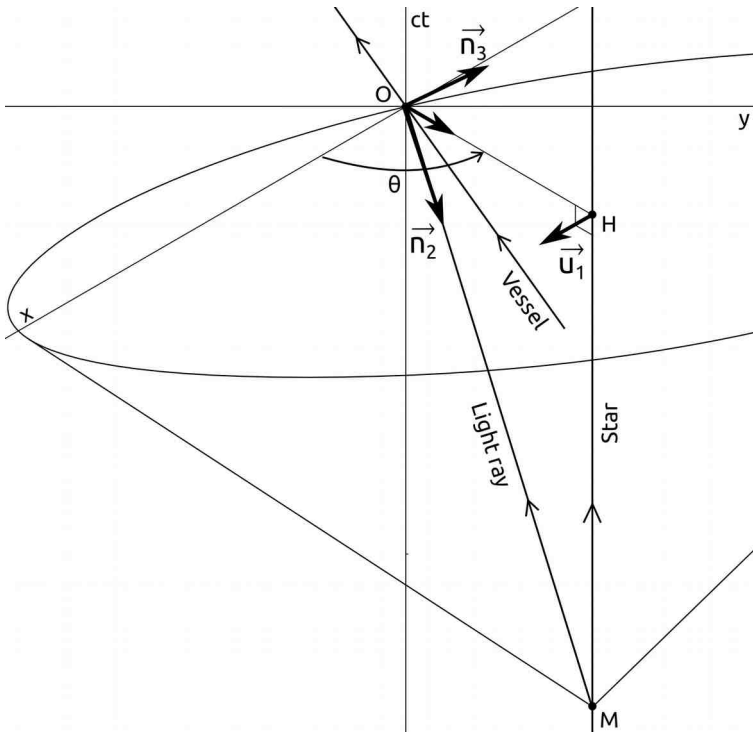
We then use the Lorentz transformation to obtain this equation in the ship's reference frame R' :

$$\begin{cases} x'=\gamma(x-\beta ct) \\ y'=y \\ ct'=\gamma(ct-\beta x) \end{cases} \quad \text{and} \quad \begin{cases} x'=\gamma\beta y' \\ \gamma y'+ct'=0 \end{cases}$$

also $\tan(\theta_a)=\frac{y'}{x'}$, then $\tan \theta_a=\frac{1}{\gamma\beta}$

In the case of a starship moving at $87\% c$, we find for $\theta=90^\circ$, $\theta_o=30^\circ$. We notice that the result does not depend on the distance D . The effect is accentuated with respect to the Newtonian formula where $\tan(\theta_o)=1/\beta$ and $\theta_o \simeq 49^\circ$.

Now let's look for any angle θ between 0° and 180° .



A unit vector parallel to \overrightarrow{OH} has for coordinates $(\cos \theta, \sin \theta, 0)$. The vector \vec{u}_1 orthogonal to the OHM plane has the coordinates $\vec{u}_1(\sin \theta, -\cos \theta, 0)$.

As collinear vector to the light beam we can choose $\vec{n}_2(\cos \theta, \sin \theta, -1)$. We verify that $\vec{n}_3(\cos \theta, \sin \theta, 1)$ is orthogonal to \vec{u}_1 and \vec{n}_2 .

Hence the world line of the light ray:

$$\begin{cases} \sin \theta x - \cos \theta y = 0 \\ \cos \theta x + \sin \theta y + ct = 0 \end{cases}$$

Using the same Lorentz transformation as the one used in the previous case, we obtain, after a somewhat long but simple calculation:

$$y' = \frac{\sin \theta}{\gamma(\beta + \cos \theta)} x'.$$

Thus the expression for $\tan(\theta_a)$, or, simpler to use, after some mathematical manipulations, detailed in exercise, the expression of $\tan(\theta_a/2)$:

$$\tan\left(\frac{\theta_a}{2}\right) = \sqrt{\frac{1-\beta}{1+\beta}} \tan\left(\frac{\theta}{2}\right)$$

For the color of the star, we give the expression of the wavelength perceived in the vessel which takes into account the transverse Doppler effect :

$$\lambda_a = \frac{1 - \beta \cos \theta_a}{\sqrt{1 - \beta^2}} \lambda$$

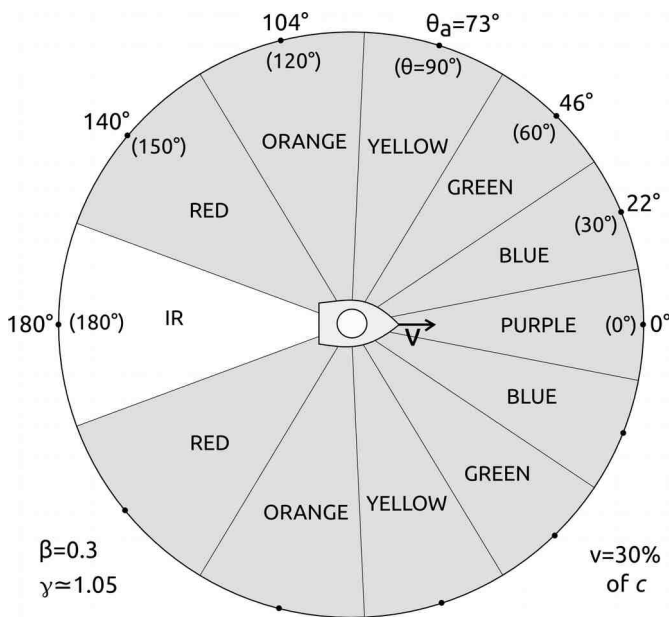
For example, for $\beta=0.3$ and $\lambda=600$ nm, we have the results in the following table which we then reported in a circular diagram.

Angles in degrees and wavelengths in nm:

θ	180	165	150	135	120	105	90	75	60	45	30	15	0
θ_a	180	160	140	121	104	87	73	59	46	34	22	11	0
λ_a	818	806	773	726	673	621	572	531	498	472	454	444	440

As the ship gains speed, the stars in the front turn

blue and those in the back red. Laterally we have all the spectral shades with an zone where the stars remain yellow. The forward hemisphere, under which we saw the stars at rest, is narrowing. Some stars present in the rear hemisphere appear in the front of the vessel, for example for $\theta=105^\circ > 90^\circ$, we have $\theta_a=87^\circ < 90^\circ$.



For even higher speeds, the stars fade in the front by passing in the UV, and in the back as they pass in the infrared. At 87% of c , only a visible ring is left in the front around 50° . However, new objects will appear, celestial objects in the infrared in the galactic frame of reference will be visible at the bow of the ship and objects in the UV will become visible at the stern.

The intensity corresponds to the energy received per m^2 and per second.

The energy comes from photons, of individual energy $e=hc/\lambda_a$. Due to the Doppler effect, the photons see, on the one hand, individually, their frequency, and thus their energy modified, and on the other hand, taken as a whole, they arrive with a different rhythm. The two effects have the same Doppler factor $\sqrt{1-\beta^2}/(1-\beta \cos \theta_a)$, hence the square in the expression of I_a . The photons shoot more frequently and violently at the front, and more slowly and softly at the back.

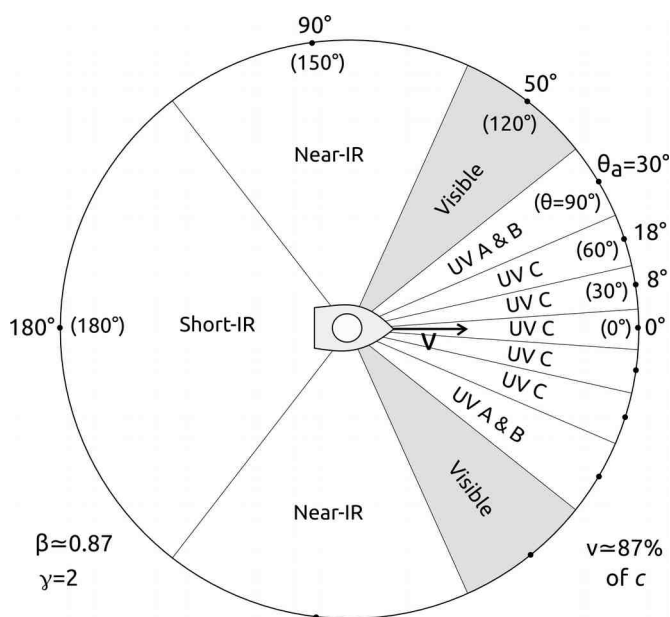
Now let's look at a group of stars, they occupy a certain area, also called a solid angle, on the celestial vault. As the ship speeds up one group of stars in the front tightens and another, in the rear, stretches. To calculate the total energy received, we must also take into account this density of stars which varies.

To find the total energy received, we integrate the light intensity on a spherical surface S of radius R , centered on the vessel. We have the following results, established in exercise:

$$E_a = \int_{\theta_a=0}^{\pi} I_a d\Omega_a = \gamma^2 (1 + \beta^2/3) E$$

$$\text{with } E = \int_{\theta=0}^{\pi} I d\Omega = 4\pi I = E(\beta=0)$$

Ω is the solid angle, it corresponds by definition to the cut surface on a unit sphere, $\Omega=S_{\perp}, R=1$.



To illustrate, at 30% of c , the frontal solid angle, of vertex angle 30° in R , reduces to 22° in R' , thus the apparent density of stars in this frontal part of the sky becomes 80% greater⁷. In addition, the photons received have a higher energy, from yellow they become blue, and moreover they are received in greater number.

At 50% of c , the stars become even more rare at the back, and 91% of the light energy comes from the front hemisphere.

At 95% of c , the sky is 13 times brighter.

Now what about the number of photons arriving on the ship? We have N photons which arrive on the ship during a proper time interval. From the galactic

⁷ ratio of the surfaces seen under the solid angles $\Omega = 2\pi(1 - \cos\theta)$.

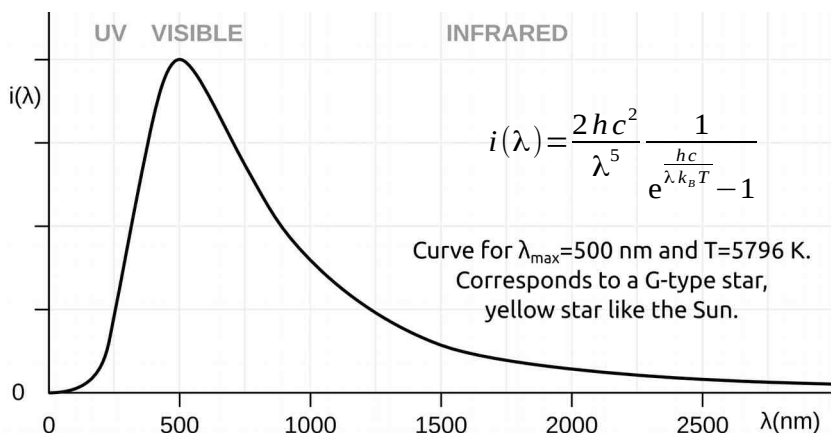
frame of reference, we observe these same photons arriving on the vessel during a relative dilated interval. Thus, the more the vessel gains speed, the more the number of photons received per second by the astronauts increases with the factor γ .

At 50% c , the vessel receives 15% more photons, and 84% of the photons come from the front hemisphere.

At 95% of c , the vessel receives 3 times more photons, the front celestial hemisphere is 26 times brighter, and the back hemisphere 350 times less. Now let's focus on a half-degree disk, which is the apparent size of the Moon or Sun as seen from Earth. This disc located at the zenith of the ship will have a luminosity 1500 times greater than that of the sky at rest. For comparison with what is observed from the Earth's ground, this luminosity is 40,000 times less than that of the Sun, and 12 times greater than that of the full Moon⁸. But beware, this central disk emits in the ultraviolet, the visible corona is located between 34 and 52°.

Of course the stars are not all the same color, the Sun is yellow, but Rigel is blue and Betelgeuse red. In addition, a star does not emit only one wavelength but a continuous spectrum given by what is called the spectrum of the black body:

8 Data: Starry sky 0.002 lux / Moon 0.25 / Sun 120,000 lux.



Thus stars of the Sun type, such as Alpha Centauri A or B, can be seen with the naked eye at the front of the ship even at 50% of c , because they also emit in the IR which shifts in the visible by Doppler effect, and, although the emitted intensity is lower in the IR, this is compensated by an increase in the perceived intensity towards the front. So, no navigation problem by finding your way in the starry sky to reach Proxima Centauri. On the other hand, towards the rear of the ship, the stars will fade much faster.

Regarding the energy and the number of total photons received the results do not change because they do not depend on the wavelength. The Doppler factor does not depend on λ and the aberration displaces all the chromatic components of a star's spectrum by the same angle. There is no dispersion, as in the phenomenon of refraction of light rays (through a prism the frequency components are deflected differently and create an iridescence in the form of a rainbow).

Exercises

$\frac{01}{10}$: resolution by numerical simulation.

1. ▲▲△ The suicidal physicist

A driver arrives at a crossroads and the traffic light is red. The driver, who is going crazy after reading a physics book, decides, instead of stopping, to increase his speed so that by Doppler effect, the light of the traffic light appears green to him.

What speed should his vehicle reach?

$\lambda_{\text{red}}=700 \text{ nm}$, $\lambda_{\text{green}}=500 \text{ nm}$

Answers p357.

2. ▲▲△ Laser sail

A Terajoule laser battery based on the ground bombs photons for 10 minutes on a sail placed in orbit. The sail reaches a speed of 20% of c .

a - What is the radiation pressure exerted on the sail depending on the light power Φ received?

b - For a constant luminous power incident on the sail in the terrestrial reference frame, will the force felt on the sail remain constant?

By what factor is it modified?

c - By what factor is the radiation pressure modified at the end of the acceleration phase?

Answers p357.

3. ▲▲△ **Optical molasses**

To slow down atoms and thus cool them we place two identical lasers face to face. If an atom placed between these two beams is stationary, it remains so, because the radiation pressures are in equilibrium.

a - Show that, for an atom moving along the axis of the lasers, a force appears that causes it to come to a standstill.

This force is similar to viscous friction, hence the name optical molasses for this phenomenon. Atomic clocks use optical molasses to cool atoms.

b - Show that, for low speeds compared to c , this force is analogous to the friction force of viscous fluids in laminar regime: $\vec{f} = -\alpha \vec{v}$.

The radiation pressure can be explained at the microscopic scale by the absorption then emission of a photon by the atom. The momentum of the atom is modified, in the direction of the laser during absorption and in a statistically isotropic manner during spontaneous emission. The atom is thus slowed down and confined. The resonance frequency of the atom is slightly higher than that of lasers.

c - As with viscous friction, we have an energy dissipation phenomenon. Explain qualitatively how the process of absorption/emission of a photon by the atom allows it to lose kinetic energy and thus to cool down.

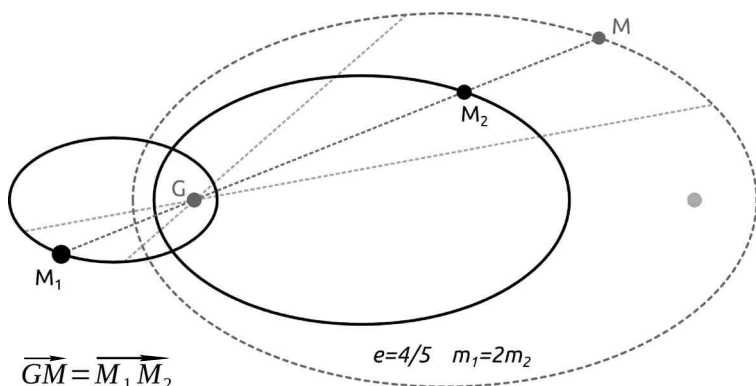
d - In the context of perfect gas, the mean kinetic energy of an atom is given by the relation $e = \frac{3}{2} k_B T$, where T is the temperature in Kelvin. Once slowed down, the atom will have a minimum kinetic energy of the order of the difference of energy between the absorbed photon and the photon emitted during de-excitation. The line width of the laser is very small compared to that of the atom, which predominates. In the extreme case, at rest, the line width of the atom is just below that of the laser. The distance between the two lines then corresponds to the width of the atomic line. The lifetime τ of the excited level of the atom is related to the energy difference by the Heisenberg uncertainty relation. From this an approximation of the temperature of the atom obtained by Doppler cooling can be deduced. Numerical application: $\tau = 27 \text{ ns}$ for a rubidium 87 atom.

e - Give the speed of an atom thus cooled.

Answers p358.

4. ▲△△ Detection of exoplanets by Doppler effect

A large number of exoplanets have been detected until now and their known number continues to increase. One method of detection, called Doppler method, or radial-velocity method, consists in observing the periodic variation of the wavelength of a star. The motion of the star is due to the presence of an exoplanet. When the star is moving towards us, and thus the planet backwards, the characteristic lines of its spectrum move towards blue, and when the star is moving away, towards red.



We consider a two-body system consisting of a star and a planet. The two masses are in a gravitational bound state. Let's conduct a Newtonian study. Each of the bodies revolves around the center of mass G of the system. We can fictitiously return to a problem with one body M of reduced mass μ which orbits around G , a fixed point of origin in the center-of-masse frame:

$$\mu = \frac{m_1 m_2}{m_1 + m_2}. \quad \text{Kepler's law for the fictive particle M:}$$

$$\frac{a^3}{T^2} = \frac{\alpha}{4\pi^2\mu} \quad \text{with} \quad \alpha = G m_1 m_2$$

a : semi-major axis of the ellipse traveled by M.

T : period of revolution around G.

We then find the trajectories of the two bodies M_1 and M_2 by applying the following homothetic factors:

$$\vec{GM}_1 = -\frac{m_2}{m_1 + m_2} \vec{GM} \quad \text{and} \quad \vec{GM}_2 = \frac{m_1}{m_1 + m_2} \vec{GM}$$

We will consider the cases of a two-body system with circular orbits and a plane of revolution that contains the long-distance observation site of the Doppler effect.

Let's take the example of a star slightly smaller than the Sun around which a giant Jupiter orbits. The Sun is a small star, a yellow dwarf, here we will take an orange dwarf of 0.8 solar mass. We will have a supermassive giant planet of 80 Jovian masses (this planet may be similar to a brown dwarf, not very luminous and not detectable by direct methods). The star in this case has a mass ten times greater than that of the planet. There are many stellar systems of this type: HD 87883, HD 4747, Epsilon Eridani, etc.

a - Determine the speed of the star on its orbit around the system's center of gravity. Show that this

speed is indeed non-relativistic.

b - Give the classical limit of the Doppler effect formula.

c - What will be the relative wavelength variation $\Delta\lambda/\lambda$ of the light emitted by the orange dwarf observed from the Earth in its plane of revolution?

Data: $G=6.67 \times 10^{-11} \text{ N.m}^2/\text{kg}^2$, $M_s=2 \times 10^{30} \text{ kg}$,
 $M_j=M_s/1000$, $d_{\text{G-Planet}}=540 \times 10^6 \text{ km}$

Answers p359.

5.√√ ▲▲▲ Calculations for the moving ruler

We detail the calculations that allow us to find the exact expression of the apparent length of the moving ruler on the photographic plate as a function of time. We rely on the notations given in the course.

a - Determine the equations of the worldlines for the E_1 and E_2 ends of the ruler.

b - We seek to express the equation of the past cone of $M(0, D, ct_M)$. We consider the vector $\vec{u}=(a, b, 1)$ with $\sqrt{a^2+b^2}=1$ and collinear with a generatrix line of the cone. Let be $C=(x, y, ct)$ a point of the cone.

We have two constraints, \overrightarrow{MC} collinear to \vec{u} and point C belongs to the ends of the worldsheet of the ruler. Deduce the apparent length L_a as a function of t .

**6.✓ ▲▲△ Velocity transformation
and aberration of the light**

a - From the Lorentz transformation determine the three components of the velocity in R' as a function of those in R .

$$\vec{v}=(v_x, v_y, v_z), \quad \vec{v}'=(v_x', v_y', v_z') \quad \text{and} \quad \vec{\beta}=\frac{\vec{u}}{c}=\frac{u}{c} \vec{i}$$

From the transformation of velocities we can quickly find the formula of the relativistic aberration of light which gives θ_o as a function of θ .

b - Give the components of the velocity of a photon that arrives in O at an angle θ with respect to Ox.

c - Give the expression of \vec{v}' and check that we have $\vec{v}' \cdot \vec{v}' = c^2$.

d - Express $\tan\theta_o$ as a function of θ .

**7.✓ ▲▲△ Composition of velocities
and accelerations. 3D generalization**

a - Two vessels move at 50% c and cross perpendicularly in O in R.

What is their relative speed?

b - In the general case of two vessels animated by velocities \vec{v}_1 and \vec{v}_2 , one does not lose in generality by taking \vec{i} co-directed with \vec{v}_1 , \vec{j} co-directed with $\vec{v}_1 \wedge \vec{v}_2$ and $\vec{k} = \vec{i} \wedge \vec{j}$.

The angle between the velocities is $\theta = \widehat{(\vec{v}_1, \vec{v}_2)}$. Express the relative velocity v' as a function of v_1 , v_2 and θ .

Numerical application for two vessels of $\gamma=2$ and trajectories that make an angle of 30° .

c - We continue the exercise *Two vessels* on page 71.

1 - Starting from the velocity \vec{v} in R , find again, with the velocity transformation laws, the velocity v' of vessel B .

2 - Establish the law of transformation of accelerations in three dimensions. From the velocity \vec{v} and acceleration \vec{a} in R , find again the acceleration a' of the starship B .

Answers on page 362.

8.√ ▲▲△ **Starry sky at the halfway point**

We start our journey to Proxima Centauri with an acceleration of one g. As we will show in the next chapter the speed is then 95% of c at mid-course (after 2 ly traveled in the galactic frame of reference). We wonder if the Sun and Proxima Centauri are at that moment visible to the naked eye from the spacecraft. In astronomy we use the apparent magnitude to determine the brightness of a star. A star of magnitude greater than 6 is invisible to the naked eye. The star Vega is taken as a reference with a magnitude of zero. A star brighter than Vega has a negative magnitude.

Magnitude formula: $M = -2.5 \log(L/L_0)$.

L and L_0 are the luminosities of the star and Vega perceived at the point of observation.

The luminosity L_V of Vega, which corresponds to the total power emitted, is expressed as a multiple of the luminosity L_S of the Sun: $L_V = 37L_S$.

Distance Vega-Sun: $D_{VS} = 25 \text{ ly}$.

For Proxima Centauri: $L_P = 5 \times 10^{-5} L_S$.

The perceived luminosity of a star decreases with distance, and is inversely proportional to the square of the distance.

a - Determine the apparent magnitude of the star Proxima Centauri from Earth. Is the star visible to the naked eye?

b - Determine the apparent magnitude of Proxima Centauri at midpoint if the spacecraft was motionless with respect to the stars. Would the star be visible to the naked eye?

c - Determine the apparent magnitude of Proxima Centauri at mid-course when the spacecraft is at 95% of c . Will the star be visible to the naked eye?

d - Determine the apparent magnitude of the Sun at mid-course if the spacecraft was stationary. Would the Sun then be visible to the naked eye?

e - Determine the apparent magnitude of the Sun at the halfway point when the spacecraft will be at 95% of c . Will the Sun then be visible to the naked eye?

f - Here you are on the exoplanet *Proxima b* orbiting the star Proxima Centauri. A well-deserved rest. Will

you see the Sun in the night sky? *Answers p366.*

9. ⁹¹₁₀ ▲▲△ **Numerical simulation of the sky**

In the analytical model of the course we have a continuous distribution of light energy to model the starry sky. Here we will have a discrete distribution of point stars. We will take $N=10,000$ stars, identical, monochromatic, and, randomly and uniformly distributed.

This numerical model allows us to better understand the perception of the sky from the moving vessel, to better understand the meaning of the integrals calculations and to verify them.

a - Uniform spherical probability law: We place stars on the celestial sphere using two angles θ , the colatitude, and φ , the longitude. These are the spherical coordinates. The positioning is analogous to the one used to find our bearings on the surface of the Earth. The colatitude is zero at the celestial North Pole, 90° at the celestial equator and 180° at the South Pole. The longitude is 0° at a meridian taken for origin and returns to it after a full 360° turn. Propose laws of probabilities Θ and Ψ which ensure a uniform distribution on the celestial sphere as a function of the continuous uniform law $U(0,1)^9$.

b - We use a spreadsheet and the function that generates a random number between 0 and 1. On the first two columns we have N values for θ and for

⁹ For the laws of probability and their simulation, see, for example, the book *Probability, Statistics and Estimation*, by the same author, on pages 109 and 118.

φ . Then we calculate for the N stars θ_a and I_a with the formulas of the course. You can thus find the values, for a speed of 50% of c , of the energy and the total number of photons received with respect to rest.

Answers on p367.

10.√√ ▲▲▲ A bit of math...

To do physics in higher education you have to be comfortable with math and I prefer to put everything on the table in the same book to be clear and avoid multiple tergiversations. Nature is logical, logic is mathematical, so let's indulge in a little trigonometry.

According to the relation between y' and x' given page 88:

θ belong to $]0, \pi[$ and $\tan \theta_a = \frac{\sin \theta}{\gamma(\beta + \cos \theta)}$ if the denominator is positive. θ_a then belong to $]0, \pi/2[$ and in this case: $\beta + \cos \theta > 0$ so $0 \leq \theta < \theta_0$ with $\theta_0 = \arccos(-\beta)$.

If the denominator is negative. θ_a belong to $] \pi/2, \pi[$ and in this case $\theta_0 < \theta \leq \pi$ then:

$$\tan(\pi - \theta_a) = \frac{\sin \theta}{-\gamma(\beta + \cos \theta)}$$

This is very complicated. The tangent function is made up of an infinity of branches, and, therefore, for one value of the tangent there are an infinity of possible angles. A traditional calculator gives the

value of the angle on the central branch on $]-\pi/2, \pi/2[$. Our star observation angle is between $-\pi$ and π , and by symmetry we restrict to $]0, \pi[$. We are then on two branches of the tangent. To solve this thorny and exciting (!) problem we prefer to have $\tan(\theta/2)$, because $\theta/2$ belongs to $]0, \pi/2[$. We stay on the same central branch whose values are given by the calculators.

a - After recalling the expressions of $\cos(a+b)$ and $\sin(a+b)$ give the expression of $\tan(a+b)$ as a function of $\tan(a)$ and $\tan(b)$.

b - Deduce $\tan(\theta)$ as a function of $\tan(\theta/2)$.

c - Solve a quadratic equation to show that

$$\tan(\theta_a/2) = \sqrt{\frac{1-\beta}{1+\beta}} \tan(\theta/2).$$

Answers p368.

11.√/ ▲▲▲ Energy distribution

We establish here the formulas giving the energy received from the starry sky in the reference frame of the vessel as a function of β .

a - Use the relationship between θ_a and θ to express $d\theta$ as a function of $d\theta_a$ and θ_a . Deduce how the solid angle $d\Omega = 2\pi \sin\theta d\theta$ transforms in the vessel's frame of reference. You will be able to express $d\Omega$ as a function of $d\Omega_a$ and θ_a . The factor gives us the star density n as a function of θ_a . Express this density at the stern and bow as a function of β , then make a numerical application for $\beta=0.5$.

b - Verify by integrating over the whole space that the number of stars remains well constant when the ship gains speed.

c - Find again the expression of E_a as a function of β of the course.

d - Determine how the energy is distributed between the front and back hemispheres of the vessel. Expression as a function of β , and numerical application for $\beta=0.5$.

Answers p369.

12.✓✓ ▲▲▲ Number of photons

The number of photons reaching the vessel every second is proportional to γ . Within the framework of the model of yellow photons uniformly emitted in the galactic frame of reference, in the moving frame of reference, the photons are more numerous and of different frequencies. They are each time less numerous and of low energy towards the rear and each time more numerous and energetic towards the front.

a - By a complete integral calculation find the factor: $N_a/N=\gamma$. You can use symbolic computation software.

b - What proportion of photons is received from the front hemisphere? Calculation as a function of β , then numerical application for $\beta=0.5$.

Answers p371.

13.✓ ▲▲▲ Power emitted by a star

To obtain the total power emitted, we integrate the luminance i on all wavelengths, solid angles and surfaces:

$$P = \int i(\lambda) d\lambda d\Omega dS$$

The expression of the luminance is given on page 93. For a black body, an infinitesimal area dS emits uniformly over a half-space, i.e. an integrated solid angle of 2π .

a - In the case of the Sun, do you find the known total emitted power of $4 \times 10^{26} \text{ W}$? The surface temperature is taken equal to $T_s = 5000 \text{ K}$ and the solar radius $R_s = 700\,000 \text{ km}$. You can estimate the integral by a numerical integration.

b - How is the power emitted by the Sun divided between visible, infrared ($> 800 \text{ nm}$) and UV ($< 400 \text{ nm}$)?

c - For Proxima Centauri, we take $T = 3000 \text{ K}$ and $R = 0.14 R_s$. We read on the Wikipedia page of Proxima Centauri that "*Its total luminosity over all wavelengths is 0.17% that of the Sun*". Does your calculation confirm this assertion?

Answers p372.

ACCELERATED MOTION

We have so far studied vessels in uniform rectilinear motion: an object animated at a constant speed and which moves along a straight line. For realistic interstellar travel the trajectory can remain rectilinear, but, on the other hand, the speed necessarily varies. We are going to be interested in uniformly accelerated rectilinear motion: the vessel has a constant acceleration, the speed constantly varies by the same amount. We can thus create an artificial gravity in the rocket: we will consider the case where the speed increases (or decreases) by 10 m/s every second.

∞ STUDY OF AN ACCELERATED FRAME

The basic principles of special relativity are stated for inertial frames of reference. Once we have a starting inertial frame of reference, all frames of reference in uniform rectilinear translation with respect to it are also inertial frames of reference. A frame of reference accelerated with respect to a frame of inertia does not belong to this set, which does not prevent the application of special relativity indirectly if we know the motion of this reference

frame with respect to an inertial reference frame, which we will name R . We proceed in the same way in Newtonian mechanics, the fundamental relationship of dynamics $\vec{F}=m\vec{a}$ is only valid in inertial frames of reference and therefore Newton's laws are used to study any type of motion in any type of frame of reference.

Classical mechanics is used to construct special relativity by using it as the limit of low speeds. In addition, the principle of additivity of the proper times on the particle worldline is added as a construction element. With this principle, we are not limited to inertial frames of reference: the particle proper frame of reference can have any motion (it is the clock hypothesis seen page 19).

Then $\tau = \int d\tau = \int \frac{dt}{\gamma}$ where τ is the proper time in the particle proper reference frame, t is the time in the inertial frame of reference and γ is expressed as a function of the instantaneous speed v of the particle in this same frame of reference.

At any time t there is always an inertial frame of reference named R' which coincides with the proper reference frame R_p . The frame R' has a constant velocity v with respect to R and, at the moment it coincides with the proper frame R_p , the particle has a zero velocity in R' . Its acceleration is a' and that in R is then $a = \frac{a'}{\gamma^3}$ (demonstrated page 69). This is

where classical mechanics comes in, indeed, the particle then has a low speed in R' between t and

$t+dt$. It is like if an accelerated vessel passed a vessel moving at constant velocity. If at the moment they are at the same level their velocities are equal, their relative velocity is zero. The vessel accelerated by the thrust of its engines then moves away slowly with respect to the speed of light and we can use classical mechanics to study the motion of the accelerated vessel from the other vessel taken as a reference.

Let's take the example of a car that first stands still at a red traffic light and then accelerates to green. From the reference frame of the road, the acceleration of the mobile is \vec{a} , but what is the acceleration felt by the passenger in the proper reference frame of his car?

According to the classical acceleration transformation formula: $\vec{a} = \vec{a}_r + \vec{a}_e + \vec{a}_c$ where we have the absolute acceleration \vec{a} in R , relative acceleration \vec{a}_r in R_p , coincident acceleration \vec{a}_e ¹⁰ and Coriolis acceleration \vec{a}_c .

Let's write Newton's second law in R' :

$$\vec{F} = m(\vec{a}_r + \vec{a}_e + \vec{a}_c) \text{ and } m\vec{a}_r = \vec{F} + \vec{F}_{ie} + \vec{F}_{ic}.$$

In an accelerated, non-Galilean frame of reference, we feel new forces, called inertial forces. Here the accelerations \vec{a}_r and \vec{a}_c are null because the passenger is motionless in his car. The driver feels a

10 Advanced remark: $\vec{a}_e = \vec{a}_R(C)$, $C(t=t_0) = M(t_0)$ and $\vec{v}_{R_p}(C) = \vec{0}$

$$\vec{a}_e = \vec{a}_R(O') + \frac{d\vec{\Omega}_{R_p/R}}{dt} \wedge \overrightarrow{O'M} + \vec{\Omega}_{R_p/R} \wedge (\vec{\Omega}_{R_p/R} \wedge \overrightarrow{O'M})$$

The coincident point C coincide instantaneously with M. For a non-rotating frame $\vec{a}_e = \vec{a}_R(O')$. For a uniformly rotating frame we obtain the centrifugal acceleration. e for *entraînement* in French.

inertial force $\vec{F}_{ie} = -m\vec{a}_e$ that pushes him to the bottom of his seat when starting. This is due to the inertial acceleration which equals that of the car: $\vec{a} = \vec{a}_e$. For the same reason, the acceleration felt by the particle in its proper frame also worth \vec{a}' , acceleration of the particle in R' .

∞ ARTIFICIAL GRAVITY

When the car accelerates at the green traffic light, it is as if a force exerted at a distance pulls the driver towards the rear of the car. Like a non-contact force, analogous in these effects to a gravitational force due to a mass placed at a distance at the back of the car. When a spaceship starts at the green traffic light at an interstellar crossroads, the passengers first in weightlessness are then pressed during the acceleration phase to the walls perpendicular to the displacement. In our case the acceleration is maintained and the vessel has a *uniformly accelerated rectilinear motion*.

The acceleration is equal to the Earth's surface gravity g , thus:

$$a = \frac{dv}{dt} = \frac{g}{\gamma^3} \quad \text{and} \quad \tau = \int \frac{\gamma^2}{g} dv = \frac{c}{g} \int \frac{d\beta}{1-\beta^2}$$

$$\text{then } \tau = \frac{c}{g} \int_0^\beta \left(\frac{1/2}{1-\beta} + \frac{1/2}{1+\beta} \right) d\beta \quad \text{and} \quad \boxed{\tau = \frac{c}{2g} \ln \left(\frac{1+\beta}{1-\beta} \right)}$$

where $v = \beta c$ is the speed reached in R after a

proper duration τ .

Let us express the distance x traveled in R as a function of v :

$$v = \frac{dx}{dt} \quad \text{then} \quad x = \int dx = \int \frac{v^3}{g} dv = \frac{c^2}{g} \int \frac{\beta}{(1-\beta^2)^{3/2}} d\beta$$

and after integration:
$$x = \frac{c^2}{g} \left(\frac{1}{\sqrt{1-\beta^2}} - 1 \right)$$

Let's calculate the galactic time t :

$$t = \int dt = \int \frac{v^3}{g} dv = \frac{c}{g} \int \frac{1}{(1-\beta^2)^{3/2}} d\beta$$

We perform the change of variable

$\beta = \sin \theta$ and we find:

$$t = \frac{c}{g} \frac{\beta}{\sqrt{1-\beta^2}}$$

We can now express the position, speed and acceleration as a function of time t :

$$\left\{ \begin{array}{l} x = \frac{c^2}{g} \left[\sqrt{1 + \frac{g^2 t^2}{c^2}} - 1 \right] \\ v = \frac{c}{\sqrt{1 + \frac{c^2}{g^2 t^2}}} \quad \text{and} \quad y = \sqrt{1 + \frac{g^2 t^2}{c^2}} \\ a = \frac{g}{\left(1 + \frac{g^2 t^2}{c^2}\right)^{3/2}} = \frac{g}{y^3} \end{array} \right.$$

We can also express the proper time τ as a function of galactic time t :

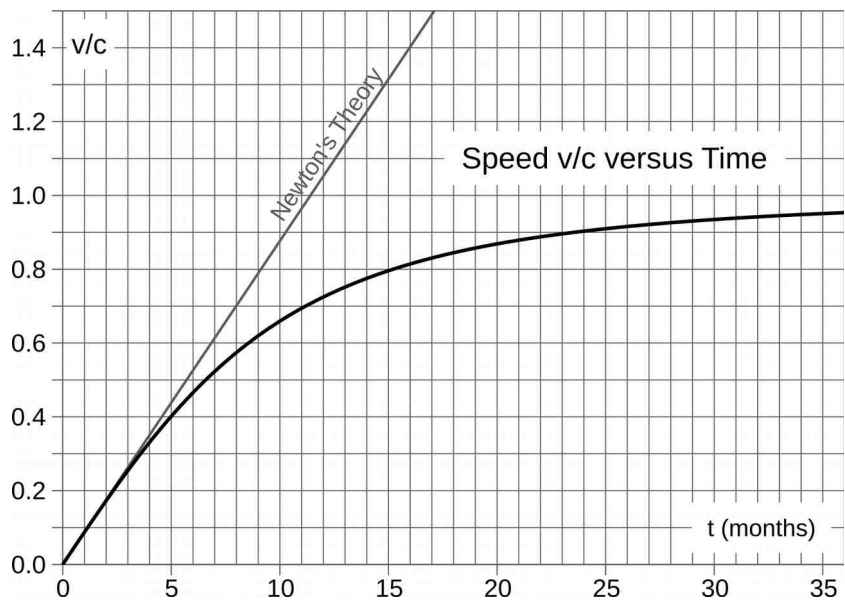
$$t = \frac{c}{g} y \beta \quad \text{then} \quad \tau = \frac{c}{g} \ln \left(\sqrt{1 + \frac{g^2 t^2}{c^2}} + \frac{gt}{c} \right)$$

$$\text{and} \quad \tau = \frac{c}{g} \operatorname{argsh} \left(\frac{gt}{c} \right)$$

$$t = \frac{c}{g} \operatorname{sh} \left(\frac{g\tau}{c} \right) \quad x = \frac{c^2}{g} \left[\operatorname{ch} \left(\frac{g\tau}{c} \right) - 1 \right]$$

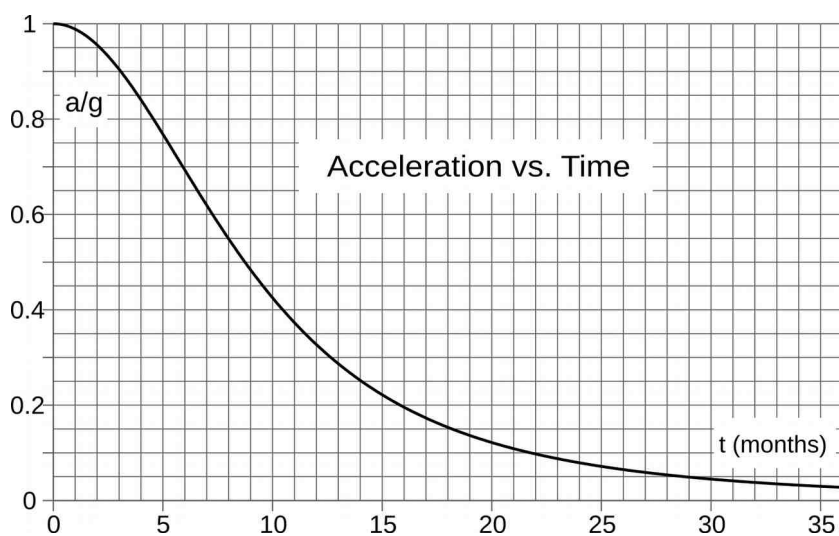
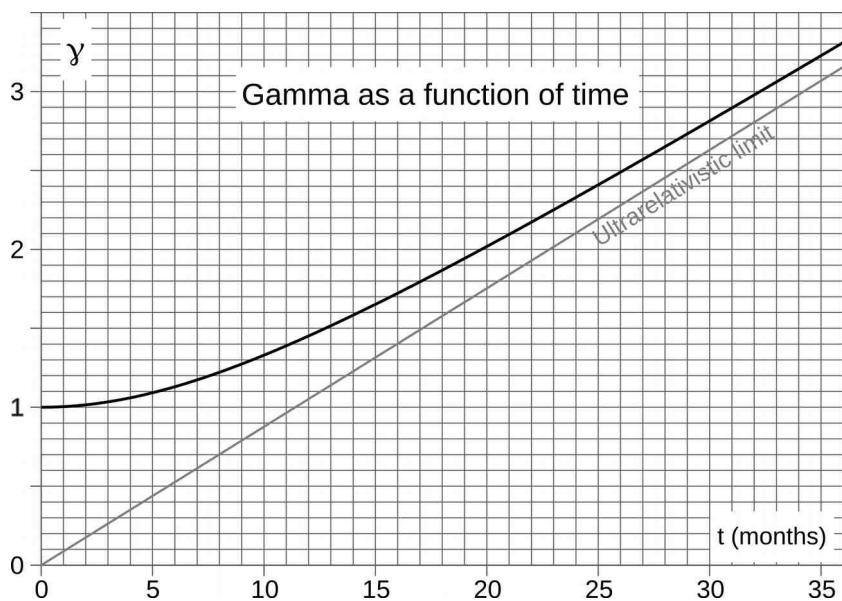
$$(ct)^2 - \left(x + \frac{c^2}{g} \right)^2 = \left(\frac{c^2}{g} \right)^2$$

Curves :

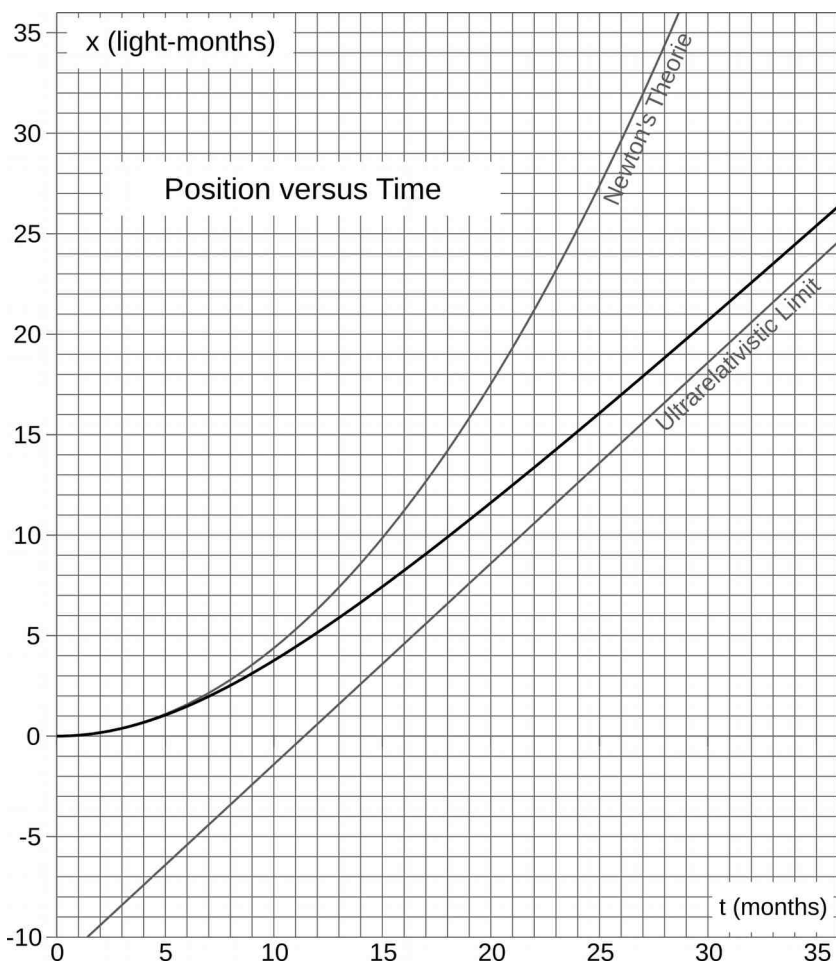


The speed tends towards the maximum speed c . For low speeds, the speed increases linearly with time, we find the classic limit $v = gt$.

Next page, the variation of the temporal dilation factor as a function of galactic time. We have a horizontal tangent at low speeds. When the speed increases, we tend towards the ultrarelativistic asymptote $\gamma \sim gt/c$, γ then varies linearly with galactic time.

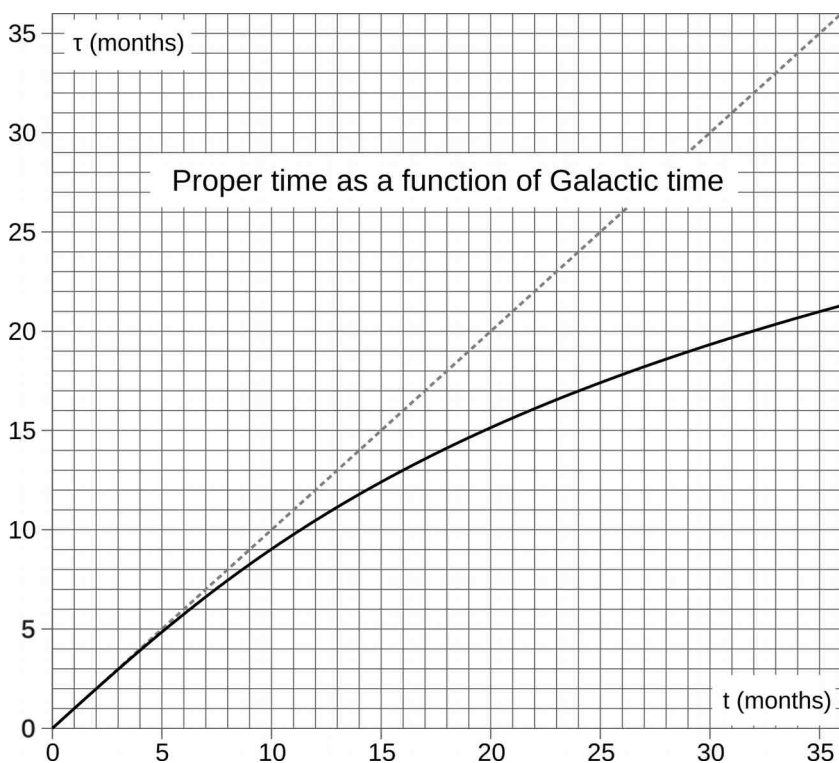


Previous page, the acceleration of the ship seen from the starting frame of reference. Although the acceleration remains constant in the proper frame, observed from the Earth, the speed reaches a ceiling and the acceleration decreases in gamma cubed. We have a horizontal tangent at low speeds, a zero infinity limit, and an inflection point at $t = c/2g$.



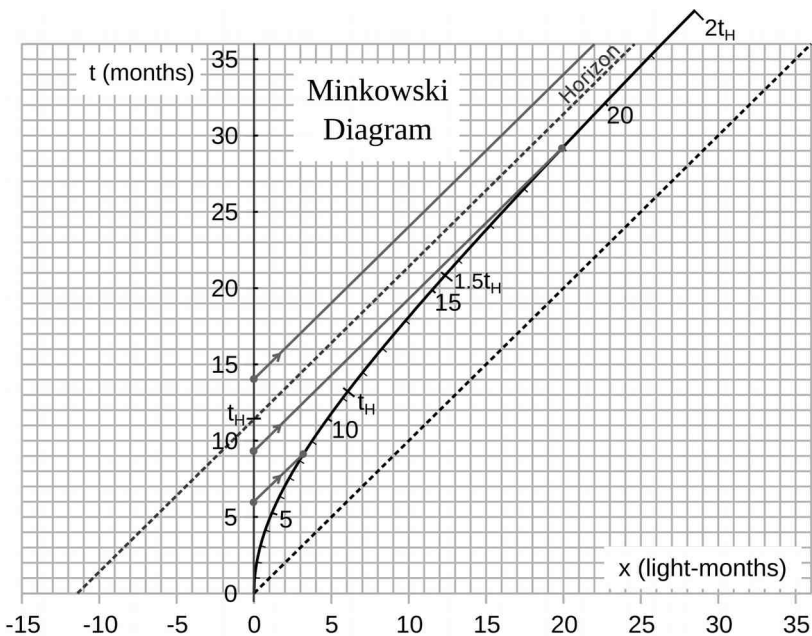
Previous page, we see, after 6 months, the position move away from the forecasts of classical mechanics. In Newton's theory we had a parabolic branch while in the context of special relativity we have a hyperbolic branch with an ultrarelativistic asymptote $x = ct - c^2/g$ where the galactic distance traveled increases linearly with time.

Below, the traveler's time accelerated according to that of those who remained on Earth:



Horizon concept:

We get the Minkowski diagram by simply reversing the x and t axes. We find that the asymptote $t=x/c+c/g$ represents a horizon. For terrestrial observers, it is impossible to communicate with the vessel after a period of time $t_{\text{lim}}=c/g$ (approximately 11.4 months). Indeed, after this period, a photon will no longer be able to reach the vessel. On the other hand, the occupants of the accelerated vessel will be able to continue to send us messages throughout their journey. They will also be able to permanently receive messages from Earth, but they will be earlier than t_{lim} .



As the proper time τ increases, the astronauts see the inhabitants of the Earth slow down their motions and freeze at the time limit t_{lim} .

∞ ROUND TRIP

We want to join an exoplanet at a distance D from our planet Earth. We will be under artificial gravity for the entire round trip. We accelerate the first half of the trip and then, after turning the ship around, decelerate to the exoplanet. We repeat the reverse procedure for the return.

First phase:
$$\frac{D}{2} = \frac{c^2}{g} \left(\frac{1}{\sqrt{1 - \beta_{\text{max}}^2}} - 1 \right)$$

Maximum speed halfway:

$$\beta_{\text{max}} = \sqrt{1 - \frac{1}{\left(1 + \frac{gD}{2c^2}\right)^2}}$$

(for $D=4$ light-years, $\beta_{\text{max}} \simeq 95\%$ and $\gamma \simeq 3$)

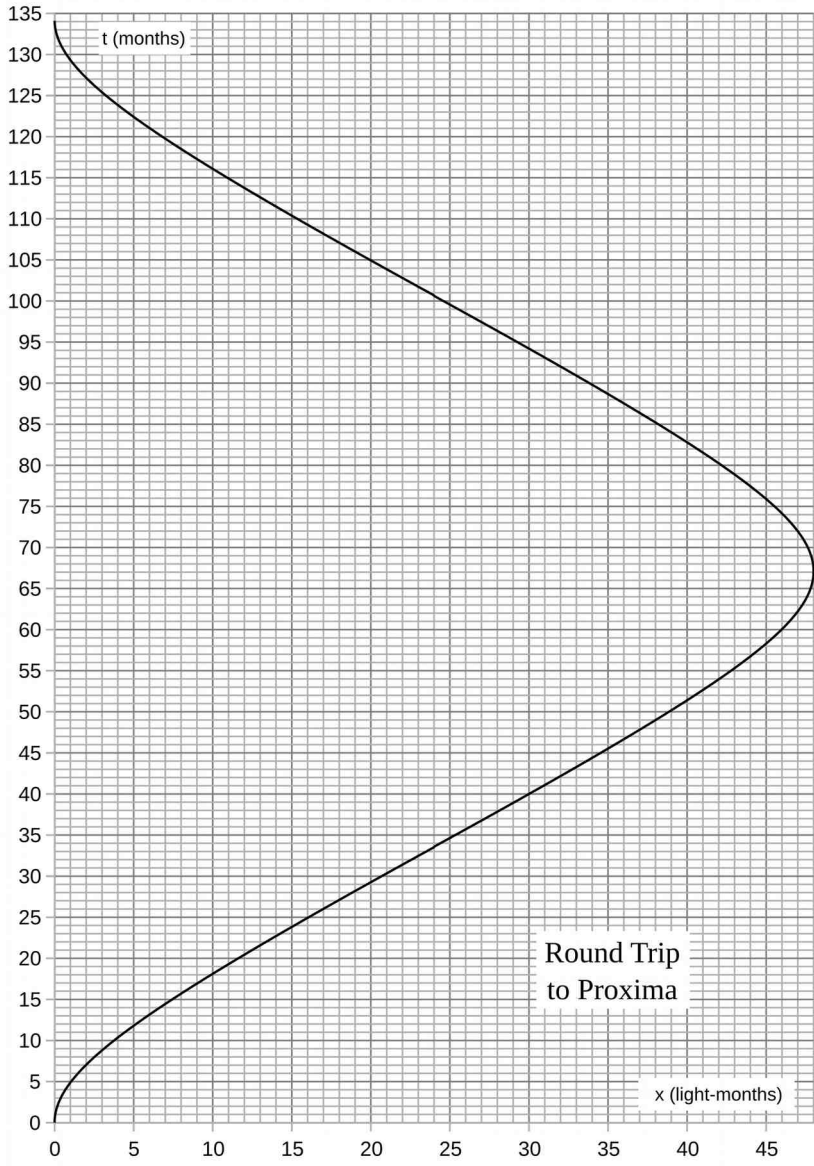
Duration T for the round trip:

$$\frac{T}{4} = \frac{c}{g} \frac{\beta_{\text{max}}}{\sqrt{1 - \beta_{\text{max}}^2}} \quad \text{and} \quad T = \frac{4c}{g} \sqrt{\left(1 + \frac{gD}{2c^2}\right)^2 - 1}$$

Proper time τ for the round trip:

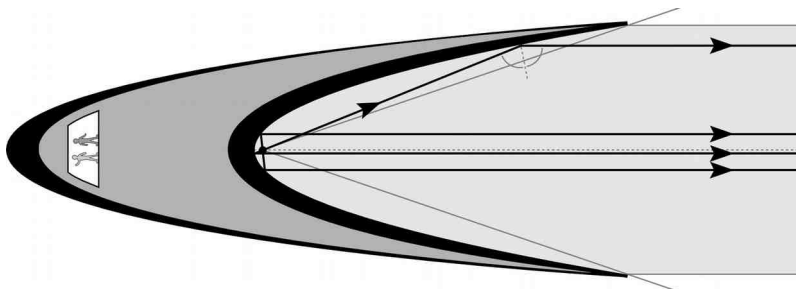
$$\tau = \frac{2c}{g} \ln \left(\frac{1 + \beta_{max}}{1 - \beta_{max}} \right) = \frac{4c}{g} \operatorname{argth} \beta_{max}$$

(for $D=4$ l.y., $T \simeq 11.2$ years and $\tau \simeq 6.84$ years)



Photon rocket:

A light beam, created by the rocket, propels it by reaction. For example, matter and antimatter, in equal parts, are placed at the focus of a parabolic mirror, and, by annihilation, produce pure energy projected backwards in a parallel beam.



Consider the following case, a particle and its antiparticle meet and create two photons which go in opposite directions. One goes backwards and the other forwards. The backward one does not contribute to the propulsion, on the other hand, the second one contributes doubly, because the reflection on the mirror is supposed to be perfect. On average, each photon transfers its impulse to the rocket. Ultra-relativistic particles are just as efficient as their mass energy converted into light.

More realistically, a photon is sometimes absorbed by the *gamma shield*. The efficiency is then 50%. Also, part of the energy of the absorbed *gamma rays* can be reused to heat a gas to a very high temperature. The thermal agitation generates a very

On the contrary, if a neutrino is created by the reaction, it carries away energy that is lost for propulsion.

The photon rocket is close to the perfect model, we can otherwise talk about an antimatter rocket.

$$e^- + e^+ \longrightarrow \gamma + \gamma \quad E_\gamma = 511 \text{ keV} \quad \sim 10 \text{ ns}$$

$$p + \bar{p} \longrightarrow 29\% \pi^+ + 29\% \pi^- + 38\% \pi^0 + 4\% K^+ K^- K^0 \dots \quad \sim \text{ps}$$

$$\begin{aligned}
 & \pi^+ \xrightarrow{26 \text{ ns}} \mu^+ \nu_\mu \xrightarrow{2.2 \mu\text{s}} e^+ \nu_e \bar{\nu}_\mu \xrightarrow{+e^-} \gamma \gamma \quad 511 \text{ keV} \\
 & \pi^- \xrightarrow{26 \text{ ns}} \mu^- \bar{\nu}_\mu \xrightarrow{2.2 \mu\text{s}} e^- \nu_e \bar{\nu}_\mu \xrightarrow{+e^+} \gamma \gamma \\
 & \pi^0 \xrightarrow{9 \times 10^{-17} \text{ s}} \gamma \gamma \quad E_\gamma = 67.5 \text{ MeV} \\
 & K^+ K^- K^0 \dots \longrightarrow \dots
 \end{aligned}$$

Proton-antiproton annihilation is more complex and creates cascades of particles. γ photons, even more energetic than for electron-positron annihilation, are created.

11 NASA proposes a rocket propelled by a positron reactor. These are annihilated with electrons in gamma photons. The heat produced heats liquid hydrogen. www.nasa.gov

Antimatter Rocket

IS

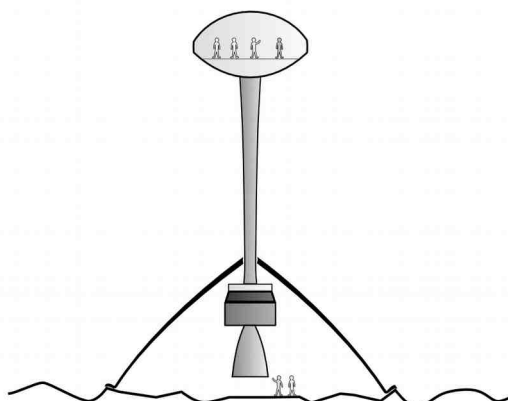
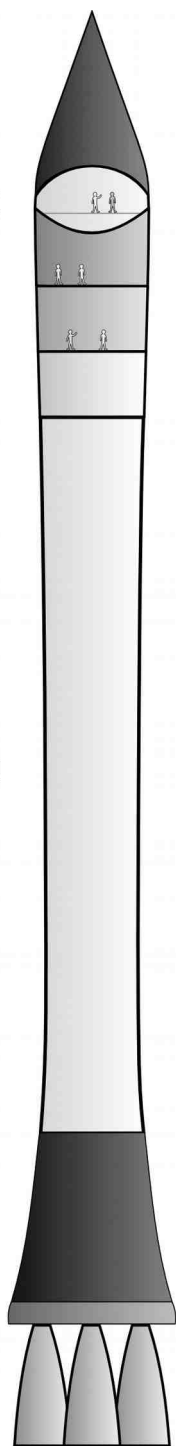
MM

TM

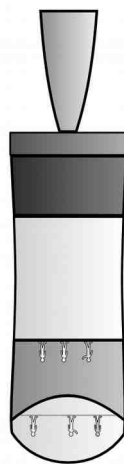
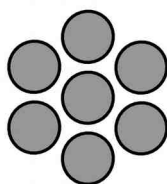
LM

AM

GS



Proxima b



Technical data :

Travel To Proxima Centauri / Distance 4.2 ly.

Traveler duration 3.3 years - Galactic 5.5 years.

Astronauts: 6.

Pressurized module: 3 / 10t / 6mx10m

Main Module - Technical Module - Leisure Module

Total height 126m / Diameter 15m / Total mass 2420t /

Payload 20t / Antimatter mass 1200t.

Antimatter: Proximum / Density 0.2 / 200 kg/m³.

Matter: Everything, except the payload, is progressively annihilated with the Proximum (shields, motors, etc).

Acceleration max 3 g / Speed max 89 % of c / γ_{\max} 2.2 /

Periods Acceleration: a_{avg} 2 g, sleep 2.8 g

Periods Speed: a_{avg} 0.3 g, sleep zero g.

Interstellar shield: 140t / Protects from the interstellar medium 0.6 proton/cm³ / vertex angle 38° / T_{\max} 498°C.

This shield is used on the first half of the course. After turning over, the motors are forward, and the radiation pressure pushes the interstellar medium away.

Gamma shield: 860t / Protects passengers and Proximum from the rays γ emitted by the motors / Armoring Pb of 20 cm, or concrete 1.2 m, reduces the flux by a factor 10⁶.

Rocket motor: efficiency 50 % / 1st phase 7 M P-2 / Thrust 10 MN / D_e 1g max 11 g/s Proximum / 2nd 1 M P-2 / 3rd 1 M P-1 Thrust 2 MN / v_e =150 000 km/s.

Comparison :

Saturn V / M=3038t / H=111m / D=10m / $M_{\text{propellant}}$ =2829t /

P_{\max} 34 MN / 1st stage 5 Motors F-1 v_e =2.6 km/s D_e =13.6 t/s

Kerosene~O₂(l) / 2nd 5 M J-2 / 3rd 1 M J-2 v_e =4.1 km/s

H₂(l)~O₂(l) / Duration 11 min 30 s from 0 to 164 km.

Exercises

1. ▲△△ Half-time

Leaving Earth, the ship reaches Proxima in a uniformly accelerated motion in two steps: the rocket cuts off its engines halfway through the journey, giving it time to turn around, and then arrives at Proxima at zero speed.

Compared to the stars considered fixed, what will be the distance traveled at half the time elapsed before the turning point? Is the result modified according to whether one considers the time of a fixed observer with respect to the stars, or that of a fixed observer with respect to the rocket? What about classical mechanics ?

We take, as usual, the following values :

$D=4 \text{ al}$, $a=g=10 \text{ m/s}^2$ and $c=3 \times 10^8 \text{ m/s}$.

Answers on page 374.

2. ▲▲△ Reality show

On January 1, 2100 at 12:00 noon, the crew of the Galaxys spaceship leaves at constant acceleration for the other end of the Milky Way.

Every day on Earth a reality show tells the adventures of the astronauts. And conversely, the astronauts also produce a daily program with the information received from the Earth during a proper

day on the spaceship. But due to time dilations, during a day on Earth we don't receive the news of a whole day lived on board the spacecraft, and vice versa. Light signals are used to transmit information.

a - Preamble: Determine the expression of position x as a function of y , and that of y as a function of τ .

b - Reality TV programs on Earth :

1- Let t_{obs} be the instant when the message corresponding to a proper time τ is received (the instant t is simultaneous to τ in the galactic reference frame, but the reception of the message due to the finite speed propagation of the wave is of course later). Illustrate the situation on a Minkowski diagram using the different worldlines (Earth / Ship / Photons).

2- Express t_{obs} as a function of τ , and τ as a function of t_{obs} .

3- Six months after their departure the astronauts send a message to Earth. How long after departure is the message received on Earth?

4- One year after departure, the daily reality shows will correspond to how much time spent in the spacecraft? Same question ten years after departure.

c - Reality show in the vessel:

1- Let τ_{obs} be the instant when the message corresponding to a terrestrial time t is received. Illustrate on a Minkowski diagram.

- 2- Express τ_{obs} as a function of t .
- 3- Six months after departure a message is sent to the astronauts. How long after their departure do they receive it?
- 4- One year after departure, the daily reality TV shows will correspond to how much time spent on Earth? Same question ten years after departure.

d - Doppler effect for an accelerating frame :

Both from the Earth and from the spacecraft a blue light signal is regularly emitted ($\lambda=400$ nm).

1- Establish the relations between the emitted frequency and the received frequency for the two reference frames, the inertial and the accelerated one.

2- After how long will the signal emitted from the Earth be perceived as red on board the vessel ($\lambda=800$ nm) ?

3- For the same time elapsed on Earth, what will be the color of the light signal received?

4- Is the Doppler effect symmetrical as in the case of inertial reference frames?

Answers on page 375.

3. ▲▲△ Head-to-head

Two vessels are traveling in opposite directions, at the same time and under the same conditions, the routes from Earth to Proxima and Proxima to Earth.

The rockets are animated with uniformly accelerated motions and complete the journey as described in this chapter.

a - Halfway, at the equidistant point, two light-years away, the ships shut down their engines to turn around. What is the galactic speed of the ships? What is their relative speed?

b - Same questions a quarter of the way.

c - Propose a technical means that would allow the ships to measure their relative speed.

d - Express the galactic speed v as a function of the proper time τ .

e - Express the relative speed v_r as a function of τ .

f - Determine the acceleration a_r of the spacecraft coming from Proxima from the point of view of the reference frame of the spacecraft coming from Earth as a function of τ .

Determine this relative acceleration at the start, halfway and a quarter of the time of the spacemen's outward journey.

Is the relative motion of the spacecrafts uniformly accelerated?

What results would we find in Newtonian mechanics?

Answers on page 379.

METRIC

A metric is used to measure distances. In relativity, the tool is generalized to space-time. We will give the metrics of the inertial frame of reference, of the uniformly accelerated frame in rectilinear translation, and of the uniformly rotating frame. We will then be able to determine the spacetime structure in our spaceship on its way to Proxima. What will be the geometric properties in the vessel? How does time flow at the different stages of the rocket?

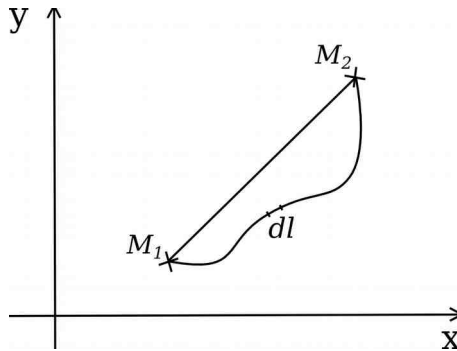
Finally, we will make a parallel with the black hole metric and thus build a bridge to general relativity.

To answer these questions we will introduce the concept of metrics through various examples.

∞ EUCLIDEAN METRIC

We measure the distance between two points. The metric can be expressed in different coordinate systems to calculate a distance, which is invariant. Let us take the case of two points M_1 and M_2 on a plane. If the coordinates of the points are Cartesian, $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$, the distance is given by:

$$L=d_{M_1M_2}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$



We can also determine the length of a curved path taken by a particle by integrating between the two points:

$$L=d_{M_1M_2}=\int_{M_1}^{M_2} dl \quad \text{with} \quad dl^2=dx^2+dy^2$$

This element dl^2 is our metric for this example.

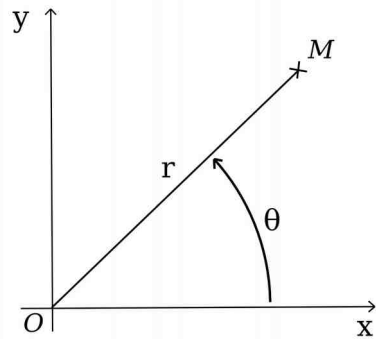
In the case where our physical problem has a central symmetry (common case, as for the motion of planets), the polar coordinates may be better adapted. We will have the same final result, but, in one case the computation can be very long, and in the other, very short. In polar coordinates these same points have the coordinates $M_1(r_1, \theta_1)$, $M_2(r_2, \theta_2)$ and $dl^2=dr^2+(r d\theta)^2$. With $x=r \cos \theta$ and $y=r \sin \theta$, we find the Cartesian metric, the steps are well equivalent.

Cartesian coordinates:

$$\begin{aligned} &M(x,y) \\ &x \in]-\infty ; +\infty[\\ &y \in]-\infty ; +\infty[\end{aligned}$$

Polar coordinates:

$$\begin{aligned} &M(r,\theta) \\ &r \in [0 ; +\infty[\\ &\theta \in [0 ; 2\pi[\end{aligned}$$



In Euclidean geometry the length of an object (like the duration of a phenomenon) is the same for all observers. Whether one carries out a translation, a rotation, or a Galilean transformation of the coordinates, the length L is invariant (done in exercise on page 159).

More generally, the laws of Newtonian mechanics are invariant according to these transformations.

This is not the case for a dilation: if $x' = kx$, $y' = ky$ and $z' = kz$ with k the dilation factor, then, $dl'^2 = dx'^2 + dy'^2 + dz'^2$, $dl' = k dl$ and $L' = kL$. The laws of physics depend on the scale, they are not the same for the infinitely small and the infinitely large.

The straight line is the shortest path between two points. We can take a rope and pull it to get a straight line. It is the path between M_1 and M_2 which minimizes L .

The Euclidean metric corresponds to a flat space: The sum of the angles of a triangle is equal to 180° ,

the ratio between the perimeter and the diameter of a circle is equal to π , and every straight line has a single parallel line passing through a point outside it.

∞ METRIC ON THE SPHERE

To better illustrate our point, let us take the case of a two-dimensional spherical space. You have to put yourself in the place of two-dimensional beings (the bidiz) who live on the surface of the sphere and are unaware of the third dimension. Euclid's postulates are no longer verified. We have simple counter-examples:

- To draw a circle, we fix a point, we attach a rope to it, and, with a tight rope, we turn around to trace it. The circle centered on the north pole and perimeter of the equator has a perimeter/diameter ratio equal to 2, a value much less than π .

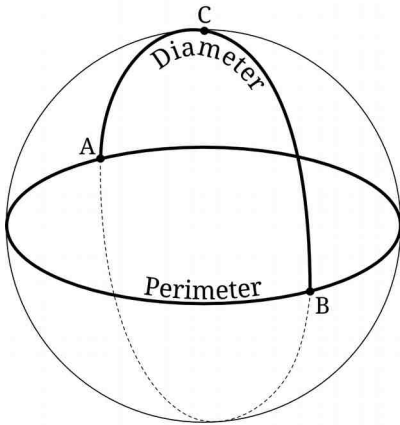
- Now let's construct a particular triangle: we have a first point at the north pole, we get a second point by joining along a straight line the equator, we turn at right angles to the east and we then follow the equator for a quarter turn, we turn at right angles to the north, and we return to the north pole to finish the triangle. We have an equilateral triangle

and all three angles are right. The sum of the angles of this triangle is 270° , a value much greater than 180° .

- Imagine yourself living on the surface of this sphere. You want to go on an adventure and discover unknown lands. You are unaware of the curvature of your 2D space, you go in a straight line, deviating neither to the right nor to the left, and finally you end up reaching your starting point from the opposite side! This is very disconcerting. The straight lines of the sphere are circles of the same radius as the sphere (the largest circles that can be drawn). For example, the line of the equator, a meridian, are straight lines for the sphere. You cannot draw parallel straight lines because they intersect. A latitude forms a circle with a radius less than that of the sphere, it is not a straight line: an airplane, to reach two cities at the same latitude, does not follow a latitude because it is not the shortest path .

We can clearly see, on these three examples, that the space on the surface of a sphere is not Euclidean. It is not a flat space but a curved space.

Geometry of the Sphere



Circle centred on C :

Perimeter P.

Diameter $D = \widehat{AB}$.

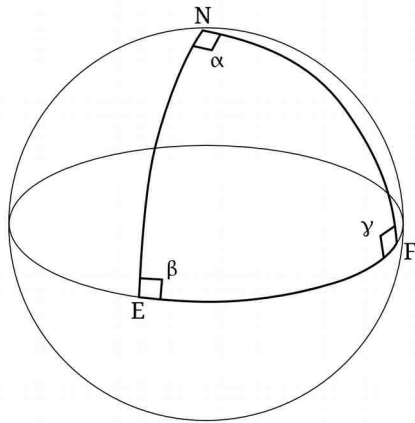
$$P / D = 2 < \pi$$

Triangle NEF :

Sum of angles:

$$\alpha + \beta + \gamma = 270^\circ$$

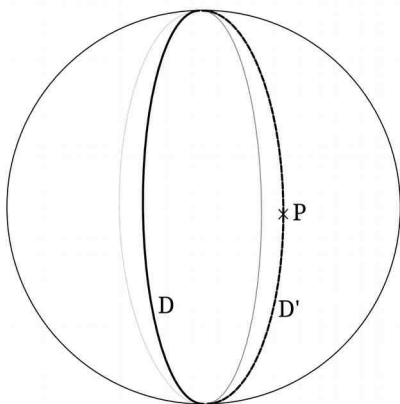
$$> 180^\circ$$



Straight line D :

All straight lines D' passing through P intersect D.

There are no parallel straight lines



Euclid's postulates are not verified.

The curvature can also be seen on the metric that bidizs would use, we give it for information¹²:

$$d l^2 = \frac{dx^2 + dy^2}{\left(1 + \frac{x^2 + y^2}{4 R^2}\right)^2}$$

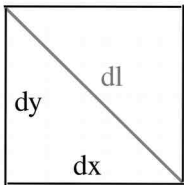
x and y are the two Cartesian coordinates internal to their two-dimensional space. Even if they don't "see" the third dimension, they could deduce it conceptually. It's a useful analogy for the little three-dimensional human beings that we are. Perhaps we ourselves live on the "surface" of a four-dimensional hypersphere, just as bidiz live on the surface of a hypercircle (a sphere for us!).

Here is a nice way to solve the problem of the edge of the Universe: if the Universe is not infinite, there should be a wall to define its limit, but what is behind the wall? If we live on the volume of a hypersphere, we have a Universe of finite volume, without border and without center.
An elegant vision allowed with a curved space.

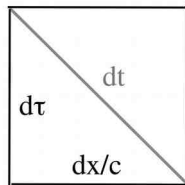
**A finite Universe
without edge
without center.**

¹² *Geometry, Relativity and the Fourth Dimension*, Ruldolf v. B. Rucker, 1977.

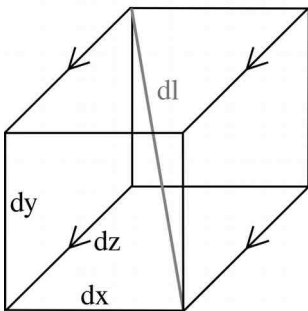
Geometries of Euclid and Minkowski



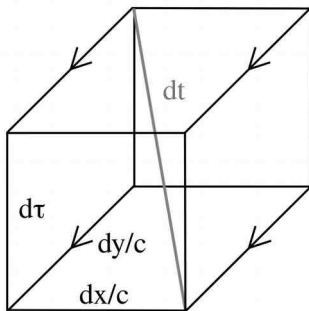
$$dl^2 = dx^2 + dy^2$$



$$d\tau^2 = dt^2 - dx^2/c^2$$

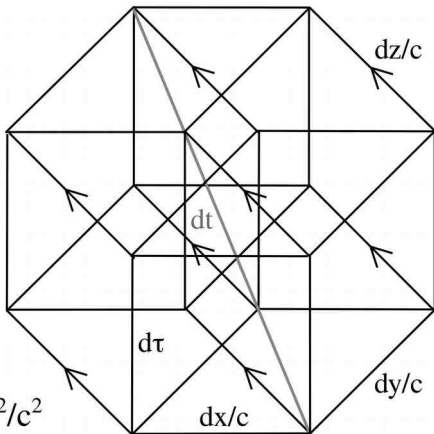


$$dl^2 = dx^2 + dy^2 + dz^2$$



$$d\tau^2 = dt^2 - dx^2/c^2 - dy^2/c^2$$

Spatiotemporal
Hypercube



$$d\tau^2 = dt^2 - dx^2/c^2 - dy^2/c^2 - dz^2/c^2$$

∞ MINKOWSKI METRIC

The time is now a coordinate integrated with the other three of space. It is the metric of special relativity. We have shown page 65 that the new invariant is:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

We discern a temporal part and a spatial part, dt and $dl^2 = dx^2 + dy^2 + dz^2$, then $T = \int dt$ and $L = \int dl$.

But this two quantities T and L are not invariant.

Straight lines, also called geodesics, maximize the proper time τ , invariant quantity:

$$\tau = \int \sqrt{dt^2 - dl^2/c^2} \quad (\text{particle : } ds^2 > 0)$$

Minkowski metric is invariant by translation, rotation and Lorentz transformation.

∞ METRIC OF AN ACCELERATING FRAME

We give the metric of the frame of reference in uniformly accelerated rectilinear translation studied in the previous chapter. This frame is not inertial and the metric is therefore necessarily different:

$$ds^2 = \left(1 + \frac{gx}{c^2}\right)^2 c^2 dt^2 - dx^2 - dy^2 - dz^2$$

We recognize a Euclidean-type spatial part, so

space is flat in the ship. Regarding the structure of space-time as a whole, we prove that this metric corresponds to a spacetime, also flat. For that it is shown that the components of the Riemann curvature tensor are all zero. This is very consistent with what we say about general relativity: in the absence of mass, spacetime is not curved¹³.

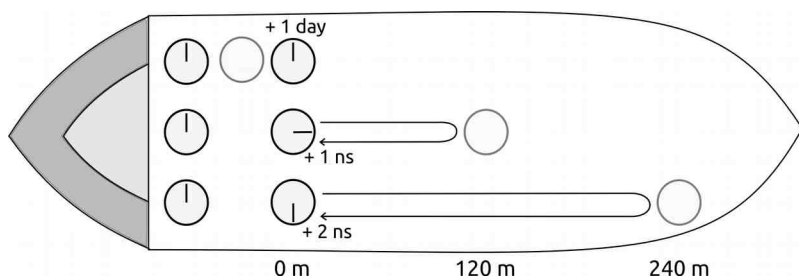
For an immobile object in the reference frame of the rocket:

$$d\tau = \left(1 + \frac{g x}{c^2}\right) dt$$

We note, for observers motionless with respect to each other in the accelerated frame of reference, that time does not flow at the same rhythm according to where one stands in the vessel. It is a phenomenon of time dilation very different from that observed between two inertial frames of reference where the clocks are in motion relative to each other. Here, the clocks are at rest in the reference solid (the rocket), they are motionless with respect to each other, and yet they do not work at the same rate and cannot be synchronized. Let us consider, in our rocket, three clocks which we will place at three different levels spaced 120 meters apart. We start by synchronizing them on the first level at the back of the ship. We leave one clock at the stern, we place the second 120 meters forward and the third at 240

¹³ It's more subtle than that. For example, gravitational waves propagate a spacetime curvature that persists even in the absence of mass.

meters at the bow (we move them slowly so as not to add another source of time dilation):



After a day we take them back down to the first level to compare the elapsed times. First observation, they are no longer at the same date, moreover the clock on the second level has turned faster and is one nanosecond ahead, the third clock has turned even faster and is two nanoseconds of advance.

The advance, of the clocks placed "higher" in the vessel, is calculated using the following expression which derives directly from the metric :

$$\Delta \tau = \frac{gH}{c^2} \Delta t$$

with $\Delta t = 1 \text{ day}$, $H = 120 \text{ m}$ and $g = 10 \text{ m/s}^2$.

We will now send photons from one floor to the other. The result will be fun, and, in addition, we will find the metric, in a simple and intuitive way, without using a mathematical arsenal. You are on the second level and you send a photon down. By the time the photon moves to the bottom, the ship has

gained speed. Speed measured in the inertial frame of reference which coincides with the accelerated frame of reference of the rocket at the time of the emission of the photon.

Put yourself in the place of the one receiving the photon at the bottom stage; it is now at a velocity v with respect to the emitter at the moment the photon was emitted. So we have a Doppler effect and as we get closer to the source, the photon "blues". The photon passes very quickly from one stage to the other and the speed of the rocket acquired over this time is very low; we will therefore only use classical formulas.

Speed acquired by the rocket : $v = gt$

$$\text{and } t = \frac{x}{c} \text{ for the photon, then } v = \frac{gx}{c}.$$

$$\text{Frequency received: } f_R = (1 + \beta) f_E = \left(1 + \frac{gx}{c^2}\right) f_E$$

We find the expected blueshift. Of course, if the photon is now sent forward, its frequency decreases, and there is a redshift:

$$f_R = \left(1 - \frac{gx}{c^2}\right) f_E \text{ and } T_R = \left(1 + \frac{gx}{c^2}\right) T_E \text{ (small variations)}$$

This result is directly related to the metric, because the clocks are motionless with respect to each other in the rocket's frame of reference, and each oscillation of the light wave can be considered as a mini-flash emitted by the clocks. For example, for an emission wavelength of 600 nm, the source clock emits 500,000,000,000 mini-flashes every second, and a clock placed 120 meters forward receives 7 less

mini-flashes during one of its own seconds (by Doppler effect the signal reddens as it rises and the frequency decreases).

The observer placed higher up thus deduces that the time flows slower on the floor below and faster on the floor above.

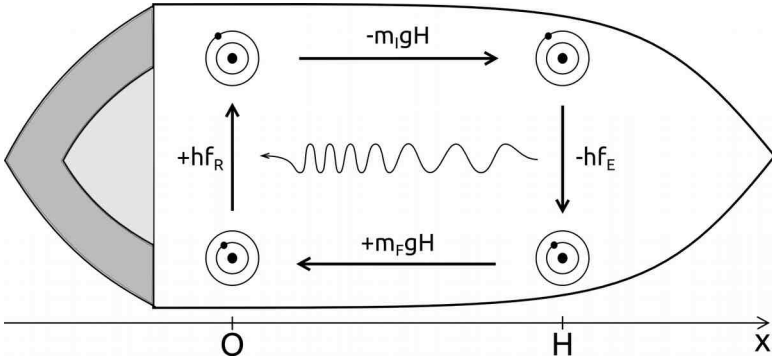
And that's not all, we can still broaden our understanding through an energetic approach. In physics we have the conservation of energy, and this fundamental law applies to special relativity by including the mass energy given by the famous formula $E=mc^2$.

We are going to move an atom from one floor to another. At the lower stage the atom is excited, we take it up in this state to the upper stage. Raising a mass requires energy from the operator. In a uniform acceleration field the energy received by an object of mass m is mgH . The energy of the atom increases by $m_I gH$, where m_I is the initial mass of the excited atom.

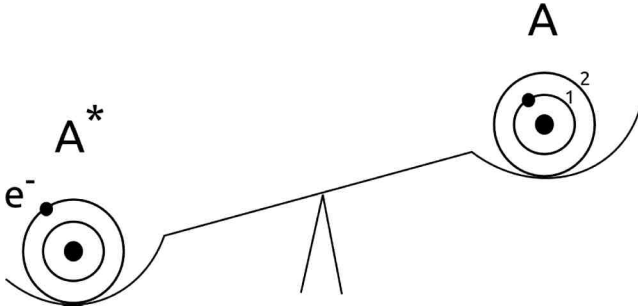
Then, the atom returns to its ground state and emits a photon of energy $e_E=hf_E$. We then go back down the atom, so the operator receives an energy $m_F gH$ where m_F is the final mass of the de-excited atom. And finally the photon of energy $e_R=hf_R$ is reabsorbed by the atom. The balance of this little game must be null because the energy must not vary:

$$-m_I gH - hf_E + m_F gH + hf_R = 0$$

$$\text{then } f_R - f_E = (m_I - m_F) \frac{gH}{h} = \frac{\Delta E}{c^2} \frac{gH}{h}$$



Atom balance:



An excited atom A^* is heavier than a de-excited atom. The difference in mass gives the energy of the emitted photon: $A^* \rightarrow A + \gamma$

$$\Delta m c^2 = (m^* - m) c^2 = E_\gamma \quad E_\gamma = \Delta E = E_2 - E_1 = h f$$

By spontaneous emission, the electron, linked to the atomic nucleus, passes from the upper level E_2 to the fundamental level E_1 by emitting a photon of energy equal to the energy difference of the electronic levels. More particles are linked, more binding energy is important and more the mass of the edifice is low.

The variation of the mass of the atom is due to the emission of the photon:

$$\text{so } \Delta E = hf_E \text{ and } f_R = f_E \left(1 + \frac{gH}{c^2} \right).$$

The received photon has a different energy than the emitted photon and we find the same expression as before. The photon gains energy when it goes down, it turns blue, and loses energy when it goes up, it reddens. The conservation of energy makes it possible to find the Doppler effect, the time dilation as a function of the position and the metric of the uniformly accelerated frame.

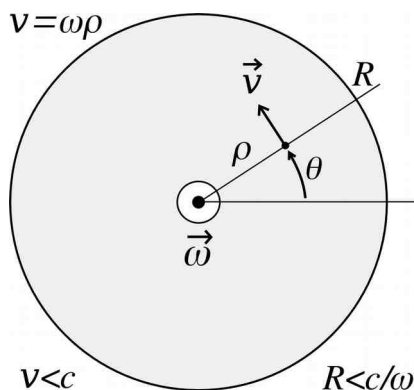
We will study the link between the uniformly accelerated reference frame and the reference frame of Schwarzschild, used for massive objects with spherical symmetry (planets, stars, black holes, etc.), in the following pages.

∞ METRIC OF A ROTATING FRAME

We are now going to approach another textbook case which can also be treated with special relativity. A case whose study opens the doors of practical applications, such as the ring laser gyroscope¹⁴ which allows orientation much more precisely than with a mechanical gyroscope or a magnetic compass. The ring laser gyro has been used in ships, submarines, airplanes and satellites since 1963.

14 Use of the Sagnac effect conceptualized in 1913.

We have a disk of radius R rotating uniformly around a fixed axis. The disc is a rigid solid¹⁵ whose speed increases linearly with the distance from the axis..



The speed is measured in an inertial reference frame R where the axis is fixed. We now place ourselves in the non-inertial frame of reference R' of the disc. Let us take a circle concentric with the axis of rotation, we measure the radius ρ with a stick of unit length. Then we begin to measure the circumference by transferring the stick as many times as necessary. For each report we use the inertial frame of reference coinciding at the location and given time. There is no contraction of the lengths radially, because the speed is perpendicular to the measured length, on the other hand in the orthoradial direction we are collinear with the speed and the length is contracted.

By dividing the perimeter of the circle by its

15 The rigidity criterion is verified for the disc in uniform rotation and the uniformly accelerated rocket: *L'espace-temps de Minkowski*, Nathalie Deruelle.

diameter, the value is greater than π , the space is curved¹⁶.

Let's determine the metric by performing the following change of coordinates¹⁷:

$$\begin{cases} t' = t \\ \rho' = \rho \\ \theta' = \theta - \omega t \\ z' = z \end{cases}$$

The metric in the inertial frame R is:

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

This standard expression given in Cartesian coordinates is also written in cylindrical coordinates, a coordinate system that facilitates calculations for this problem which has an axis of symmetry:

$$ds^2 = c^2 dt^2 - d\rho^2 - \rho^2 d\theta^2 - dz^2$$

The interval becomes in R' , removing the z coordinate for simplicity:

$$ds'^2 = ds^2 = c^2 dt'^2 - d\rho'^2 - \rho'^2 (d\theta' + \omega dt')^2$$

from where, by removing the prime symbols to lighten:

$$ds^2 = \left(1 - \frac{\rho^2 \omega^2}{c^2}\right) c^2 dt^2 - 2 \rho^2 \omega dt d\theta - d\rho^2 - \rho^2 d\theta^2$$

16 It is a new pseudo-paradox of special relativity, presented in 1909 by Ehrenfest as an internal contradiction of the theory. If we accept that the space for an observer of the disk is non-Euclidean, the contradiction disappears.

17 Detailed articles: *Space geometry of rotating platforms: an operational approach*, and, *The relativistic Sagnac effect: two derivations*, Guido Rizzi and Matteo Luca Ruggiero (2008).

By calculating the components of the Riemann curvature tensor (done in the next chapter) we find that all the components are zero. The spacetime of the uniformly rotating disk is therefore flat¹⁸. We are well within the framework of special relativity, there is no spacetime curvature, no mass present¹⁹, and the spacetime is well flat.

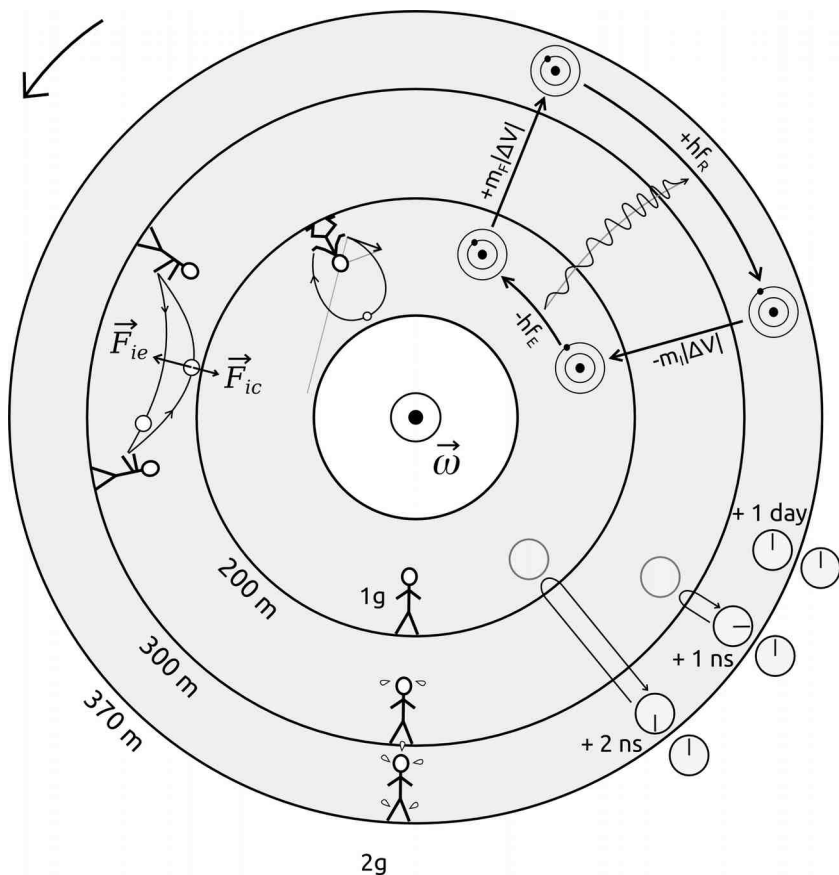
Special relativity applies in flat spacetime: a change of coordinates allows us to find the standard Minkowski metric again. In general relativity, in the presence of gravitation, this is only possible locally around an event: orders zero and one can always coincide with an inertial frame of reference (Minkowskian spacetime), on the other hand, this is no longer possible for order two, this is where the spacetime curvature is expressed.

We can create an artificial gravity with a rotating circular platform. The advantage, compared to the rocket continuously accelerated by the thrust of its reactors, is zero energy to spend. Once the disk in rotation, by conservation of energy, the disk keeps its kinetic moment, and gravity is maintained indefinitely for the occupants. On the other hand, the created gravity is not uniform, and, in addition to the centrifugal force that simulates gravity, there is the

18 You will have noticed the subtlety encountered here: space is curved and spacetime is flat.

19 As with the uniformly accelerated rocket, there is no mass present which creates a gravitational field and curves spacetime. The mass of the rocket, or of the disc, is here totally negligible and does not influence the metric. We are talking about test mass.

Coriolis force that complicates the motion of the astronauts.



To minimize these two drawbacks, the radius of the centrifuge must be large enough. The centrifugal acceleration gives: $g = \omega^2 \rho$ and $\Delta g / g = \Delta \rho / \rho$. For a variation in artificial gravity of less than 1% between the feet and the head, a radius of about 200 meters is required. And the corresponding angular speed of rotation is two revolutions per minute:

$$\omega = 2\pi f \quad \text{and} \quad f = \frac{1}{2\pi} \sqrt{\frac{g}{\rho}}.$$

The Coriolis acceleration is written $\vec{a}_c = 2\vec{\omega} \wedge \vec{v}_r$. When the astronauts run around the wheel, they feel heavier running in the same direction as the centrifuge and lighter running in the opposite direction, it is not very disturbing. On the other hand, if they bend up and down, they can be pushed sideways by the Coriolis force, which can be annoying.²⁰. Let's calculate: $a_c/g = 2v_r/\omega\rho = 2v_r/\sqrt{g\rho}$, for a speed of 20 km/h, $a_c/g \simeq 24\%$. This is not negligible, but we can consider it reasonable.

Now let's look at the time dilation. For an observer at rest:

$$d\tau = \sqrt{1 - \frac{\rho^2 \omega^2}{c^2}} dt \simeq \left(1 - \frac{\rho^2 \omega^2}{2c^2}\right) dt$$

For observers who are immobile in respect to each other, time does not flow at the same pace. A set of rest clocks at different points on the disk cannot be synchronized. The farther away from the axis, the slower the clocks go.

We place, according to the same protocol as for the rocket, a first clock at $\rho=370$ m, a second at $\rho=300$ m, and a third at $\rho=200$ m.

$$\text{We find: } \Delta\tau = \frac{(\rho_2^2 - \rho_1^2)\omega^2}{2c^2} \Delta t.$$

20 Funny video: www.voyagepourproxima.fr/ManegeTournant.mp4

After a day we bring the clocks back down to a radius of 370 meters: the one at 300 meters advances one nanosecond and the one at 200 meters advances two nanoseconds. Here, the advances do not vary linearly with the distance. The gravity is 1.5 g at 300 m and 1.85 g at 370 m, a good exercise to build muscle and stay young!

We take back our excited atom. We count the work received by the atom at each step. We mount it from $\rho_1=300$ m to $\rho_2=200$ m. The atom then gains a potential energy:

$$w_I = -\Delta e_{p_I} = \int m_I g(\rho) d\rho = m_I \omega^2 \int \rho d\rho = \frac{1}{2} m_I \omega^2 (\rho_2^2 - \rho_1^2)$$

It emits the photon: $w_E = -e_E = -h f_E$

It goes up: $w_F = -\Delta e_{p_F} = \frac{1}{2} m_F \omega^2 (\rho_1^2 - \rho_2^2)$

It receives the photon: $w_R = e_R = h f_R$

We perform the energy balance:

$$\frac{1}{2} m_I \omega^2 (\rho_2^2 - \rho_1^2) - h f_E - \frac{1}{2} m_F \omega^2 (\rho_2^2 - \rho_1^2) + h f_R = 0$$

$$\text{and we obtain: } f_R = f_E \left(1 + \frac{\omega^2 (\rho_1^2 - \rho_2^2)}{2 c^2} \right)$$

The photon turns blue as it moves away from the axis of rotation. We always have the same phenomenon, the photon reddens as it goes up and blues as it goes down.

∞ SCHWARZSCHILD METRIC

For comparison, we give the metric of spacetime around a massive object with spherical symmetry. It is the Schwarzschild metric of general relativity which replaces Newton's force of gravity to calculate the orbits of celestial bodies. For example, it can be used for studying the motion of the space station in the gravitational field generated by the Earth. In order to respect the central symmetry, the metric is given in spherical coordinates:

$$ds^2 = \left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{2GM}{rc^2}\right)} - r^2 d\theta^2 - r^2 \sin^2 \theta d\varphi^2$$

M is the mass of the central body (planet, star or black hole). This mass creates a gravitational field and spacetime is curved. There is no global coordinate change that makes this metric Minkowskian. Gravitation and spacetime curvature are absent in the special relativity.

Appears in the metric a quantity with the same units as a radius, this characteristic distance of the system is called Schwarzschild radius:

$$\text{we define } r_s = \frac{2GM}{c^2}.$$

As for the accelerated frame in special relativity, we have an event horizon, here located in r_s .

For an object at rest we obtain the temporal part:

$$d\tau = \sqrt{1 - \frac{2GM}{rc^2}} dt$$

The further we move away from the massive object, the lower the curvature. At great distance, space can be approximated as flat, and, according to the equivalence principle of general relativity, we must find the form of the metric of the uniformly accelerated rocket:

$$d\tau \simeq \left(1 - \frac{GM}{rc^2}\right) dt \quad \text{for } r \gg r_s.$$

For example, for the Earth, the radius r_s is about 9 millimeters. On the Earth ground, about 6370 km away, the approximation is extremely good²¹.

With $r = r_0 + x$ and $r_0 \gg r_s$:

$$d\tau_{r_0} = \left(1 - \frac{GM}{r_0 c^2}\right) dt \quad \text{and} \quad d\tau_{r_0+x} = \left(1 - \frac{GM}{r_0 c^2} \left(1 - \frac{x}{r_0}\right)\right) dt$$

$$\text{gives } d\tau_{r_0+x} = \left(1 + \frac{GMx}{r_0^2 c^2}\right) d\tau_{r_0}$$

The form is the same as for the uniformly accelerated rocket:

$$d\tau = \left(1 + \frac{gx}{c^2}\right) dt.$$

²¹ Also, we can forget the Earth's rotation because the ground speed can be neglected in front of the escape velocity (geocentric reference frame).

We find the equivalence principle when:

$$g = \frac{GM}{r_0^2}.$$

Here we have also the highest clocks faster and the ascending photons that redden. On the Earth ground, over a height of 100 meters, the time lag reaches 0.9 nanoseconds in 24 hours²². Result close to that obtained in the rocket²³. Locally, nothing allows astronauts to differentiate the artificial gravity field created by the acceleration of the rocket, from a natural gravity generated by a mass. On the other hand, over a sufficiently large space domain, they could differentiate the two situations: the space of the uniformly accelerated rocket is Euclidean while that of the massive celestial body is not²⁴.

22 In the case of the space station, even if the 110 meters beam can be maintained directed towards the Earth with a tidal stabilization, the clocks remain synchronized. At the level of the station, the gravity field is still 90% of the one on the ground, but there is no redshift, because during the rotation around the Earth, the external part goes slightly faster than the internal part and the effect is perfectly compensated. This is the principle of equivalence, for the astronauts everything happens as if there was no more gravitation (they are in weightlessness) because they are in free fall.

23 In both cases we have clocks at rest in relation to each other, which become desynchronized. For the rocket, by changing the reference frame, we can consider that it is a Doppler effect. This is not possible for gravitation and we speak of a redshift or blueshift.

24 Also in the rocket the proper acceleration is inversely proportional to the horizon distance, while for the massive object it varies with the square of the distance to the center of the body. The equivalence principle is only true very locally.

Exercises

1. ▲▲▲ Euclidean metric

$$d l^2 = dx^2 + dy^2 + dz^2$$

Show that the Euclid metric is invariant by translation, rotation and a Galilean transformation.

Answers p381.

2. ▲▲▲ Rapidity

1 - Show that the standard Lorentz transformation can be written:

$$\begin{cases} ct' = ct \operatorname{ch} \varphi + x \operatorname{sh} \varphi \\ x' = ct \operatorname{sh} \varphi + x \operatorname{ch} \varphi \\ y' = y \\ z' = z \end{cases}$$

We used hyperbolic trigonometry and φ is the *rapidity*.

2 - Show that, for two successive Lorentz transformations in the same direction, the rapidities are additive.

Answers p382

3. ▲▲▲ Rindler metric²⁵

$$d s^2 = r^2 d \tau^2 - dr^2 - dy^2 - dz^2$$

1 - What are the invariances of the Rindler

²⁵ W. Rindler, *Relativity*, Oxford Univ. Press, 2^d Ed, 2006, 430 pages.

coordinate system by rotation and Lorentz transform?

2 - Show that this coordinate system corresponds to that of a uniformly accelerating reference frame.

3 - Show that the following change of coordinates makes it possible to find a Minkowskian metric:

$$\begin{cases} ct = r \operatorname{sh} \tau \\ x = r \operatorname{ch} \tau \end{cases}$$

Deduce the change of coordinates between the frame of reference (x, t) of the uniformly accelerated rocket and the galactic frame of reference (x', t') .

Draw on a Minkowski diagram, in the inertial frame R' , the set of coordinate lines for x and t .

Answers p382

4.⁰¹₁₀ ▲▲△ **Free fall in the rocket**

In our uniformly accelerated rocket, to pass the time during this trip of a few years, we have fun throwing objects at each other. Whether you drop a ball with no initial speed, or throw it to your partner, we call this motion of the object free fall, because it is not subjected to any force. We explained that the acceleration of the rocket generates artificial gravity. This is locally equivalent to a uniform gravity field, but, given the metrics of the accelerated frame, we suspect that the trajectory of an object in

free fall will be modified. We will approach the question in two phases: a first qualitative approach and then a complete computation.

1 - We take two clocks initially synchronized and stationary in the same place. As in the course, one will stay in the same place, and the second one will be moved and brought back to the starting point. You play the following game: At the start both clocks indicate zero. You have the mobile clock that you can move as you wish. The only constraint is that at one minute exactly as indicated on the fixed clock, your clock will have to be back, placed very quietly next to it. The challenge is to get the greatest possible time on your clock. How do you have to move it to win?

Variation of the game: Previously the starting point was the end point. If now the finish point, while remaining at the same level, is different, how do we proceed to maximize the time on our clock?

2 - The path followed by a free particle to go from the initial event E_i to the final event E_f maximizes its proper time:

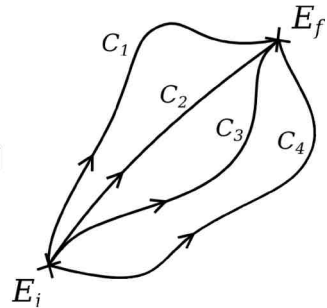
$$\tau = \int_{E_i}^{E_f} d\tau = \int_C \sqrt{g(x) - \frac{v^2}{c^2}} dt \quad \text{with} \quad g(x) = \left(1 + \frac{ax}{c^2}\right)^2$$

$$\text{Lagrangian: } L(x, v) = \sqrt{g(x) - \frac{v^2}{c^2}}$$

An infinity of possible paths C links E_i to E_f .

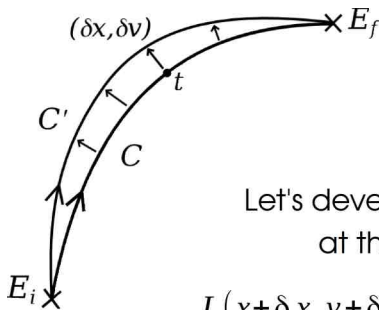
Which one extremes τ ?

We know that for the extremal path, a small variation of the parameters x and v does not modify, at order one, the proper time.



It is a simple mathematical property: at the maxima and minima of a function the slope is zero.

Suppose that C is the optimal path and consider C' infinitely close. At given t , we pass from C to C' by small variations of x and v :



$$x_{C'} = x_C + \delta x$$

and

$$v_{C'} = v_C + \delta v$$

Let's develop the Lagrangian at the first order:

$$L(x + \delta x, v + \delta v) = L(x, v) + \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \delta v$$

Thus: $\int_{C'} L(x + \delta x, v + \delta v) dt$

$$= \int L(x, v) dt + \int \left(\frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial v} \delta v \right) dt = \tau + \delta \tau$$

For the searched path $\delta \tau = 0$.

a- Continue the reasoning and establish the equation of motion of an object in free fall. Show that this equation, at the start of the throw and at low speeds, gives the equation of free fall in

Newtonian mechanics.

Finally, how will you move your clock to win?

b- Demonstrate the following conservation law:

$$L - \frac{\partial L}{\partial v} v = \text{cst}$$

We consider the case of a release from rest. Find the expression of position, velocity and acceleration as a function of $g(x)$. How does g vary during the fall? Show that the falling velocity reaches a maximum and then cancels on the horizon. What is the maximum falling speed? At what distance from the horizon?

c- Perform a numerical simulation to plot position, velocity and acceleration curves as a function of time. When is the maximum speed reached? When does the object reach the horizon for an observer of the rocket?

d- Proper time: Give the expression of the proper time. In its proper reference frame, when does the object reach the horizon? Suppose that the object is a mini auxiliary rocket that leaves the mother ship in free fall. What will happen to the occupant of the mini-rocket when it reaches the horizon? This small rocket is very fast, the pilot decides to ignite the engine to return to the main ship, will he succeed? You can illustrate the situation on two Minkowski diagrams (galactic and rocket frames).

e- Local Minkowskian observer: The coordinate system of the accelerated rocket is not Minkowskian. The velocity previously determined in a non-Minkowskian metric is called the coordinate velocity. This coordinate system has been constructed in a non-inertial frame of reference and the assumptions of special relativity do not apply directly to it. This reference frame is nevertheless very useful and necessary for the occupants of the rocket, but the speed of light is not fixed at c . This is why we will consider a new observer, an inertial one. At each instant and position of the object in free fall, we consider the Minkowskian reference frame coinciding with that of the rocket:

$$c^2 d\tau^2 = c^2 dt_{Mink}^2 - dx^2$$

For example, imagine two rockets fixed relatively to each other and uniformly accelerated. All of a sudden, one of them cuts its engine, its reference frame becomes inertial, and for some time it coincides with the rocket still accelerated. Thus an observer in the rocket which cut its engine is minkowskian, and he can observe the fall of the object. What speed will he measure for the falling object? What will be the velocity of the falling object at the horizon for a Minkowskian observer?

3 - Analogy with the fall into a black hole:

a- The Schwarzschild coordinate system is that of an outside observer at the black hole. We can

compare the radial fall of an object towards a black hole with the vertical fall of an object observed by the occupant of a uniformly accelerated rocket:

$$d\tau^2 = g(r) dt^2 - \frac{dr^2}{c^2 g(r)} \quad \text{with} \quad g(r) = 1 - \frac{2GM}{rc^2}$$

$$\tau = \int L(r, v) dt \quad \text{and} \quad L(r, v) = \sqrt{g(r) - \frac{1}{g(r)} \frac{v^2}{c^2}}$$

Describe the velocity profile of a falling body, dropped without initial velocity, to the horizon of the black hole $r_H = r_s = 2GM/c^2$. You will draw curves for speed and acceleration as a function of r .

What is the maximum speed reached? At what distance from the horizon?

b- Perform a numerical simulation to plot position, speed and acceleration curves as a function of time. When is the maximum speed reached? When does the object reach the horizon for an observer outside the black hole?

c- Proper time: Give the expression of the proper time. In its proper reference frame, when does the object reach the horizon? Suppose the object is a spacecraft in free fall. What will happen to the occupant of the spacecraft when he reaches the horizon? This rocket is very fast and powerful, the pilot decides to start the reactor to leave the black hole, will he succeed?

d- Local Minkowskian observer: The Schwarzschild coordinate system is not Minkowskian. We

have previously determined the coordinate velocity and *coordinate acceleration* in this coordinate system. This coordinate system is very convenient and useful but the speed of light is not fixed at c . That is why we will consider a new observer, him inertial. At each instant and position of the falling object, we consider the Minkowskian frame motionless with respect to the black hole and coinciding with the Schwarzschild frame of reference:

$$c^2 d\tau^2 = c^2 dt_{Mink}^2 - dr_{Mink}^2$$

Which speed is measured in this way for the object in free fall? What will be the speed of the falling object for a Minkowskian observer at the horizon?

e- Comparison to experimental data:

In 2018, a study²⁶ of the measurements made by the XMM-Newton probe, which observed a super-massive black hole of 40 million solar masses, shows a wind of matter in free fall towards the black hole that reaches relativistic speeds:

$$v \sim 0.3c \quad \text{for } r \sim 20 R_s$$

$$v \sim 0.1c \quad \text{for } r \sim 200 R_s$$

Do these results seem consistent with those found in the exercise?

Answers p384.

²⁶ *An ultrafast inflow in the luminous Seyfert PG1211+143*, 2018, K.A.Pounds, C.J.Nixon, A.Lobban and A.R.King. University of Leicester, United-Kingdom.

5. ▲△△ Fall of a blue ball

We release from rest a blue ball into the uniformly accelerated rocket and watch it fall in free fall. What will be the color of the ball perceived during its fall by the astronauts of the rocket?

Answers p404.

6. ▲▲△ Trajectory of a ray of light in the Einstein's Elevator

Albert Einstein proposes a thought experiment in his book *Relativity* written in 1916. We imagine a portion of empty space infinitely distant from all masses. We have at our disposal a large box in which an observer evolves in weightlessness. A hook makes it possible to exert a constant force on the box by means of a rope, which is then animated by a rectilinear translation motion uniformly accelerated. The observer thus experiments an artificial gravity.

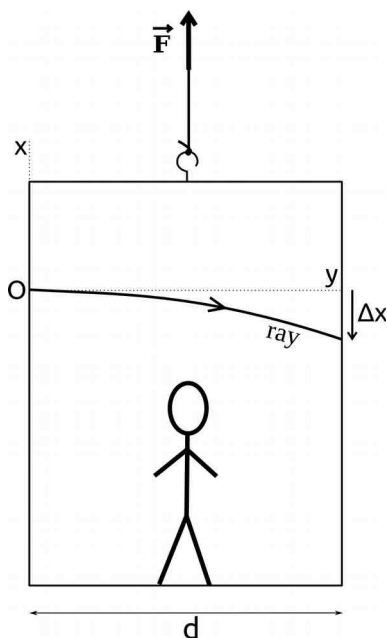
Compared to the immobile box, constituting an inertial frame of reference, the trajectory of a light ray of speed c is rectilinear. On the other hand, in the box accelerated by the traction of the rope, a light ray, here, initially perpendicular to the direction of motion, will take a curved trajectory. Let's quote Einstein: "*It can easily be shown that the path of the same ray of light is no longer a straight line*".

1 - Newtonian approximation:

We consider the speed of light constantly equal to c , and the rectilinear trajectory, in the Galilean frame of reference which initially coincides with the box.

For a constant acceleration box a , determine Δx .

Express the equation of the trajectory $y(x)$ and of the light speed $v(x)$ in the accelerated frame.



2 - Special Relativity:

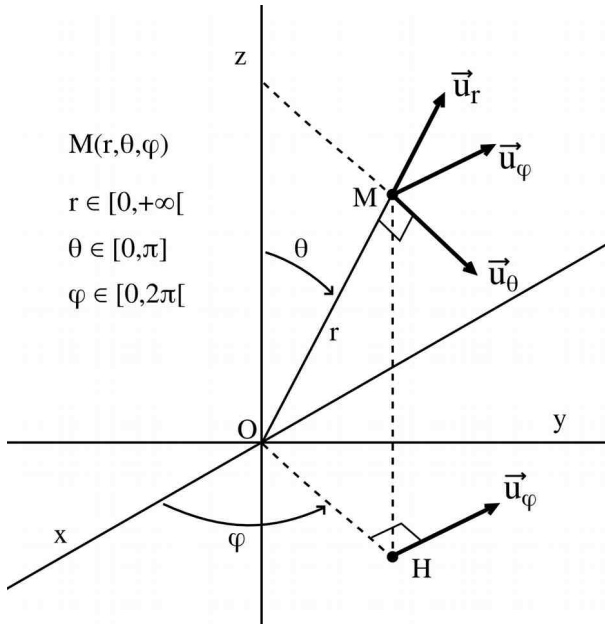
We answer the same questions as above. For that, we first consider the equation of the light ray worldline in an inertial reference frame. Then, with the appropriate change of coordinates, we obtain the equation of the worldline in the non-inertial box.

3 - Drawing of curves.

Answers p404.

7. ▲▲△ Spherical coordinate system

Spherical coordinate system definition:



1 - Conversions between spherical and rectangular coordinates.

2 - Express the position vector $\vec{r} = \overrightarrow{OM}$ and the infinitesimal element vector $d\vec{r} = \overrightarrow{MM'}$ between M and M' infinitely close.

3 - Find by integration the surface and the volume of a sphere.

4 - Definition of plane angles and solid angles: from an observation point O , we observe an object. The extensions of the periphery of the object cuts an arc on the circle unit of center O . The length of this arc

gives the value of the angle in radians under which we see the object. In 3D space the circle is replaced by a sphere unit on which a surface is cut out. The area of this surface gives the solid angle in steradians under which we see the object.

a- From which solid angle do we see the whole space? The starry sky on a clear night? A room from one of its corners?

b- Calculate the solid angle of an angle cone α .

c- What is the probability that a star is in the plane of the ecliptic within ten degrees?

Answers p407.

FOUR-VECTORS

We have introduced special relativity through the Minkowski spacetime: events space with its metric²⁷. We can extend this points space to build more complex elements such as vectors or tensors.

The following presentation is a bit formal but necessary for a full understanding of relativity. We will continue to rely on a geometrical vision as soon as possible.

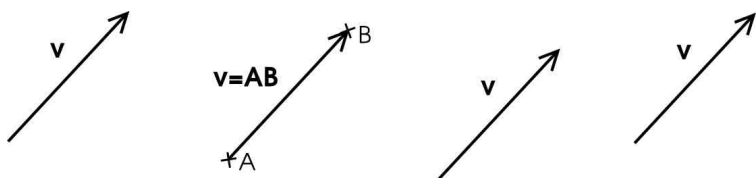
The elements of a vector space E are vectors, noted in this book with bold letters : \mathbf{v} .

If we need to specify that we are in a Euclidean vector space, we will use the classic notation with arrows : \vec{v} .

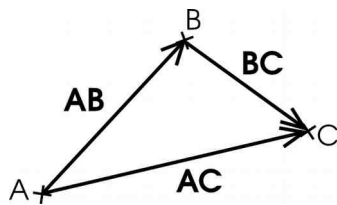
In the case of the Minkowski space, we can clarify the context by talking about four-vectors noted with tildes : $\tilde{\mathbf{v}}$.

²⁷ We considered the standard Minkowski metric of an inertial frame $ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$ in an orthonormal Cartesian coordinate system. While keeping an inertial reference frame, the form of the metric can be different. For example, in cases where the metric is expressed in a non-orthonormal or non-Cartesian coordinate system. We then speak of Minkowskian metric. When the change of coordinates gives a non-inertial frame of reference (as for our rocket and the rotating disk) the special relativity is applied by adding metric effects (page 229).

In general, a vector can be uniquely defined from two points (or events) in our space (or spacetime):

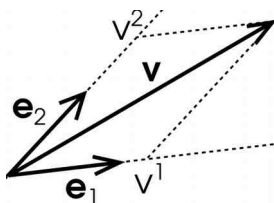


Vector space is affine and with a third point we have the relation $\mathbf{AC} = \mathbf{AB} + \mathbf{BC}$:



By multiplying by a real we have a new vector $k \mathbf{AB}$ and the vector is directed $\mathbf{BA} = -\mathbf{AB}$. Any linear combination of E vectors is a new E vector.

We express a \mathbf{v} vector in a basis. The basis vectors are denoted \mathbf{e}_i and form a spanning and generating set of E.



For a vector space of dimension n :

$$\mathbf{v} = v^1 \mathbf{e}_1 + v^2 \mathbf{e}_2 + \dots + v^n \mathbf{e}_n = \sum_{i=1}^n v^i \mathbf{e}_i = v^i \mathbf{e}_i$$

We use Einstein summation convention, the summation is implied for a repeated index up and down. The v^i are the components of \mathbf{v} expressed with the basis vectors $(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n)$.

Scalar product of two vectors \mathbf{a} and \mathbf{b} :

$$\mathbf{a} \cdot \mathbf{b} = (a^i \mathbf{e}_i) \cdot (b^j \mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j a^i b^j$$

We define the components of the metric tensor \mathbf{g} such as: $g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$.

$$\text{so } \mathbf{a} \cdot \mathbf{b} = g_{ij} a^i b^j.$$

For example, for $n=2$, we have:

$$\mathbf{a} \cdot \mathbf{b} = g_{11} a^1 b^1 + g_{12} a^1 b^2 + g_{21} a^2 b^1 + g_{22} a^2 b^2$$

The scalar product²⁸ is commutative and the components of the metric tensor are symmetrical:

$$g_{ij} = g_{ji}$$

We can write the components of the metric tensor in a matrix.

For example, for $n=3$ in the basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$:

$$\mathbf{g} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

We have a second way to project a vector. The first

²⁸ In math, we talk about bilinear form, it associates to two vectors a number, called scalar.

components, given above, are obtained parallel to the basis vectors. We can obtain a new set of v_i components with orthogonal projections:

$$\mathbf{v} \cdot \mathbf{e}_i = (v^j \mathbf{e}_j) \cdot \mathbf{e}_i = g_{ij} v^j = v_i$$

We then have a new basis associated with these new components: $\mathbf{e}^i = g^{ij} \mathbf{e}_j$. The g^{ij} are calculated from the g_{ij} with: $g_{ik} g^{kj} = \delta_i^j$ where δ_i^j is the Kronecker delta, null, if the indices are different, and, equal to one, if they are equal.

We then have a new writing :

$$\mathbf{v} = v_i \mathbf{e}^i$$

Lower-index objects are covariant quantities, while upper-index objects are contravariant quantities.

For example, the components v_i are covariants and the basis vectors \mathbf{e}^i are contravariants. The components g_{ij} are two times covariants and the tensor g^{ij} is two times contravariants. We will see the precise meaning and importance of this vocabulary at the moment of the change of basis.

The metric tensor allows us to switch between these two types of quantities.

In the end, we can have four different writings for the scalar product:

$$\mathbf{a} \cdot \mathbf{b} = g_{ij} a^i b^j = a^i b_i = a_i b^i = g^{ij} a_i b_j$$

Orthogonal vectors: $\mathbf{a} \cdot \mathbf{b} = 0$.

In the case of orthogonal bases:

$$\text{if } i \neq j \text{ then } g_{ij} = 0.$$

For example, for $n=2$: $\mathbf{a} \cdot \mathbf{b} = g_{11} a^1 b^1 + g_{22} a^2 b^2$

$$\text{and } \mathbf{g} = \begin{pmatrix} g_{11} & 0 \\ 0 & g_{22} \end{pmatrix}.$$

Vectors, tensors and scalars are essential mathematical objects for physics. The laws of nature are expressed using equations constructed from these three types of objects, because if we change the basis, the laws keep the same form. The new basis is associated with new coordinates used to realize a translation, a rotation or a change of Galilean or inertial reference frame. We will study the change of coordinates later.

Following this somewhat abstract interlude, let us approach different practical cases.

∞ EUCLIDEAN VECTOR SPACE

Newton's laws and all classical mechanics is built with vectors, scalars and tensors.

Newton's second law:

$$\vec{F} = m \vec{a} ,$$

Kinetic power:

$$P_k = \frac{dE_k}{dt} = \frac{d}{dt} \left(\frac{1}{2} m \vec{v} \cdot \vec{v} \right) = P = \vec{F} \cdot \vec{v} ,$$

Angular momentum:

$$\frac{d\vec{O}}{dt} = \frac{d}{dt} (m \vec{r} \wedge \vec{v}) = \vec{r} \wedge \vec{F} .$$

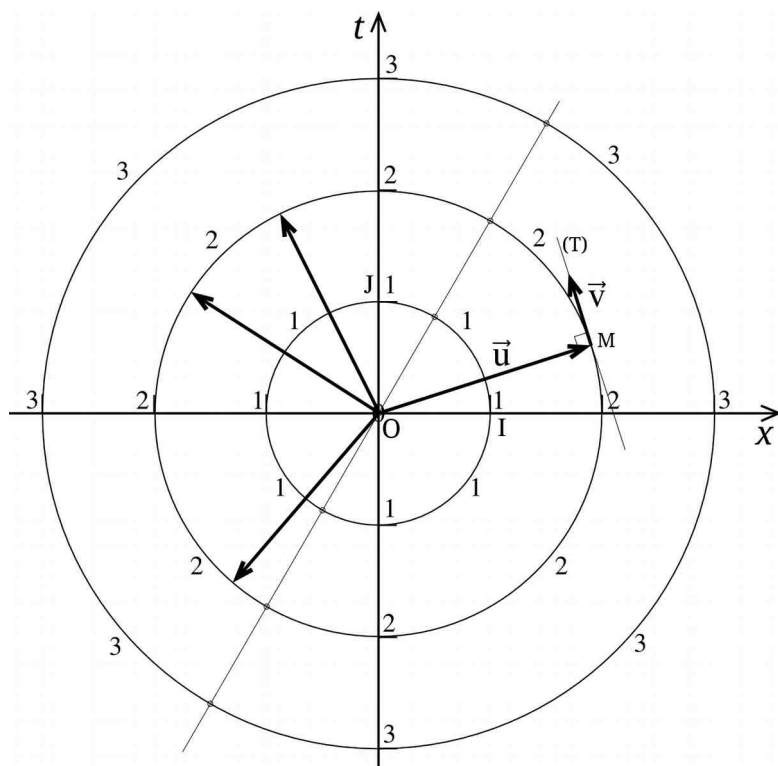
All these laws keep the same form by translation, rotation and Galilean transformation. The use of vectors assures us that.

In Euclidean geometry the scalar product of a vector with itself can only be positive or zero, we can then define a **norm**:

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

The norm is positive definite:

- $\vec{v} \cdot \vec{v} \geq 0$.
- $\vec{v} \cdot \vec{v} = 0$ if and only if $\vec{v} = \vec{0}$.



In Euclidean geometry, the norm of a vector is represented by its length and this length is independent of the chosen basis. Starting from O , all the ends of vectors of the same norm are placed on the same circle (we have represented four vectors of norm 2).

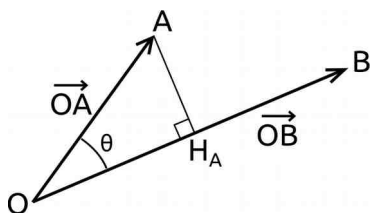
A property of the circle: if we draw a radius OM , the tangent (T) is always perpendicular to (OM) . We thus obtain a pair of orthogonal vectors:

$$\vec{u} \cdot \vec{v} = 0.$$

For a set of concentric circles of radii multiple of unity, a line through O intersects the circles at a set of equidistant points.

Geometric determination of the scalar product:

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\widehat{\vec{a}, \vec{b}})$$



$$\vec{OA} \cdot \vec{OB} = OA \times OB \times \cos \theta$$

$$= \pm OH_A \times OB$$

$$= \pm OH_B \times OA$$

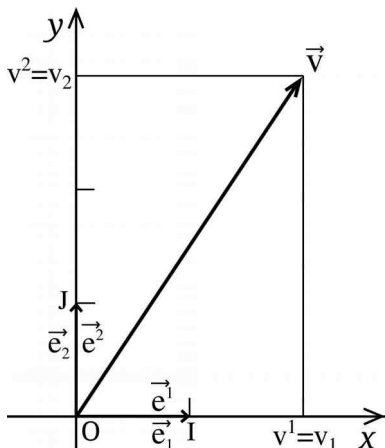
$$\vec{a} \cdot \vec{b} = \vec{OA} \cdot \vec{OB} = (\vec{OH} + \vec{HA}) \cdot \vec{OB} = (\vec{c} + \vec{n}) \cdot \vec{b} = \vec{c} \cdot \vec{b} + \vec{n} \cdot \vec{b}$$

In the end, if we find an orthogonal vector \vec{n} , the dot product comes down to that of two collinear vectors and the value is the product of their radii:

$$\vec{a} \cdot \vec{b} = \vec{c} \cdot \vec{b} = \pm R_c \times R_b$$

The sign is positive if the two collinear vectors are in the same direction, and negative if they are in opposite directions. We have two equivalent options, find a vector orthogonal to \vec{a} or to \vec{b} .

• Orthonormal Cartesian bases:



We can always go back to an orthonormal Cartesian basis:

$$\vec{e}_i \cdot \vec{e}_j = \delta_{ij}.$$

For example, for $n=2$, we have in this case:

$$g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\vec{a} \cdot \vec{b} = a^1 b^1 + a^2 b^2$$

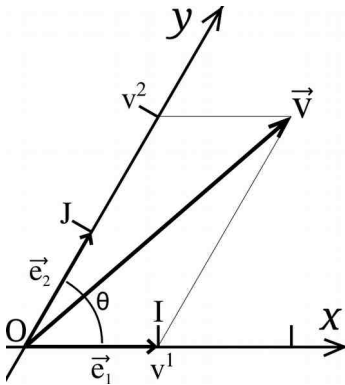
and for the norm:

$$v = \sqrt{\|\vec{v}\|} = \sqrt{(v^x)^2 + (v^y)^2}$$

The covariant and contravariant components are then identical. The same applies to the bases.

◦ Normal and non-orthogonal Cartesian bases:

Case for a vector of the plane (2-vector) :



We know the contravariants components of \vec{v} in the covariant base:

$$\vec{v} = v^1 \vec{e}_1 + v^2 \vec{e}_2 = \vec{e}_1 + 2 \vec{e}_2$$

$$\text{with } g_{ij} = \begin{pmatrix} 1 & \cos \theta \\ \cos \theta & 1 \end{pmatrix}$$

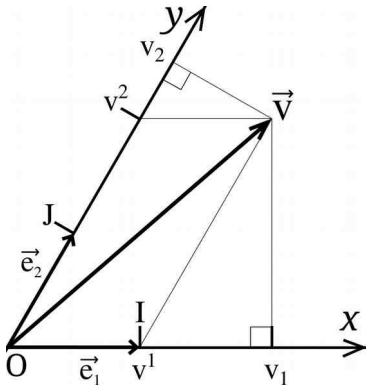
$$\text{and } \theta = \frac{\pi}{3}.$$

Let's determine the covariant components of \vec{v} :

$$v_i = g_{ij} v^j = g_{i1} v^1 + g_{i2} v^2$$

$$v_1 = g_{11} v^1 + g_{12} v^2 = v^1 + \cos \theta v^2 = 2 = \vec{v} \cdot \vec{e}_1$$

$$v_2 = g_{21} v^1 + g_{22} v^2 = \cos \theta v^1 + v^2 = \frac{5}{2} = \vec{v} \cdot \vec{e}_2$$



We now have two possible decompositions for \vec{v} :

$$\vec{v} = \vec{e}_1 + 2\vec{e}_2 = 2\vec{e}^1 + \frac{5}{2}\vec{e}^2$$

Let us determine the metric tensor components in the contravariant base:

$$g_{i1} g^{1j} + g_{i2} g^{2j} = \delta_i^j$$

$$g_{11} g^{11} + g_{12} g^{21} = 1 \quad \text{then} \quad g^{11} + \cos \theta g^{21} = 1$$

$$g_{11} g^{12} + g_{12} g^{22} = 0 \quad \text{and} \quad g^{12} = -\cos \theta g^{22}$$

$$g_{21} g^{12} + g_{22} g^{22} = 1 \quad \text{and} \quad \cos \theta g^{12} + g^{22} = 1$$

$$g_{21} g^{11} + g_{22} g^{21} = 0 \quad \text{and} \quad g^{21} = -\cos \theta g^{11}$$

$$\text{so: } g^{11} = g^{22} = 1/\sin^2 \theta$$

$$\text{and } g^{12} = g^{21} = -\cos \theta / \sin^2 \theta$$

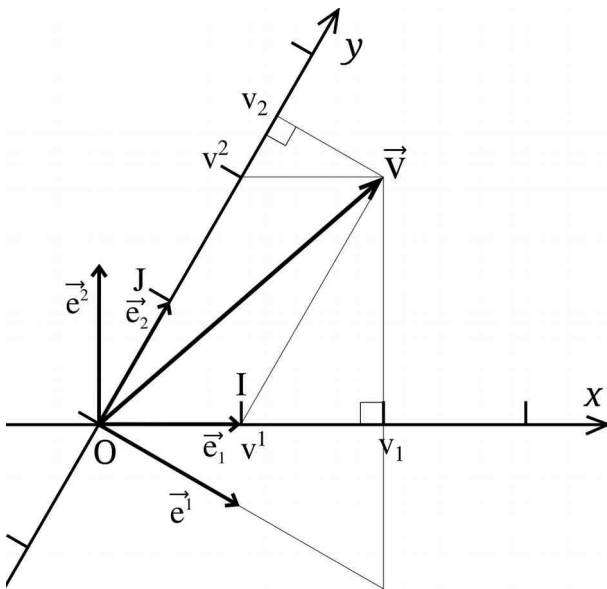
$$\text{Metric : } g^{ij} = \frac{1}{\sin^2 \theta} \begin{pmatrix} 1 & -\cos \theta \\ -\cos \theta & 1 \end{pmatrix}$$

Let's find the contravariant basis:

$$\vec{e}^i = g^{ij} \vec{e}_j = g^{i1} \vec{e}_1 + g^{i2} \vec{e}_2$$

$$\text{so } \vec{e}^1 = g^{11} \vec{e}_1 + g^{12} \vec{e}_2 = \frac{\vec{e}_1 - \cos \theta \vec{e}_2}{\sin^2 \theta} = \frac{4}{3} \left(\vec{e}_1 - \frac{1}{2} \vec{e}_2 \right)$$

$$\vec{e}^2 = g^{21} \vec{e}_1 + g^{22} \vec{e}_2 = \frac{-\cos \theta \vec{e}_1 + \vec{e}_2}{\sin^2 \theta} = \frac{4}{3} \left(-\frac{1}{2} \vec{e}_1 + \vec{e}_2 \right)$$



Now, if you are a math teacher in middle school and when studying non-orthogonal coordinate systems a pupil asks you, "Why do we project along parallels and not perpendiculars?" you will know what to answer. The pupil is absolutely right, both types of projections are possible and even complementary.

∞ MINKOWSKI VECTOR SPACE

We will establish the new physical laws of special relativity based on four-vectors. For the formulas, we will be inspired by Newton's mechanics via the low speed limit.

We note the components of an event E with indices from 0 to 3 :

$$\tilde{X} = x^\mu (x^0, x^1, x^2, x^3)$$

$$x^0 = ct, \quad x^1 = x, \quad x^2 = y, \quad \text{and} \quad x^3 = z$$

$$\tilde{v} = \widetilde{OE} = x^\mu(E) - x^\mu(O) \quad ^{29}$$

$$\text{For the scalar product: } \tilde{a} \cdot \tilde{b} = g_{\mu\nu} a^\mu b^\nu.$$

With the Minkowski metric:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We will show that this metric gives back the triangle of times.

$$\text{We have: } \tilde{a} \cdot \tilde{b} = a^0 b^0 - a^1 b^1 - a^2 b^2 - a^3 b^3.$$

²⁹ Vectors, or tensors, are regularly misidentified with their components. In general, this does not lead to confusion.

For the spatial part, we recognize a Euclidean scalar product, we can then write:

$$\vec{a} \cdot \vec{b} = a^0 b^0 - \vec{a} \cdot \vec{b}.$$

The scalar product of a vector \vec{v} with itself can be positive, zero or negative:

$$\vec{v} \cdot \vec{v} = (v^0)^2 - \|\vec{v}\|^2.$$

Contrary to the Euclidean case, the Minkowskian scalar product of a vector with itself is not always positive. Moreover, $\vec{v} \cdot \vec{v} = 0$ does not imply $\vec{v} = \vec{0}$. There is no norm for a vector in Minkowski space. The quantity $\vec{v} \cdot \vec{v}$ is sometimes called pseudo-norm³⁰.

In Euclidean space the length of a vector, represented on an orthonormal coordinate system, corresponds to its norm, and the vectors of the same norm, starting from the same point, are distributed on the same circle. This is no longer the case on a Minkowski diagram: two vectors can have the same pseudo-norm and not appear with the same length.³¹ The 4-vectors of the same pseudo-norm are distributed on hyperbolas.

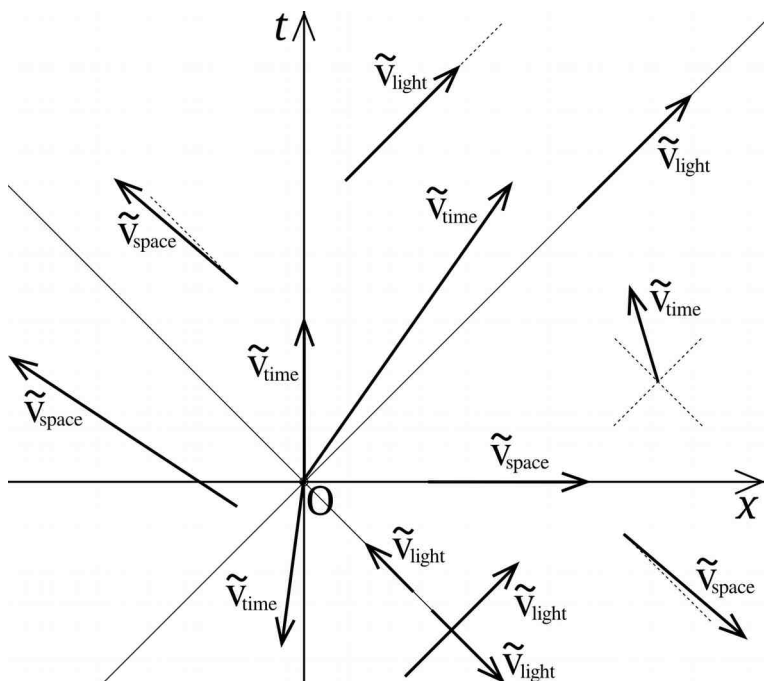
30 Term used and debatable: this term refers to the Euclidean norm without taking up all its principles. Contrary to the norm, the pseudo-norm does not have the same units as the vector (the square root is missing). We could consider the quantity: $k = \sqrt{|\vec{v} \cdot \vec{v}|}$ where k is the parameter of the hyperbola associated with the 4-vector. We could name k , the timelike or spacelike norm depending on the case (as in Euclidean where R is the parameter of the circle and the norm of the vector). We will use the term *intensity* for the k of a four-vector.

31 We represent the two-dimensional Euclidean space on a sheet of paper which is itself a 2D Euclidean physical object. On the other hand, using a Euclidean sheet to represent Minkowski's plane requires an effort of abstraction.

We have three kinds of 4-vectors :

- timelike : $\tilde{v} \cdot \tilde{v} > 0$
- lightlike : $\tilde{v} \cdot \tilde{v} = 0$
- spacelike : $\tilde{v} \cdot \tilde{v} < 0$

The light-like vectors are on the light cones associated with the world-lines of photons. The time-like vectors are in the cone (towards the vertical), and the space-like vectors towards the outside of the cone.



Depending on the sign of the time component, a four-vector can point towards the future or the past. This property and that of the time, light or space-like

kind do not depend on the inertial frame of reference considered.

When the scalar product of two vectors is null we have orthogonal vectors:

$$\tilde{a} \cdot \tilde{b} = 0$$

This property of orthogonality is also valid in all inertial frames of reference.

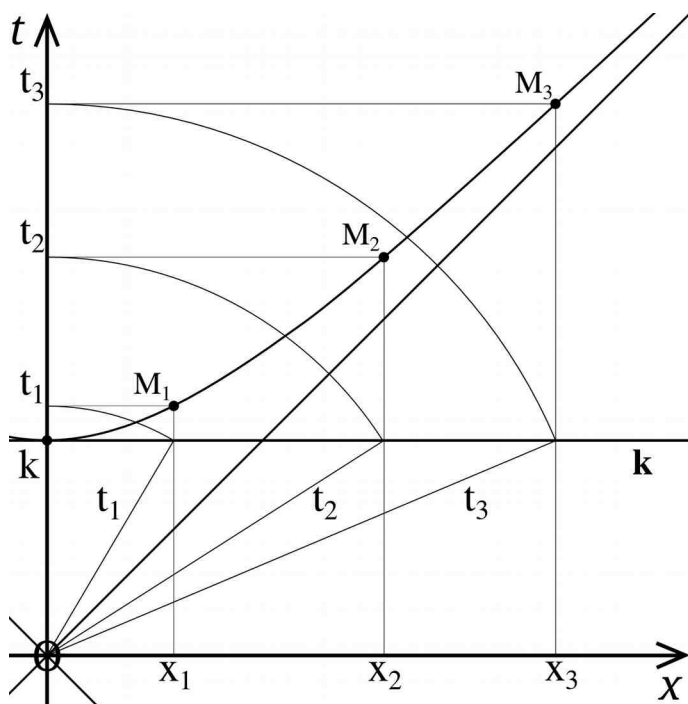
Again, the situation is not as intuitive as in Euclidean, it is not because two vectors are orthogonal that they appear perpendicular on a diagram.

We have two types of hyperbolas, those time-like, internal to the light cone, of equations $t^2 - x^2 = k^2$ (to simplify we have set $c=1$), and the external ones, space-like, of equations $t^2 - x^2 = -k^2$ ³².

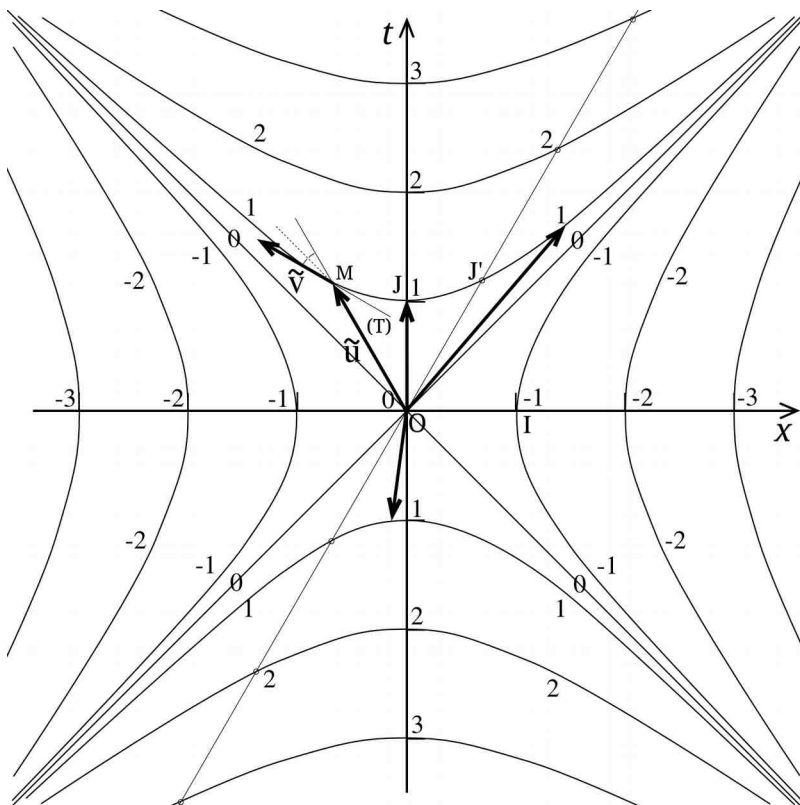
k defined as positive.

32 "*Space and Time*", Hermann Minkowski, lecture delivered at Cologne on 21st September 1908.

We easily find again the hyperbolas by a construction with *the triangle of times*:



Plot of an internal hyperbola of parameter k . For a given x it corresponds to a value of t which forms a right-angled triangle with k : $t^2 = k^2 + x^2$. For a 4-vector position x^μ , time-like, k corresponds to a proper time τ . For an external hyperbola, k is represented by a vertical line and it is x which is placed at the hypotenuse: $x^2 = k^2 + t^2$.



A hyperbolic geometry: vectors of the same pseudo-norm, that start in O , end on the same pair of hyperbolas. We have represented four 4-vectors which have the same pseudo-norm 1, they join the unit hyperbola on one or the other of these two branches. The time-like hyperbolas are indexed by k and the space-like hyperbola by $-k$.

A property of the hyperbola: if we plot a radius OM , the tangent (T) is always symmetrical, with respect to the bisectors, at (OM) . We thus obtain a pair of orthogonal vectors: $\tilde{u} \cdot \tilde{v} = 0$.

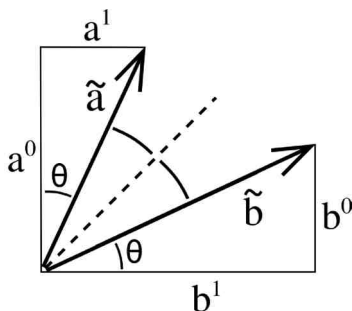
For a set of hyperbolas with the same center O , the same orthogonal axes, and parameters multiple of the unit, a straight line passing through O cuts the hyperbolas into a set of equidistant points.

In **2D**, in Minkowski's plane:

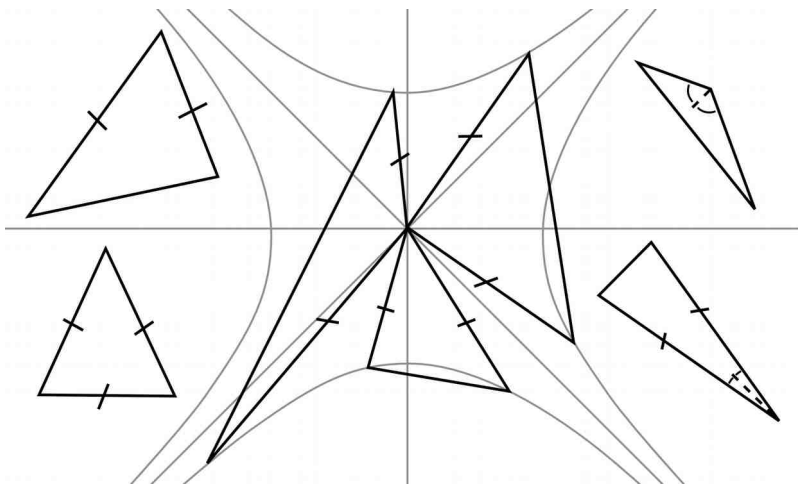
$$\tilde{a} \cdot \tilde{b} = 0 \Rightarrow a^0 b^0 = a^1 b^1$$

Two orthogonal 4-vectors are symmetrical with respect to the photon worldlines:

$$\tan \theta = \frac{a^1}{a^0} = \frac{b^0}{b^1}$$



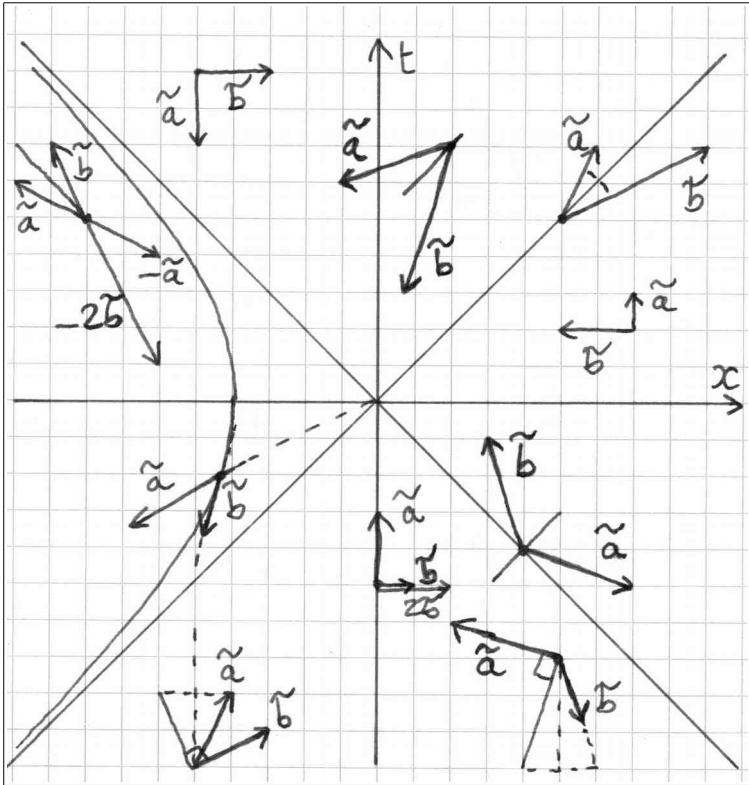
Triangles



Four isosceles triangles, one equilateral triangle, one right-angled triangle and one isosceles right triangle. All these triangles keep their properties by 90° rotation and change of scale.

Examples of 4-vectors orthogonal

For all pairs represented: $\tilde{a} \cdot \tilde{b} = 0$



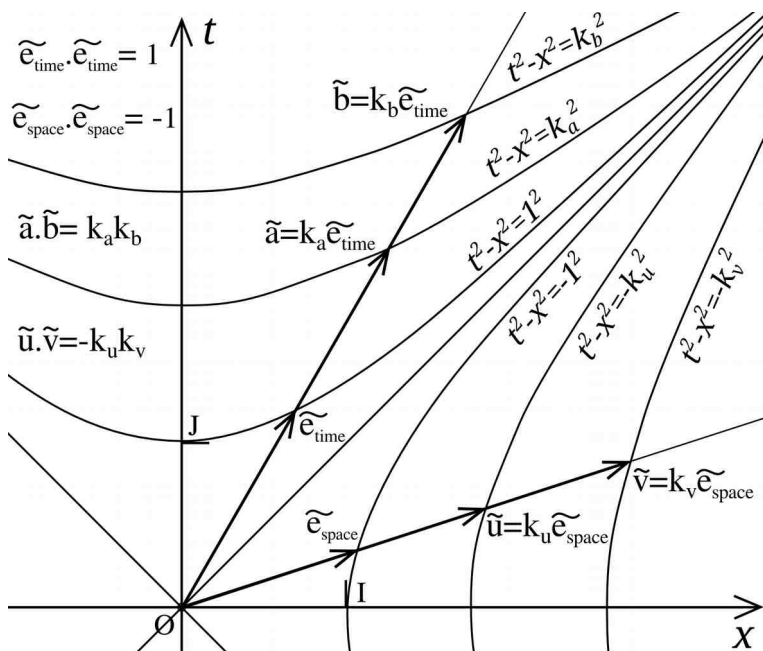
By taking the opposite of one of the vectors of the pair, or by multiplying it by a constant, the pair remains orthogonal.

Geometrical methods:

- Use of the hyperbola.
- Symmetry with respect to the photon worldlines.
- Passage through the Euclidean: two perpendicular vectors and we take the symmetry with respect to the vertical of one of them.

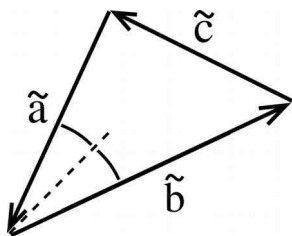
Case of 4-vectors collinear :

Two examples, the pair (\tilde{a}, \tilde{b}) and the pair (\tilde{u}, \tilde{v})



Pythagorean theorem in Minkowski space:

$\tilde{a} + \tilde{b} = \tilde{c}$ with \tilde{a} and \tilde{b} orthogonal.



$$k_a^2 - k_b^2 = \pm k_c^2$$

k : parameter of the
hyperbola /
magnitude / intensity
of the 4-vectors.

◦ Geometric determination of the scalar product

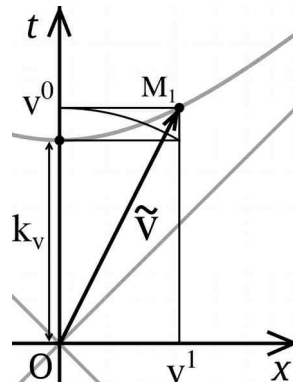
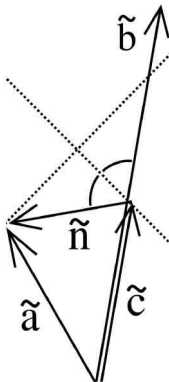
To evaluate $\tilde{a} \cdot \tilde{b}$ in the space of Minkowski:

- We break down one of the two four-vectors as the sum of an orthogonal vector and a collinear vector to the second one.

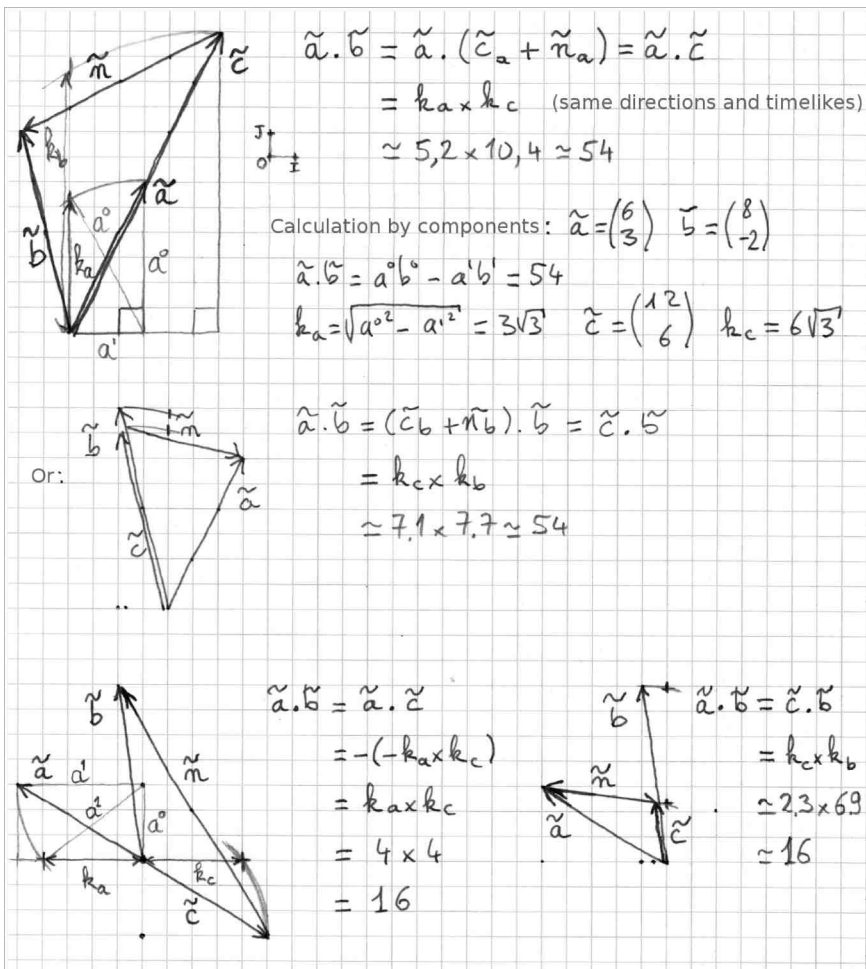
$$\tilde{a} \cdot \tilde{b} = (\tilde{c} + \tilde{n}) \cdot \tilde{b} = \tilde{c} \cdot \tilde{b} + \tilde{n} \cdot \tilde{b}$$

- We determine with a compass the parameters of the hyperbolas of the two collinear vectors obtained.
- The scalar product is the product of the two parameters: $\tilde{a} \cdot \tilde{b} = \tilde{c} \cdot \tilde{b} = \pm k_c \times k_b$.

The sign is positive if the two collinear vectors are timelike and in the same direction, or, if they are spacelike and in opposite directions. In other cases the sign is negative.



**Examples of geometric determination
of the scalar product :**

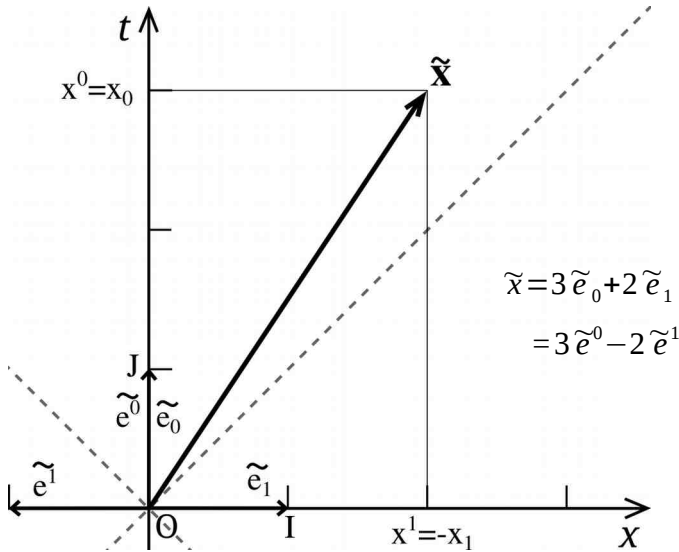


◦ Orthogonal bases

We can always go back to an orthogonal base such as $\tilde{e}_\mu \cdot \tilde{e}_\nu = 0$ for $\mu \neq \nu$.

• Reference frame R

Let's look at the case of the contravariant and covariant components on a Minkowski diagram:



Let's check, on this particular case, the general formulas:

$$g_{\mu\nu} = \tilde{e}_\mu \cdot \tilde{e}_\nu, \quad \tilde{x} = x^\mu \tilde{e}_\mu,$$

$$x_\mu = g_{\mu\nu} x^\nu, \quad \tilde{e}^\mu = g^{\mu\nu} \tilde{e}_\nu \quad \text{and} \quad \tilde{x} = x_\mu \tilde{e}^\mu.$$

We have well, by graphically calculating scalar products : $\tilde{e}_0 \cdot \tilde{e}_1 = (\tilde{e}_0 \cdot s(\tilde{e}_1))_{Euclid} = 0 = g_{10}$.

Also $\tilde{e}_0 \cdot \tilde{e}_0 = \tilde{e}_0 \cdot \tilde{e}_0 = 1$, $\tilde{e}_1 \cdot \tilde{e}_1 = -\tilde{e}_1 \cdot \tilde{e}_1 = -1$ then

$g_{00}=1$ and $g_{11}=-1$. \tilde{e}_0 pseudo-norm worth 1 and \tilde{e}_1 pseudo-norm worth -1.

$$\text{2D metric : } g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For the covariant components :

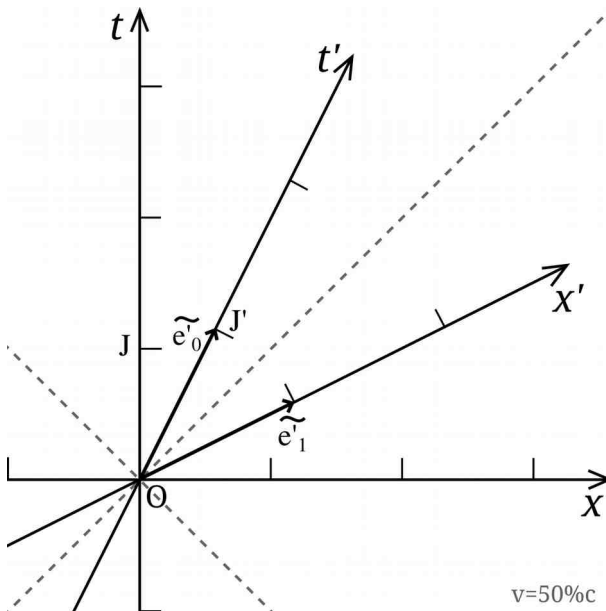
$$x_0 = g_{00}x^0 + g_{01}x^1 = x^0 \quad \text{and} \quad x_1 = g_{10}x^0 + g_{11}x^1 = -x^1$$

$$\tilde{e}^0 = g^{00}\tilde{e}_0 + g^{01}\tilde{e}_1 = \tilde{e}_0 \quad \text{and} \quad \tilde{e}^1 = g^{10}\tilde{e}_0 + g^{11}\tilde{e}_1 = -\tilde{e}_1$$

$$\tilde{x} = x_0\tilde{e}^0 + x_1\tilde{e}^1 = x^0\tilde{e}_0 + x^1\tilde{e}_1 = x_0\tilde{e}_0 - x_1\tilde{e}_1$$

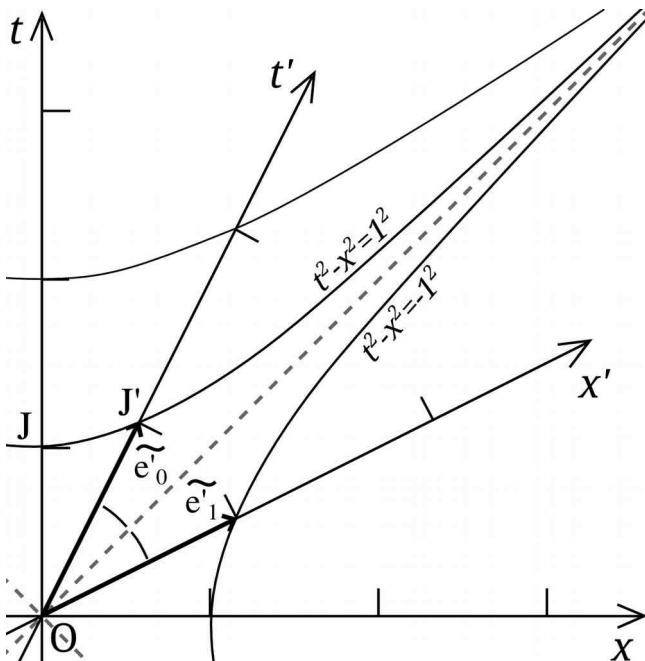
• Reference frame R'

Let's now take the case of the inertial frame R' seen from R :



An unwarned Euclidean glance would naively see a non-orthogonal coordinate system, and, basis vectors longer than one. It is not so, the basis vectors are well orthogonal because they are symmetrical with respect to the worldline of a photon, and, besides, the time vector of the bases of R' is along the unit hyperbola and, therefore, of pseudo-norm 1, the space vector is along the hyperbola corresponding to a pseudo-norm -1. The metric is thus the same as for R , which is to be expected because there is no privileged inertial frame of reference:

$$g'_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$



$$x'_0 = x'^0, \quad x'_1 = -x'^1, \quad \tilde{e}'^0 = \tilde{e}'_0 \quad \text{and} \quad \tilde{e}'^1 = -\tilde{e}'_1.$$

The diagram shows a coordinate system with a vertical t -axis and a horizontal x -axis. A dashed line $t=x$ and a solid line $t'=x'$ intersect at the origin O . A hyperbola $t^2-x^2=2^2$ is centered at O . Points $J(1,1)$, $J'(1,1)$, $H(2,2)$, and $E(3,3)$ are marked. Vectors \tilde{e}_0 and \tilde{e}_1 are shown, and the vector \tilde{x} is the sum of \tilde{e}_0 and \tilde{e}_1 .

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∞ CHANGE OF COORDINATES

We can switch from a system of n coordinates x^i to a new system of n coordinates x'^i , where each of the n coordinates x'^i depend on the n coordinates x^i :

$$x'^i(x^1, \dots, x^2, \dots, x^n)$$

We have a function with n variables. For a function f with two variables, we add the variations in both directions :

$$df(x, y) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

When we move from $M(x, y)$ to $M'(x+dx, y+dy)$, infinitely close, the function f varies by df .

$$\text{The generalization gives : } df(x^i) = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

$$\text{Then } dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \quad \text{and} \quad dx^i = \frac{\partial x^i}{\partial x'^j} dx'^j.$$

$$\text{We note : } \Lambda_j^i = \frac{\partial x'^i}{\partial x^j} \quad \text{and} \quad \Lambda_j^i = \frac{\partial x^i}{\partial x'^j}.$$

These two tensors are used to switch from one coordinate system to the other, they are the change of basis matrices. The superscript indices correspond to the rows and the subscript indices to the columns.

Let's do the product of the two matrices³³ :

$$\Lambda^i_k \Lambda^k_j = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i_j.$$

The matrices are inverse to each other :

$$\Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = I$$

The covariant components of a vector are transformed according to Λ , and the contravariant components according to Λ^{-1} . This is where the famous name comes from. The same is true for the base vectors :

$$v'^i = \Lambda^i_j v^j \quad v^i = \Lambda^j_i v'^j \quad v_i = \Lambda^j_i v'_j \quad v^i = \Lambda^i_j v'^j$$

$$\mathbf{e}'_i = \Lambda^j_i \mathbf{e}_j \quad \mathbf{e}^{'i} = \Lambda^i_j \mathbf{e}^j \quad \mathbf{e}_i = \Lambda^j_i \mathbf{e}'_j \quad \mathbf{e}^i = \Lambda^i_j \mathbf{e}^{'j}$$

We can easily verify that the scalar product of two vectors is invariant by basis change :

$$\mathbf{A} \cdot \mathbf{B} = A_i B^i = \Lambda^j_i A'_j \Lambda^i_k B'^k = \delta^j_k A'_j B'^k = A'_j B'^j$$

Also if two n-vectors are equal, they are still equal after changing the coordinate system:

$$\mathbf{A} = \mathbf{B} \Rightarrow \Lambda^i_k A^k = \Lambda^i_k B^k \Rightarrow A'^i = B'^i \Rightarrow \mathbf{A} = \mathbf{B}$$

33 Some additional mathematical tools :

$$\text{If } x'(x, y) \text{ and } y'(x, y) \text{ then } \frac{\partial x'}{\partial y'} = \frac{\partial x'}{\partial x} \frac{\partial x}{\partial y'} + \frac{\partial x'}{\partial y} \frac{\partial y}{\partial y'}.$$

$$\text{Generalized : } \frac{\partial x'^i}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} \text{ and } \frac{\partial f}{\partial x'^j} = \frac{\partial f}{\partial x^k} \frac{\partial x^k}{\partial x'^j}.$$

Fundamental properties for constructing physical laws, whether in classical mechanics, special relativity or general relativity.

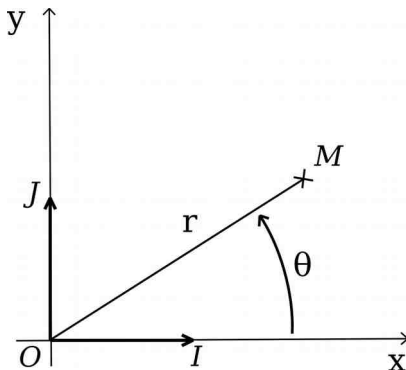
Let's look for the new metric:

$$g'_{ij} = \mathbf{e}'_i \cdot \mathbf{e}'_j = \Lambda_i^k \mathbf{e}_k \cdot \Lambda_j^l \mathbf{e}_l = \Lambda_i^k \Lambda_j^l g_{kl}$$

In general, the change of basis matrix is applied as many times as there are indices on a tensor. For example, on the Riemann curvature tensor :

$$R'^{\alpha}_{\beta\gamma\delta} = \Lambda^{\alpha}_{\mu} \Lambda_{\beta}^{\nu} \Lambda_{\gamma}^{\rho} \Lambda_{\delta}^{\lambda} R^{\mu}_{\nu\rho\lambda}$$

◦ Rotation in Euclidean geometry :



$$\begin{cases} x(r, \theta) = r \cos \theta \\ y(r, \theta) = r \sin \theta \end{cases}$$

$$x'^1 = x(x^1 = r; x^2 = \theta)$$

$$x'^2 = y$$

$$\Lambda^1_1 = \frac{\partial x}{\partial r} = \cos \theta$$

$$\Lambda^1_2 = \frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\Lambda^2_1 = \frac{\partial y}{\partial r} = \sin \theta$$

$$\Lambda^2_2 = \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\text{then } \Lambda = \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}$$

$$\Lambda_1^1 = \frac{\partial r}{\partial x} = \frac{x}{r} = \cos \theta \quad \text{because} \quad r = \sqrt{x^2 + y^2}$$

$$\Lambda_1^2 = \frac{\partial \theta}{\partial x} = -\frac{\sin \theta}{r} \quad \text{as} \quad \frac{\partial \theta}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial \theta}{\partial y} \frac{\partial y}{\partial \theta} = 1$$

$$\Lambda_2^2 = \frac{\partial \theta}{\partial y} = \frac{1/x}{1+y^2/x^2} = \frac{\cos \theta}{r} \quad \Lambda_2^1 = \frac{\partial r}{\partial y} = \sin \theta$$

$$\text{finally : } \Lambda^{-1} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\frac{\sin \theta}{r} & \frac{\cos \theta}{r} \end{pmatrix}$$

$$\text{we well have } \Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = I.$$

$$\mathbf{e}_1 = \vec{e}_r = \Lambda_1^1 \mathbf{e}'_1 + \Lambda_1^2 \mathbf{e}'_2 = \cos \theta \vec{i} + \sin \theta \vec{j}$$

$$\mathbf{e}_2 = \vec{e}_\theta = \Lambda_2^1 \mathbf{e}'_1 + \Lambda_2^2 \mathbf{e}'_2 = -r \sin \theta \vec{i} + r \cos \theta \vec{j}$$

The basis $(\vec{e}_r, \vec{e}_\theta)$ is orthogonal and not normalized.
For an orthonormal basis we have the unit vectors as follows $\vec{e}_r = \vec{u}_r$ and $\vec{e}_\theta = r \vec{u}_\theta$.

$$\text{Metrics : } g'_{ij} = \begin{pmatrix} \vec{i} \cdot \vec{i} & \vec{i} \cdot \vec{j} \\ \vec{j} \cdot \vec{i} & \vec{j} \cdot \vec{j} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{and } g_{ij} = \Lambda^k_i \Lambda^l_j g'_{kl} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\text{for example } g_{22} = \Lambda^1_2 \Lambda^1_2 g'_{11} + \Lambda^2_2 \Lambda^2_2 g'_{22} + 0 + 0 = r^2$$

Invariant length element :

$$dl^2 = \vec{dl} \cdot \vec{dl} = dx^i dx^i = g_{ij} dx^i dx^j = dx^2 + dy^2$$

$$dl^2 = dx_i dx^i = g_{ij} dx^i dx^j = dr^2 + r^2 d\theta^2$$

Vector components : $\vec{v}(v^x, v^y)$

$$v^1 = v^r = \Lambda_1^1 v'^1 + \Lambda_2^1 v'^2 = \cos \theta v^x + \sin \theta v^y$$

$$v^2 = v^\theta = \Lambda_1^2 v'^1 + \Lambda_2^2 v'^2 = -\frac{\sin \theta}{r} v^x + \frac{\cos \theta}{r} v^y$$

we will have $\vec{v} \cdot \vec{v} = g_{ij} v^i v^j = (v^x)^2 + (v^y)^2 = g'_{ij} v'^i v'^j$

◦ **Lorentz transformation :**
$$\begin{cases} ct'(ct, x) = \gamma(ct - \beta x) \\ x'(ct, x) = \gamma(x - \beta ct) \end{cases}$$

$$x'^0 = ct'(x^0 = ct; x^1 = x) \quad x'^1 = x'$$

$$\Lambda^0_0 = \frac{\partial ct'}{\partial ct} = \gamma$$

$$\Lambda^0_1 = \frac{\partial ct'}{\partial x} = -\gamma\beta$$

$$\Lambda^1_0 = \frac{\partial x'}{\partial ct} = -\gamma\beta$$

$$\Lambda^1_1 = \frac{\partial x'}{\partial x} = \gamma$$

$$\text{then } \Lambda = \Lambda^\mu_\nu = \begin{pmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{pmatrix}$$

Inverse standard Lorentz boost :
$$\begin{cases} ct = \gamma(ct' + \beta x') \\ x = \gamma(x' + \beta ct') \end{cases}$$

Then : $\Lambda^{-1} = \Lambda_\nu^\mu = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix}$ and $\Lambda \Lambda^{-1} = \Lambda^{-1} \Lambda = I$.

Basis vectors :

$$\tilde{e}_0 = \tilde{e}_t = \Lambda^0_0 \tilde{e}'_0 + \Lambda^1_0 \tilde{e}'_1 \quad \text{and} \quad \tilde{e}_t = \gamma(\tilde{e}_{t'} - \beta \tilde{e}_{x'})$$

$$\tilde{e}_0 = \tilde{e}_t = \Lambda^0_0 \tilde{e}'_0 + \Lambda^1_0 \tilde{e}'_1 \quad \text{and} \quad \tilde{e}_x = \gamma(-\beta \tilde{e}_{t'} + \tilde{e}_{x'})$$

$$\text{also} \quad \tilde{e}_{t'} = \gamma(\tilde{e}_t + \beta \tilde{e}_x) \quad \text{and} \quad \tilde{e}_{x'} = \gamma(\beta \tilde{e}_t + \tilde{e}_x)$$

For the Minkowski diagrams, we find the results given on page 42 and following. On a Euclidean sheet of paper the vector $\tilde{e}_{t'}$ appears longer than \tilde{e}_t :

$$\|\tilde{e}_{t'}\|_{Euclid} = \gamma \sqrt{1 + \beta^2}.$$

$$\text{Apparent angle : } (\widehat{\tilde{e}_t, \tilde{e}_{t'}})_{Euclid} = \arctan \beta.$$

$$\text{Metrics : } g_{\mu\nu} = \begin{pmatrix} \tilde{e}_t \cdot \tilde{e}_t & \tilde{e}_t \cdot \tilde{e}_x \\ \tilde{e}_x \cdot \tilde{e}_t & \tilde{e}_x \cdot \tilde{e}_x \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{e}_{t'} \cdot \tilde{e}_{t'} = \gamma^2 (\tilde{e}_t \cdot \tilde{e}_t + 2\beta \tilde{e}_t \cdot \tilde{e}_x + \beta^2 \tilde{e}_x \cdot \tilde{e}_x) = 1$$

$$\text{and so on, hence } g'_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu g_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The metric remains the same.

$$\begin{aligned} \text{The invariant } ds^2 : \quad ds^2 &= g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dx^2 \\ &= g'_{\mu\nu} dx'^\mu dx'^\nu = c^2 dt'^2 - dx'^2 \end{aligned}$$

$$\text{Vector components : } \tilde{v}(v^t, v^x)$$

$$v'^0 = v^{t'} = \Lambda^0_0 v^0 + \Lambda^1_0 v^1 = \gamma(v^t - \beta v^x)$$

$$v'^1 = v^{x'} = \Lambda^1_0 v^0 + \Lambda^1_1 v^1 = \gamma(-\beta v^t + v^x)$$

We find the Lorentz transformation that applies to any four-vector.

$$\text{Also : } \tilde{v} \cdot \tilde{v} = g_{\mu\nu} v^\mu v^\nu = (v^t)^2 - (v^x)^2 = (v^{t'})^2 - (v^{x'})^2$$

And the scalar product is well invariant :

$$\begin{aligned} \tilde{u} \cdot \tilde{v} &= g_{\mu\nu} u'^\mu v'^\nu = u'^0 v'^0 - u'^1 v'^1 - u'^2 v'^2 - u'^3 v'^3 \\ &= \gamma^2 (u^0 - \beta u^1)(v^0 - \beta v^1) - \gamma^2 (u^1 - \beta u^0)(v^1 - \beta v^0) - u^2 v^2 - u^3 v^3 \\ &= \gamma^2 (1 - \beta^2) u^0 v^0 + 0 + 0 - \gamma^2 (1 - \beta^2) u^1 v^1 - u^2 v^2 - u^3 v^3 \\ &= u^0 v^0 - u^1 v^1 - u^2 v^2 - u^3 v^3 = g_{\mu\nu} u^\mu v^\nu \end{aligned}$$

For all 4-vectors we have the standard Lorentz transformation :

$$\begin{cases} v^{t'} = \gamma(v^t - \beta v^x) \\ v^{x'} = \gamma(v^x - \beta v^t) \\ v^{y'} = v^y \\ v^{z'} = v^z \end{cases}$$

The change of basis lambda matrices :

$$\Lambda = \Lambda_{\nu}^{\mu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\Lambda^{-1} = \Lambda_{\nu}^{\mu} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

∞ FOUR-VELOCITY

After building a new geometry of space and time, let us build the new physics associated with it. The position vector and universal time have been replaced by the four-vector \tilde{x} . What about the other physical quantities introduced by Newton: velocity, acceleration, momentum, energy, force, etc?

First of all, we are looking for quantities that transform according to Lorentz's transformation, then we will establish laws that give back the classical mechanics at low speeds, and of course, the supreme criterion, the experimental verification will finalize the selection.

We will construct the covariant velocity from the four-vector x^μ . We resume the classical approach which allows to build a vector tangent to the trajectory of an object. For two infinitely close events on a worldline, we have the infinitesimal 4-vector :

$$d\tilde{x} = \widetilde{E E'} = \tilde{x}(E') - \tilde{x}(E).$$

To define the velocity, simply divide by the duration, just as infinitesimal, which separates these two events. Of course, in Newton's mechanics, there is no hesitation to have, on the other hand, in special relativity, we have the duration dt measured in the same frame of reference as the dx^μ , or, the duration $d\tau$ measured in the proper reference frame of the moving object. No hesitation because $d\tau$ is the only

duration invariant by the Lorentz transformation³⁴,
hence the expression of the four-vector velocity :

$$\tilde{u} = \frac{d\tilde{x}}{d\tau} \quad \text{and} \quad u^\mu = \frac{dx^\mu}{d\tau}$$

For the three spatial components, we find well the
classical velocity \vec{v} at low speeds :

$$\tilde{u} = (\gamma c, \gamma \vec{v})$$

$$\text{with } \gamma = \frac{1}{\sqrt{1-\beta^2}}, \quad \beta = \frac{v}{c}, \quad v = \|\vec{v}\|,$$

$$\gamma(v) = \frac{dt}{d\tau}, \quad v^i = \frac{dx^i}{dt} \quad \text{and} \quad \vec{v} = (v^1, v^2, v^3).$$

This four-velocity transforms well according to the
Lorentz transformation given on page 205, which
was not the case for the classical velocity (easy to
convince oneself by looking at the relations on page
362).

For example, along the x axis : $u^x = \frac{dx}{d\tau} = \gamma v^x$.

To think about relativity, it seems logical to reason
with the velocity provided by this same theory, and
not with that of Newton. But as with the notion of
absolute space and absolute time, habits are
tenacious, and it must be noted that Newton's
velocity makes resistance.

34 $d\tau$ is obtained by doing the scalar product of two four-
vectors, it is therefore invariant by the Lorentz transfor-
mation : $d\tilde{x} \cdot d\tilde{x} = g_{\mu\nu} dx^\mu dx^\nu = c^2 dt^2 - dl^2 = c^2 d\tau^2$

"You can't go faster than the speed of light" we hear. Everything would then happen as if there were a forbidden zone from c to infinity. We don't like the prohibitions, and neither does nature, it seems to realize everything that is possible. So, not supporting limits, in this supposedly inaccessible zone, we put strange particles, *tachions*, particles that would always have been faster than light... except that these tachions violate causality, a basic principle in physics.

Let's think differently, let's use the right definition for velocity, the one that respects the symmetries of spacetime. When you give each time more energy to a particle to accelerate it, it gains speed and its velocity tends towards infinity :

$$v_{Newton} = \frac{dx}{dt} \rightarrow c, \quad \gamma \rightarrow \infty \quad \text{and} \quad v_{Einstein} = \frac{dx}{d\tau} \rightarrow \infty.$$

The prohibited zone no longer exists!

Let's take again the example of the journey for Proxima. From the Earth the astronaut travels 4 ly, his journey lasts 3 years, and 5 years for the Earthlings. Sometimes I hear "but he goes faster than light!". He is going well, slower than light, he arrives after a ray of light, and in the ship's frame of reference he has traveled a distance of only 2.4 ly. But it is interesting to note that the person finally refers to the covariant velocity $u = \Delta x / \Delta \tau = 4/3 c$, and, in terms of covariant velocity, that of light is infinite. Finally, we are not so limited as that, at speeds close to c we find

ourselves on the other side of the galaxy very quickly. For example, an ultra-relativistic electron can travel 100,000 ly in one year (in its own frame of reference!).

The temporal component of \tilde{u} is always positive, the four-velocity is always directed towards the future.

Let's calculate the pseudo-norm :

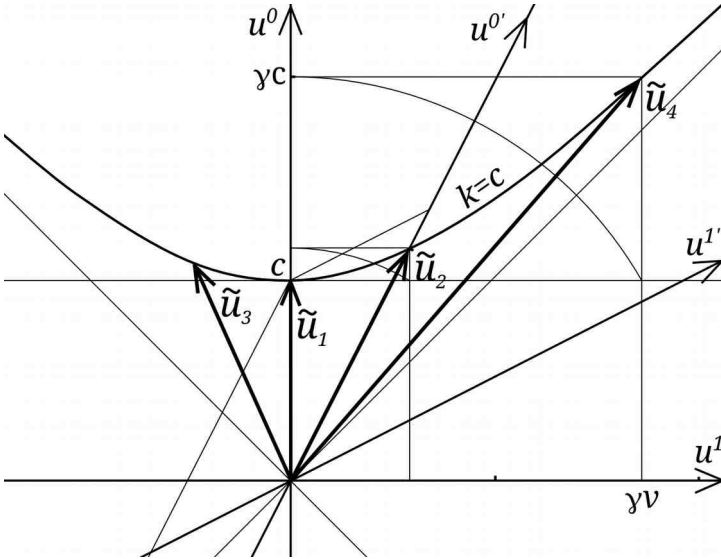
$$\tilde{u} \cdot \tilde{u} = \gamma^2 c^2 - \gamma^2 v^2 = c^2 > 0$$

The 4-velocity is a time-like vector whose end is located on the upper branch of the c parameter hyperbola. The 4-velocity cannot be null. For a particle at rest there is only the time component which corresponds, in a way, to the speed of the flow of time.

$$\text{Particle at rest : } \tilde{u} = (c, \vec{0}).$$

$$\text{Particle in motion : } \tilde{u} = \gamma c (1, \vec{\beta}).$$

Minkowski Diagram for the 4-velocity :



\tilde{u}_1 : relativistic velocity of an object at rest in R .
The vector is vertical.

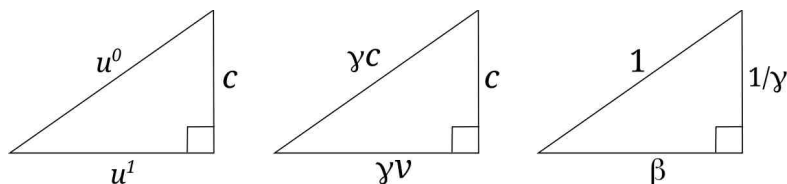
\tilde{u}_2 : 4-velocity of an object moving to the right.
The tip is on the hyperbola of parameter c .
The corresponding gamma is 1.15 and $v=50\%c$.

\tilde{u}_3 : 4- velocity of an object moving to the left.

\tilde{u}_4 : The more gamma increases, the closer the velocity vector gets to the asymptote and the light cone.

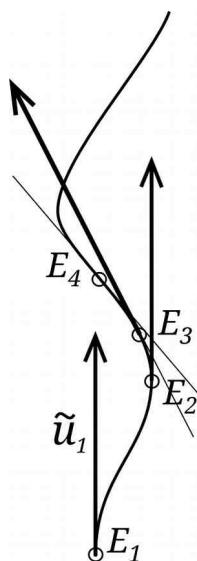
We have built the frame where the particle 2 is motionless. By projecting the tip of \tilde{u}_1 into R' , we obtain a particle 1 that moves to the left at 50 % of c .

The velocity triangle : $\tilde{u} \cdot \tilde{u} = (u^t)^2 - (u^x)^2 = c^2$



(Triangles for $\gamma=2$ and $\beta=\sqrt{3}/2$)

Here is the worldline of a particle. The velocity is always tangent to the worldline and contained in the future light cone. In E_1 the tangent is vertical, the particle is at rest, then it starts moving to the right, slows down and stops further to the right in E_2 . It resumes its motion to the left, accelerates and reaches its maximum speed at the point of inflection in E_4 .



∞ FOUR-ACCELERATION

The approach is of course quite similar:

$$\tilde{w} = \frac{d\tilde{u}}{d\tau} \quad \text{and} \quad w^u = \frac{du^u}{d\tau}$$

As for the 4-velocity, we do not use the classical notations so that the differences appear without ambiguity : \tilde{w} for the 4-acceleration and \vec{a} for the Newton acceleration.

To begin with, we have a nice property, 4-velocity and 4-acceleration are orthogonal vectors :

$$\frac{d}{d\tau}(\tilde{u} \cdot \tilde{u}) = 0 = \frac{d\tilde{u}}{d\tau} \cdot \tilde{u} + \tilde{u} \cdot \frac{d\tilde{u}}{d\tau} \quad \text{then} \quad \tilde{u} \cdot \tilde{w} = 0.$$

As we have established the link between \tilde{u} and \vec{v} , we are going to make the link between \tilde{w} and \vec{a} . There, however, the link will be much less immediate and the calculations are longer:

$$\tilde{w} = \frac{d\tilde{u}}{d\tau} = \left(\frac{d\gamma}{d\tau} c, \frac{d\gamma}{d\tau} \vec{v} + \gamma \frac{d\vec{v}}{d\tau} \right)$$

after calculation $\frac{d\gamma}{dt} = \frac{\gamma^3}{c^2} \vec{a} \cdot \vec{v}$ with $\vec{\beta} = \frac{\vec{v}}{c}$

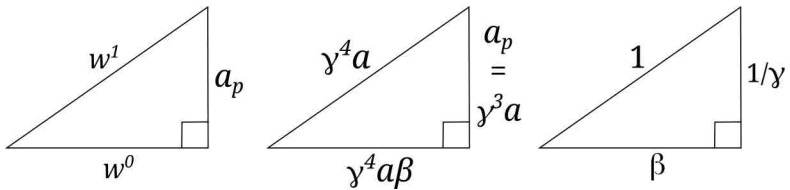
we have $\tilde{w} = (\gamma^4 \vec{a} \cdot \vec{\beta}, \gamma^4 (\vec{a} \cdot \vec{\beta}) \vec{\beta} + \gamma^2 \vec{a})$

Now let's determine the pseudo-norm of \tilde{w} . The scalar product is the same in all inertial frames of reference. We then place ourselves in the inertial frame of reference which coincides at a given

moment with the proper frame of reference. In this coinciding reference frame, by definition, $\vec{v}=\vec{0}$ at $t=0$. Thus $\tilde{w}=(0, \vec{a}(0))$ and $\tilde{w} \cdot \tilde{w} = -a_p^2$, where a_p is the acceleration felt in the proper frame of reference. All inertial observers will agree on the value of the proper acceleration a_p . The 4-acceleration is a space-like vector, in accordance with the orthogonality with the 4-velocity.

In the Minkowski plane $(w^0)^2 - (w^1)^2 = -a_p^2$ and \tilde{w} is placed on a space-like hyperbola of parameter a_p .

The acceleration triangle:



For one-dimensional motion : $\tilde{w} = \gamma a_p (\pm \beta, \pm 1)$

Generally speaking, one can always place oneself locally in an inertial reference frame that contains the worldline in a Minkowski plane coinciding on a portion. We then have an osculating hyperbola that allows us to determine the proper acceleration.

◦ A look back on the trip to Proxima

We are on a particular case of rectilinear motion at constant proper acceleration, where the worldline of the rocket corresponds with the hyperbola of parameter g .

We will elegantly retrieve the expressions of the page 116.

In the coinciding inertial reference frame $\tilde{w}=(0, g)$.

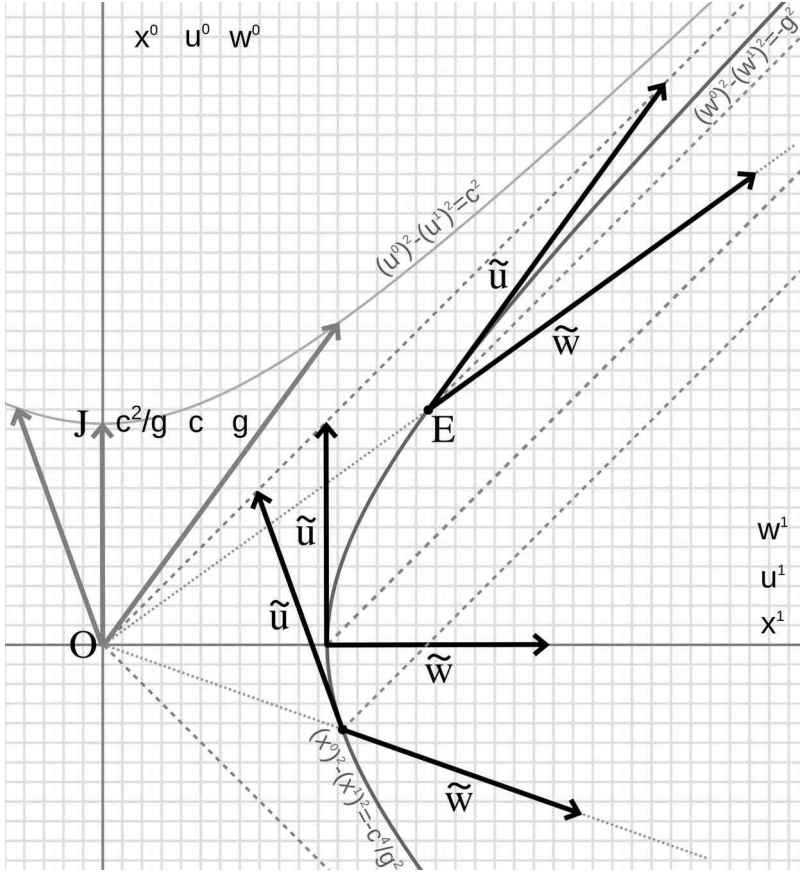
We perform a Lorentz transformation to obtain the coordinates of this same acceleration in the terrestrial frame of reference:

$$\tilde{w}=(\gamma\beta g, \gamma g),$$

$$\text{as } \gamma\beta g = \gamma^4 \vec{a} \cdot \vec{\beta} \quad \text{we have} \quad a(t) = \frac{dv}{dt} = \frac{g}{\gamma^3}$$

after integration we find the expressions for $v(t)$ and $x(t)$.

Voyage to Proxima :



We have represented the Minkowski diagrams for the three four-vectors \tilde{x} , \tilde{u} and \tilde{w} . We have made an appropriate choice of units so that the hyperbolas correspond: OJ worth c^2/g for the 4-position, c for the 4-velocity and g for the 4-acceleration. We study the uniformly accelerated motion in its generality, both for positive and negative t : in the latter case \vec{v} and \vec{a} are in opposite directions, the rocket decelerates, and $\tilde{w} = (-\gamma \beta g, \gamma g)$. For this motion, the rocket worldline is a

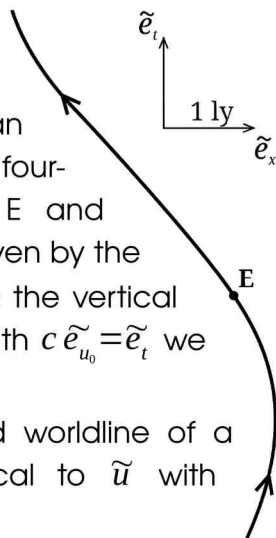
hyperbole branch of equation $c^2 t^2 - x^2 = -c^4 / g^2$ which coincides here with the space-like hyperbole branch of \tilde{w} . The hyperbola branch of \tilde{u} is simply rotated by 90° . For any event E of our worldline, \tilde{u} and \tilde{w} are as it should be symmetrical with respect to the bisectors, but, in this particular situation, they appear, moreover, of the same length on our Euclidean sheet. Indeed we have in this case $\tilde{u}/c = \gamma(1, \beta)$ and $\tilde{w}/g = \gamma(\pm\beta, 1)$. The drawing is very simple, for any event E, you draw the line (OE), \tilde{w} corresponds with \overline{OE} , and \tilde{u} is the symmetrical with respect to the photon worldline. Although the 4-acceleration remains constantly on the spacelike hyperbola of parameter g, on the diagram, the Euclid's length of the relativistic acceleration \tilde{w} increases with γ , while that of the classical acceleration \tilde{a} decreases in γ^3 .

◦ Geometric determination of 4-acceleration

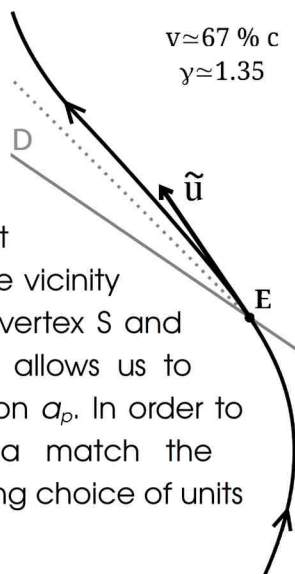
• From the worldline :

All the information is available in this line. For any E event we can determine the four-velocity and four-acceleration. \tilde{u} is tangent at E and directed towards the future. β is given by the arctangent of the angle between the vertical and \tilde{u} . By adapting the scales with $c \tilde{e}_{u_0} = \tilde{e}_t$ we can carry out the plot.

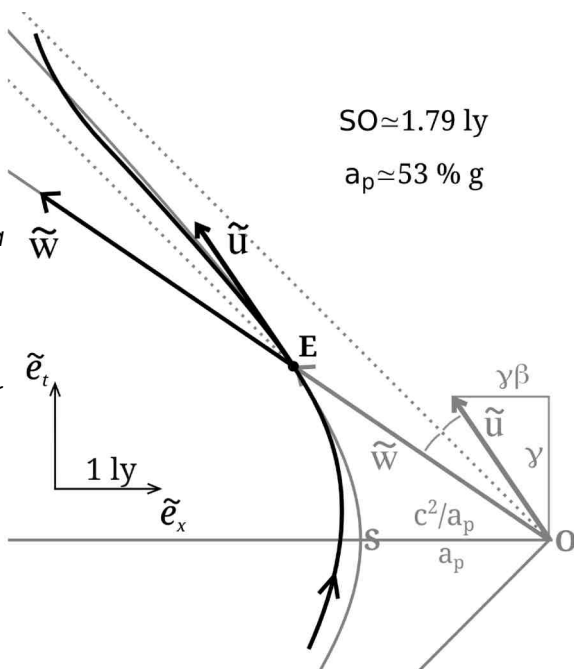
We then placed in E the dotted worldline of a photon. The line D is symmetrical to \tilde{u} with respect to the photon.



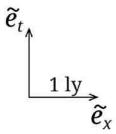
As \tilde{w} is orthogonal to \tilde{u} its end is necessary on D. As the worldline continues below \tilde{u} , there is acceleration and \tilde{w} is upwards. We place the osculating hyperbola that best coincides with the worldline in the vicinity of E. The distance between the vertex S and the center O of the hyperbola allows us to determine the proper acceleration a_p . In order to make the osculating hyperbola match the acceleration we have the following choice of units $a_p \tilde{e}_{w_1} = c^2 / a_p \tilde{e}_x = \tilde{SO}$.



For any event of a world line, there is always a tangent hyperbola unique that gives the proper acceleration.



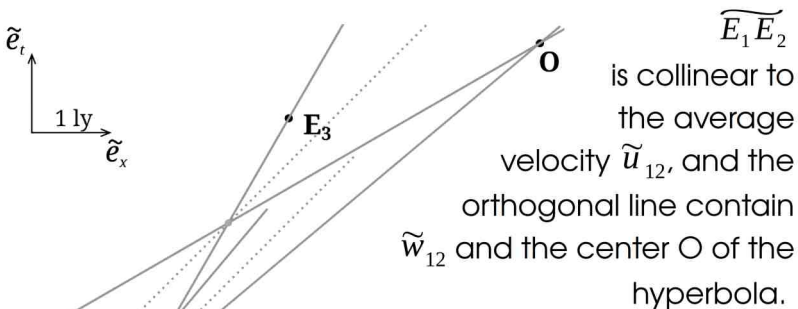
• From three close events :



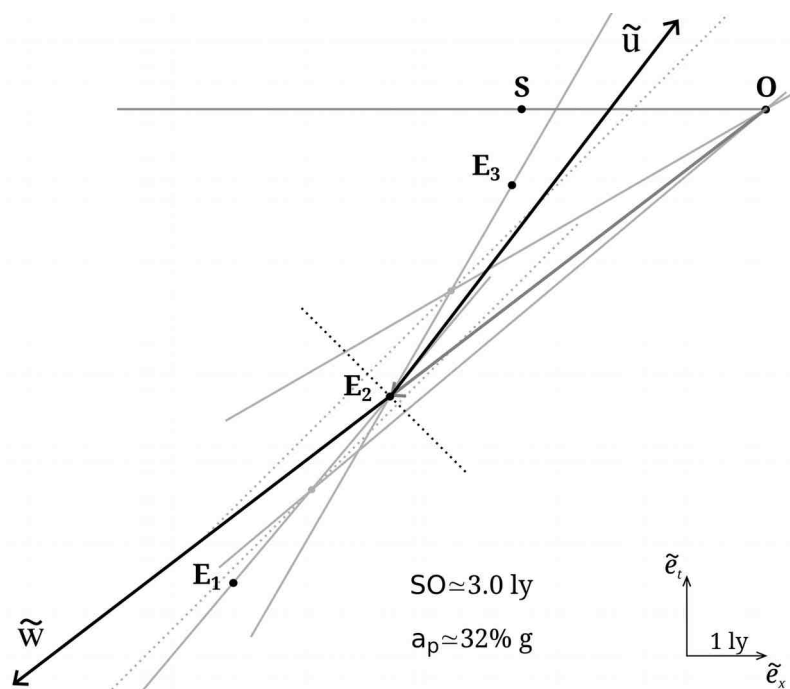
• E_2

• E_1

Previously, it was not easy to determine the tangent hyperbole. Here, from three events we will find the osculating hyperbola optimized for the midpoint E_2 . We know that three points determine a single hyperbola. The approach is the same in Euclidean geometry, if you have three points of a circle you find the osculating circle using two perpendicular bisectors whose intersection provides the center of the circle. Here again the tangents are orthogonal to the radii.



We proceed in the same way with the pair of events (E_2, E_3) . The intersection of the two orthogonal lines gives the center. We then check that the pseudo-norms of $\widetilde{OE_1}$, $\widetilde{OE_2}$ and $\widetilde{OE_3}$ are equal. We then have \widetilde{OS} , the parameter k_x of the hyperbola, and the proper acceleration $a_p = -c^2 / \widetilde{SO}$.



∞ MASS-ENERGY EQUIVALENCE

Let us look for the relativistic equivalent of the Newton's second law. In classical mechanics :

$$m \vec{a} = \vec{F} \quad \text{or} \quad \frac{d\vec{p}}{dt} = \vec{F}$$

with the momentum $\vec{p} = m \vec{v}$

We will also need the kinetic power theorem :

$$P_k = \frac{dE_k}{dt} = \vec{F} \cdot \vec{v}$$

◦ Four-momentum

The mass is a property specific to a particle, it does not depend on the frame of reference. It thus seems natural to consider the four-vector $\tilde{p} = m \tilde{u}$.

For the 4-momentum we keep the letter p because contrary to the 4-velocity or the 4-acceleration, this one has been directly adopted in the scientific mores. Its spatial part is commonly called *momentum* and the 4-vector as a whole can be called the 4-momentum or more precisely the 4-vector energy-momentum: $\tilde{p} = (m \gamma c, m \gamma \vec{v})$.

$$\boxed{\tilde{p} = (E/c, \vec{p})} \quad \text{with} \quad \boxed{E = m \gamma c^2} \quad \text{and} \quad \boxed{\vec{p} = m \gamma \vec{v}}$$

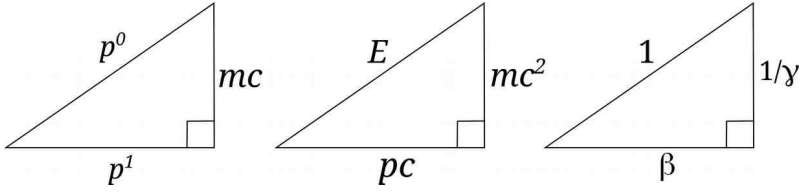
The temporal component shows a quantity with the units of an energy. Let's find out what this energy corresponds to. In the coinciding reference frame $\tilde{p} = (mc, \vec{0})$ and $\tilde{p} \cdot \tilde{p} = m^2 c^2$. In the observational frame $\tilde{p} \cdot \tilde{p} = E^2/c^2 - \vec{p}^2$. In the proper frame, where the particle is at rest, $\tilde{p} \cdot \tilde{p} = E_r^2/c^2$, then $E_r = mc^2$.

A completely new notion, absent in classical mechanics, appears, an energy is associated with the mass of an object. Even at rest, a particle has an energy, it is an energy of mass.

When the particle is in motion :

$$m^2 c^2 = E^2/c^2 - \vec{p}^2 \quad \text{and} \quad E^2 = (mc^2)^2 + (pc)^2.$$

The Energy-Momentum Triangle :



E corresponds to the total energy of the particle, which includes its mass energy and its kinetic energy:

$$E^2 = m^2 c^4 + p^2 c^2 = m^2 c^4 + m^2 \gamma^2 v^2 c^2 = m^2 \gamma^2 c^4$$

$$\text{and we find: } E = m \gamma c^2$$

For the kinetic energy: $E_k = E - E_r$.

At low speeds:

$$E = m(1 - \beta^2)^{-1/2} c^2 \simeq mc^2 + \frac{1}{2} m v^2$$

We find again the classical expression of kinetic energy.

For a massless particle, like a photon, $E = pc$,
 $\tilde{p} = (p, \vec{p})$ and $\tilde{p} \cdot \tilde{p} = 0$.

◦ **Four-force**

For the 4-force \tilde{g} we suggest :

$$\frac{d\tilde{p}}{d\tau} = \tilde{g}$$

Equation covariant with respect to the Lorentz transformation. In the classical limit, the temporal

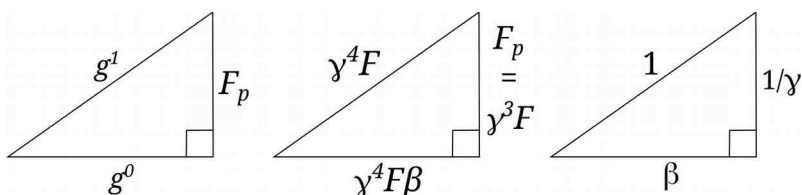
part gives back the kinetic power theorem, and the spatial part gives the Newton's second law:

$$\frac{d\vec{p}}{d\tau} = m\vec{w} = (\gamma^4 \vec{F} \cdot \vec{\beta}, \gamma^4 (\vec{F} \cdot \vec{\beta}) \vec{\beta} + \gamma^2 \vec{F}) = \vec{g}$$

The link between 4-force and Newton's force is not obvious. Classically, the force \vec{F} is collinear and has the same direction as acceleration \vec{a} , in relativity it is the case for \vec{g} and \vec{w} .

Pseudo-norm : $\vec{g} \cdot \vec{g} = -F_p^2$ with $F_p = ma_p$.

Force Triangle :



For one-dimensional motion : $\vec{g} = \gamma F_p (\pm \beta, \pm 1)$.

For the spatial part : $\frac{d\vec{p}}{d\tau} = \vec{g}$ and $\frac{d\vec{p}}{dt} = \frac{\vec{g}}{\gamma}$.

We have the spatial part \vec{g} of the 4-force, and on the other hand the classical force \vec{F} , the Newton's second law then takes the following form:

$$\frac{d\vec{p}}{dt} = \frac{\vec{g}}{\gamma} = \gamma^3(\vec{F} \cdot \vec{\beta})\vec{\beta} + \gamma\vec{F} = \vec{f}$$

The relationship between \vec{g} and \vec{F} is not simple and we find that they are not collinear. Within the limit of low speeds, we find Newton's second law $m\vec{a} = \vec{F}$.

Most often, to build relativity, the third force \vec{f} is used. When one injects, in Newton's law, the relativistic momentum instead of the classical one, it is the force that appears. This force \vec{f} is commonly used as an equivalent of the classical force at the relativistic level. This standard force has a definition similar to that of classical mechanics, but it is not the spatial part of a covariant four-vector.

In Newtonian mechanics the force is independent of the inertial frame of reference $\vec{F}' = \vec{F}$, in relativity it is also the case for the four-force $\vec{g}' = \vec{g}$. On the other hand, we have in general $\vec{f}' \neq \vec{f}$ and $\vec{g}' \neq \vec{g}$.

◦ Conservation of momentum and energy

For an *isolated* system, $\tilde{g} = \tilde{0}$ and the momentum-energy four-vector is constant. For a set of particles, the total momentum is the sum of the individual momenta, and the same applies to the energy :

$$\tilde{p} = \sum \tilde{p}_i, \quad E = \sum E_i \quad \text{and} \quad \vec{p} = \sum \vec{p}_i$$

These quantities are then conserved:

$$\tilde{p} = \tilde{cst}, \quad E = cste \quad \text{and} \quad \vec{p} = \vec{cst}$$

For example, during a collision, the particles may change in nature and number, but whatever happens there will always be conservation of these three quantities: they will have the same values before and after the impact. We can consider an isolated system in three situations: no force is exerted on the system, the sum of the forces is zero, or, as in a collision, the interaction being very brief, the 4-momentum of the system has no time to vary significantly. The forces internal to the system do not intervene in these balances.

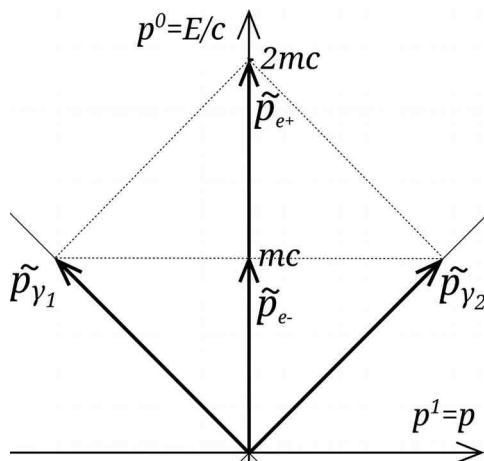
• Annihilation of an electron with a positron

Two gamma photons are produced :

$$e^- + e^+ \rightarrow 2\gamma \quad \text{with} \quad \tilde{p}_{e^-} + \tilde{p}_{e^+} = \tilde{p}_{\gamma_1} + \tilde{p}_{\gamma_2}$$

We take the case where the electron and the positron have the same velocities (opposite directions). In the frame of reference where the

particle and the antiparticle are at rest, we have the following Minkowski diagram of momentums-energies:



We have at least two photons produced by annihilation. It is not possible that only one photon is produced because a photon cannot be at rest and its momentum cannot be annulled to respect the conservation of the momentum in the considered frame of reference. If two photons are created, they necessarily have the same energy and they go in opposite directions. The energy of a photon corresponds to the mass energy of an electron (or what is the same of a positron). Photons thus have energies of 511 keV. They are very energetic photons, as a comparison the visible photons have an energy of the order of eV.

We study in exercise the collision of two protons with the creation at the threshold of a proton-antiproton pair.

Summary

Quantity	Classical Physics	Links / Standards	Special Relativity
position	$\vec{r} = (x, y, z)$		$\tilde{x} = (ct, \vec{r})$ $\tilde{x} \cdot \tilde{x} = c^2 \tau^2$
velocity	$\vec{v} = \frac{d\vec{r}}{dt}$	$\vec{u} = \gamma \vec{v}$ $\gamma = \frac{dt}{d\tau}$	$\tilde{u} = \frac{d\tilde{x}}{d\tau}$ $\tilde{u} = (\gamma c, \vec{u})$ $\tilde{u} \cdot \tilde{u} = c^2$
momentum	$\vec{p} = m \vec{v}$	$\vec{p} = m \gamma \vec{v}$	$\tilde{p} = m \tilde{u}$ $\tilde{p} = (E/c, \vec{p})$ $\vec{p} = m \vec{u}$
acceleration	$\vec{a} = \frac{d\vec{v}}{dt}$	$w^0 = \gamma^4 \vec{a} \cdot \vec{\beta}$ $\vec{w} = \gamma^4 (\vec{a} \cdot \vec{\beta}) \vec{\beta} + \gamma^2 \vec{a}$	$\tilde{w} = \frac{d\tilde{u}}{d\tau}$ $\tilde{w} = (w^0, \vec{w})$ $\tilde{w} \cdot \tilde{w} = -a_p^2$ $\tilde{u} \cdot \tilde{w} = 0$
force	$\vec{F} = m \vec{a}$	$\vec{f} = \frac{d\vec{p}}{dt}$ $\vec{g} = \gamma \vec{f}$	$\vec{g} = \frac{d\vec{p}}{d\tau}$ $\vec{g} = m \tilde{w}$ $\vec{g} = (g^0, \vec{g})$
energy	$\frac{dE_k}{dt} = \vec{F} \cdot \vec{v}$ $E_k = \frac{1}{2} m v^2$	$\frac{dE}{dt} = \vec{f} \cdot \vec{v}$	$\gamma \frac{dE}{d\tau} = \vec{g} \cdot \tilde{u}$ $E = \gamma m c^2$ $E_k = E - m c^2$

electro -magnetic field	$\vec{F}_E = q \vec{E}$ $\vec{F}_B = q \vec{v} \wedge \vec{B}$	Lorentz force : $\vec{f} =$ $q(\vec{E} + \vec{v} \wedge \vec{B})$ $\vec{g} = \gamma \vec{f}$	$\tilde{g} = F \tilde{j}$ $\tilde{j} = q \tilde{u}$ $\vec{g} =$ $q(\gamma \vec{E} + \vec{u} \wedge \vec{B})$
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The standard definition \vec{f} for force is widely used by the scientific community, which summarizes relativity in a few equations:

$$\vec{p} = m \gamma \vec{v} \quad \vec{f} = \frac{d\vec{p}}{dt} \quad \vec{f}_L = q(\vec{E} + \vec{v} \wedge \vec{B})$$

$$\frac{dE}{dt} = \vec{f} \cdot \vec{v} \quad E = \gamma m c^2 = T + m c^2.$$

Taught directly in this way it is fast and effective, but at the same time, if the student wants to deepen the concepts it will be necessary for him to enlarge his view in order to have a clear vision and avoid confusion. Moreover, in our book we put forward a geometrical perspective which is mainly based on the approach of Hermann Minkowski. These are of course the covariant quantities that are naturally represented in a diagram and are simply transformed with the Lorentz boost.

For the electromagnetic field, the quantities are detailed in exercise on page 252.

∞ NON-INERTIAL REFERENCE FRAMES

As we know how to do in Newtonian mechanics, we must also learn to apply special relativity in non-inertial frames.

Let us recall the approach in classical mechanics. Newton's laws are verified in Galilean frames and by a change of frame of reference we find their new expressions in any moving frame:

$$m \vec{a}_r = \vec{F} + \vec{F}_{ie} + \vec{F}_{ic}$$

Everything happens as if we had new forces, called inertial or fictitious. One may wonder if these forces really exist. Indeed, these forces are not related to fundamental interactions but to the change in the frame of reference. Nevertheless, the driver and passengers of a car experience these different dynamic effects as real during the acceleration phases, such as a sudden start, more or less tight bends and braking strokes.

Classical mechanics give an interpretation of these effects in terms of forces: coincident forces and Coriolis forces.

It goes without saying that special relativity must allow all these effects to be found. At low speeds, they must be equivalent. We will have new effects that will appear with increasing speed. But also at low speeds, for precise measurements and for the behavior of light which is now included in the

theoretical framework. The interpretation is however very different.

In special relativity, there are no inertial forces but *metric effects*. By a non-inertial change of frame, we deviate from the Minkowskian metric and a free particle follows a geodesic which modifies its initially rectilinear and uniform motion to follow a curved and accelerated trajectory.

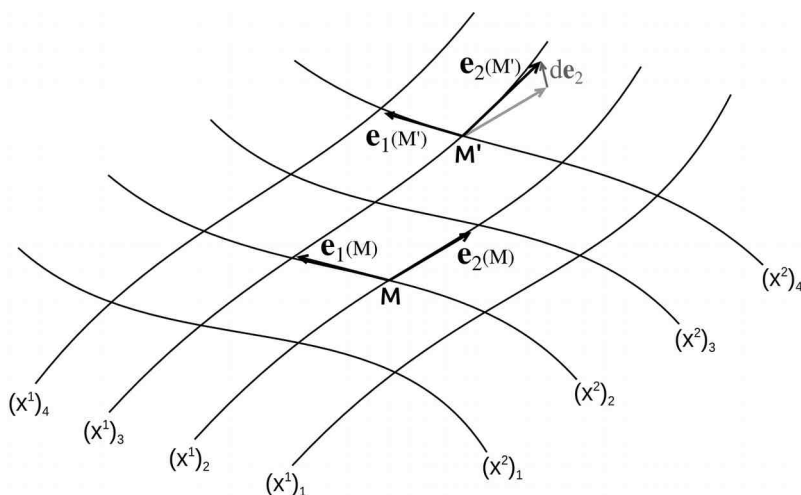
For example, when the car accelerates at a green traffic light, it is not an inertial force that puts you against the seat, but a metric modification that puts you in free fall towards the back of the car (as in the uniformly accelerated rocket). At the same time, the watches of the passengers in the back of the car are slow with respect to those in the front. Quite the opposite when you brake, the metric modification makes you plunge in free fall towards the windshield. In a turn, the metric change causes you to fall towards the outside of the bend, the watches will also go out of sync and Euclid's postulates will no longer be verified.

In special relativity, the notion of inertial force is replaced by that of metric effect. We have previously studied the two particular cases of the uniformly accelerated reference frame and the uniformly rotating frame and we will now focus on the general case³⁵.

35 Here we make the analogy between classical mechanics and special relativity, but historically we are rather used to the analogy made with general relativity. In this analogical framework, during a brake

◦ Coordinate lines, local basis and connections

Here we complete our description of a vector space. These are very general mathematical concepts that can be used in all scientific fields.



Coordinate lines are obtained when one coordinate varies and all others are fixed.

At each point of this network we have a local basis with the basis vectors tangent to the lines. When we go from M to M' infinitely close, we have a small variation of the basis vectors :

stroke, we say that everything happens as if a gravitational field was pulling you forward. This gravitational field is of course fictitious. If it were real, at the same time as you brake, a gigantic massive wall of infinite size would have to appear in front of the car to justify such a gravitational field! In general relativity, the gravitational field creates an additional metric effect, spacetime is then curved, and the gravitational field is very real (it exists in all observation frames of reference).

$$d\mathbf{e}_i = \frac{\partial \mathbf{e}_i}{\partial x^j} dx^j = \Gamma_{ij}^k \mathbf{e}_k dx^j$$

This variation can be projected on the starting basis. The quantities Γ_{ij}^k allow to encode the variation of the local basis at this point. We will call *connection* the object Γ_{ij}^k . For a global basis, which does not depend on the point, all the components of the connection are null.

The connection is symmetrical on the last two indices:

$$\Gamma_{ij}^k \mathbf{e}_k = \frac{\partial \mathbf{e}_i}{\partial x^j} = \frac{\partial}{\partial x^j} \left(\frac{\partial \mathbf{M} \mathbf{M}'}{\partial x^i} \right) = \frac{\partial^2 \mathbf{M} \mathbf{M}'}{\partial x^j \partial x^i} = \frac{\partial^2 \mathbf{M} \mathbf{M}'}{\partial x^i \partial x^j} = \Gamma_{ji}^k \mathbf{e}_k$$

The metric contains all the information about space. We can establish the expression of the connection coefficients according to the metric:

$$g_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j \quad dg_{ij} = \partial_k g_{ij} dx^k = (d\mathbf{e}_i) \cdot \mathbf{e}_j + \mathbf{e}_i \cdot (d\mathbf{e}_j)$$

$$g_{ij,k} dx^k = (\Gamma_{ir}^l \mathbf{e}_l dx^r) \cdot \mathbf{e}_j + \mathbf{e}_i \cdot (\Gamma_{jn}^m \mathbf{e}_m dx^n)$$

$$g_{ij,k} = g_{lj} \Gamma_{ik}^l + g_{im} \Gamma_{jk}^m$$

$$g_{ij,k} + g_{ki,j} - g_{jk,i}$$

$$= g_{lj} \Gamma_{ik}^l + g_{im} \Gamma_{jk}^m + g_{li} \Gamma_{kj}^l + g_{km} \Gamma_{ij}^m - g_{lk} \Gamma_{ji}^l - g_{jm} \Gamma_{ki}^m$$

$$g_{ij,k} + g_{ki,j} - g_{jk,i} = 2g_{im} \Gamma_{jk}^m$$

$$g^{ni} (g_{ij,k} + g_{ki,j} - g_{jk,i}) = 2g^{ni} g_{im} \Gamma_{jk}^m$$

$$\text{Finally : } \Gamma_{jk}^i = \frac{1}{2} g^{il} (g_{lj,k} + g_{kl,j} - g_{jk,l})$$

◦ Covariant derivative

Variation of a vector \mathbf{A} when moving from M to M' : $d\mathbf{A} = \mathbf{A}(M') - \mathbf{A}(M)$. In the Minkowski basis, or in a Cartesian basis, we are in particular cases where the basis is global, the basis does not depend on the point and only the variations on the components are to be taken into account.

In the general case: $d\mathbf{A} = d(A^i \mathbf{e}_i) = d(A^i) \mathbf{e}_i + A^i d\mathbf{e}_i$.

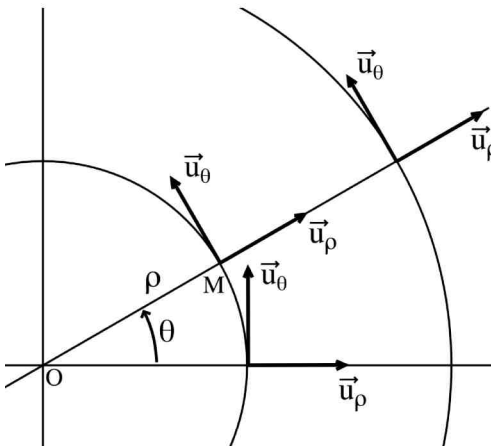
$$d\mathbf{A} = \partial_j A^i dx^j \mathbf{e}_i + \Gamma_{ij}^k A^i dx^j \mathbf{e}_k = (\partial_j A^i + \Gamma_{kj}^i A^k) dx^j \mathbf{e}_i$$

Notations: $D_j A^i = A^i_{;j} = \partial_j A^i + \Gamma_{kj}^i A^k$, $DA^i = A^i_{;j} dx^j$

The capital D makes it clear that all variations have been taken into account. For inertial frames of reference, the connections are null in the Minkowski basis, and ∂_μ was our covariant derivative. In non-inertial frames D_μ is the covariant derivative.

◦ Illustration on an example

Let's take the case of polar coordinates.



The basis depends on the point.

It rotates with the angle θ and remains unchanged when ρ varies:

$$\vec{u}_\rho(\theta) \text{ and } \vec{u}_\theta(\theta)$$

In classical mechanics, we

usually take unit vectors.

$$\text{Basis Variations : } \frac{d\vec{u}_\rho}{d\theta} = \vec{u}_\theta \quad \text{and} \quad \frac{d\vec{u}_\theta}{d\theta} = -\vec{u}_\rho.$$

$$\text{Then : } \vec{OM} = \vec{r} = \rho \vec{u}_\rho \quad \text{gives} \quad \vec{v} = \dot{\rho} \vec{u}_\rho + \rho \dot{\theta} \vec{u}_\theta \quad \text{and}$$

$$\vec{a} = (\ddot{\rho} - \rho \dot{\theta}^2) \vec{u}_\rho + (\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta}) \vec{u}_\theta$$

We can retrieve this result with the metric and the connections :

$$ds^2 = g_{ij} dx^i dx^j = dl^2 = d\rho^2 + \rho^2 d\theta^2 \quad dl^2/dt^2 = g_{ij} v^i v^j$$

$$\vec{e}_\rho = \vec{u}_\rho \quad \vec{e}_\theta = \rho \vec{u}_\theta \quad \vec{OM} = \rho \vec{e}_\rho + \theta \vec{e}_\theta \quad \vec{v} = \frac{d\vec{l}}{dt} = (\dot{\rho}, \dot{\theta})$$

$$g_{22,1} = 2\rho \quad \Gamma^1_{11} = 0 \quad \Gamma^2_{22} = 0 \quad \Gamma^2_{11} = 0$$

$$\Gamma^1_{22} = -\frac{1}{2} g^{11} g_{22,1} = -\rho \quad \Gamma^2_{12} = \frac{1}{2} g^{22} g_{22,1} = \frac{1}{\rho} \quad \Gamma^1_{12} = 0$$

$$d\vec{v} = (\partial_j v^i + \Gamma^i_{kj} v^k) dx^j \vec{e}_i$$

$$\vec{a} = (\partial_j v^i + \Gamma^i_{kj} v^k) \dot{x}^j \vec{e}_i = \partial_t v^i \vec{e}_i + \Gamma^i_{kj} v^k \dot{x}^j \vec{e}_i$$

$$\vec{a} = \dot{v}^1 \vec{e}_1 + \dot{v}^2 \vec{e}_2 + \Gamma^1_{22} v^2 \dot{x}^2 \vec{e}_1 + \Gamma^2_{12} v^1 \dot{x}^2 \vec{e}_2 + \Gamma^2_{21} v^2 \dot{x}^1 \vec{e}_2$$

$$\text{then } \vec{a} = \ddot{\rho} \vec{e}_\rho + \ddot{\theta} \vec{e}_\theta - \rho \dot{\theta} \dot{\theta} \vec{e}_\rho + \frac{1}{\rho} \dot{\rho} \dot{\theta} \vec{e}_\theta + \frac{1}{\rho} \dot{\theta} \dot{\rho} \vec{e}_\theta.$$

We have a new method that uses the metric to account for local basis variations using connections.

• Geodesics

Geodesics are the worldlines followed by free particles. These curves, the equivalent of Euclid's straight lines, maximize proper time.

On a geodesic, the proper acceleration is zero.

Let us take up again the building of special relativity for non-inertial frames of reference:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = g_{\mu\nu} u^\mu u^\nu d\tau^2, \quad u^\mu = \frac{dx^\mu}{d\tau} \quad \text{and} \quad p^\mu = mu^\mu.$$

With the covariant derivative, we can generalize the Newton's second law :

$$\frac{d\vec{p}}{dt} = \vec{F} \quad \text{and} \quad \vec{a} = \frac{\vec{F}}{m} - \vec{a}_e - \vec{a}_c \quad \text{becomes} \quad \frac{D\vec{p}}{D\tau} = \vec{g}$$

$$\text{Equations of Motion:} \quad \frac{du^\mu}{d\tau} = \frac{g^\mu}{m} - \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta.$$

$$\text{For the geodesics equation:} \quad g^\mu = 0.$$

The metric effects, equivalent to the classical forces of inertia, are expressed through the connections, which themselves reflect the variations of the metric in a non-inertial frame.

$$\text{In classical mechanics:} \quad \frac{dv^i}{dt} = \frac{F^i}{m} - \Gamma^i_{jk} v^j v^k.$$

• Classical limit

In the classical case we already noticed that the mass of the particle did not play a role: $\vec{a} = -\vec{a}_e - \vec{a}_c$. For the calculation of the acceleration \vec{a} from the velocity \vec{v} , we have two kinds of terms, those which involve the variation of the coordinates only, and the others for the variations of the basis:

$$\vec{a} = \vec{a}_{coord} + \vec{a}_{base} \quad \text{and} \quad \vec{a}_{coord} = -\vec{a}_e - \vec{a}_c - \vec{a}_{base}$$

These are the three terms on the right that are expressed using connections.

Uniformly accelerated frame :

→ *Mechanics of Newton :*

$$\vec{a}_r = -\vec{a}_e = \frac{d^2 \overrightarrow{OM}}{dt^2} = -\vec{a}_{R'}(O)$$

$$R : \text{rocket}, \quad \vec{a}_{R'}(O) = \frac{d^2 \overrightarrow{O'O}}{dt^2} = a \vec{i} \quad \text{and} \quad \ddot{x} = -a.$$

→ *Special relativity :* as demonstrated in the exercise on page 243, the non-zero connection components are $\Gamma^1_{00} = \frac{g'}{2}$ and $\Gamma^0_{10} = \Gamma^0_{01} = \frac{g'}{2g}$ with $g(x) = \left(1 + \frac{ax}{c^2}\right)^2$.

Then:

$$\frac{du^1}{d\tau} = \frac{d^2 x}{d\tau^2} = -\Gamma^1_{00} u^0 u^0 = -\frac{a}{c^2} \left(1 + \frac{ax}{c^2}\right) \gamma^2 c^2 = -\gamma^2 a \left(1 + \frac{ax}{c^2}\right)$$

We find the classical limit: $\ddot{x} = -a$.

Rotating frame :

→ *Mechanics of Newton :* $\vec{a}_r = -\vec{a}_e - \vec{a}_c$

$$\vec{a} = \omega^2 \overrightarrow{HM} - 2\vec{\omega} \wedge \vec{v} = -\omega^2 \rho \vec{u}_\rho - 2\omega \vec{u}_z \wedge (\dot{\rho} \vec{u}_\rho + \rho \dot{\theta} \vec{u}_\theta)$$

$$\vec{a} = (\ddot{\rho} - \rho \dot{\theta}^2) \vec{u}_\rho + (\rho \ddot{\theta} + 2\dot{\rho} \dot{\theta}) \vec{u}_\theta = \omega^2 \rho \vec{u}_\rho - 2\omega \dot{\rho} \vec{u}_\theta + 2\omega \rho \dot{\theta} \vec{u}_\rho$$

→ *Special relativity :* $\tilde{u} = \gamma(c, \dot{\rho}, \dot{\theta}, \dot{z})$

Only non-zero connections :

$$\Gamma^1_{00} = -\frac{\rho \omega^2}{c^2} \quad \Gamma^1_{02} = \Gamma^1_{20} = -\frac{\rho \omega}{c} \quad \Gamma^1_{22} = -\rho$$

$$\Gamma^2_{10} = \Gamma^2_{01} = \frac{\omega}{\rho c} \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\rho}$$

Then :

$$\begin{aligned} \frac{du^1}{d\tau} \tilde{e}_1 + \frac{du^2}{d\tau} \tilde{e}_2 \\ = (-\Gamma^1_{00} u^0 u^0 - 2\Gamma^1_{02} u^0 u^2 - \Gamma^1_{22} u^2 u^2) \tilde{e}_1 \\ + (-2\Gamma^2_{10} u^1 u^0 - 2\Gamma^2_{12} u^1 u^2) \tilde{e}_2 \end{aligned}$$

$$\begin{aligned} \frac{d\gamma^{\dot{\rho}}}{d\tau} \tilde{e}_{\rho} + \frac{d\gamma^{\dot{\theta}}}{d\tau} \tilde{e}_{\theta} \\ = (-\rho \omega^2 \gamma^2 + 2\rho \omega \gamma^2 v^{\theta} + \rho \gamma^2 (v^{\theta})^2) \tilde{e}_{\rho} \\ + (-2\omega \gamma^2 v^{\rho} - 2\gamma^2 v^{\rho} v^{\theta}) \tilde{e}_{\theta} / \rho \end{aligned}$$

We find the classical limit :

$$\ddot{\rho} \vec{u}_{\rho} + \ddot{\theta} \rho \vec{u}_{\theta} = (-\rho \omega^2 + 2\rho \omega \dot{\theta} + \rho \dot{\theta}^2) \vec{u}_{\rho} + (-2\omega \dot{\rho} - 2\dot{\rho} \dot{\theta}) \vec{u}_{\theta}$$

We now understand how particles move in a non-inertial frame of reference. Special relativity gives us a new interpretative and experimental framework where metric effects take the place of the inertial forces of the old Newtonian framework.

In a flat space-time and a non-inertial frame of reference, a free particle maximizes its proper time by following a curved trajectory.

This is not simply a new point of view, but a generalization to massless particles, and, as a correction, with modified experimental measurements.

The classical notion of force is abandoned in favor of a relativistic description in terms of space-time geometry. Here, it is the concept of force of inertia that becomes useless, we follow the same kind of approach in general relativity, where geometry makes the concept of gravitational force disappear.

No need for the space to be curved, for a free particle to have a curved trajectory.

◦ Lagrangian approach

The geodesic equations are found with the Lagrange equations. The approach is explained in the exercise on page 160. We are looking for geodesics that extremes proper time :

$$c^2 \tau = \int g_{\mu\nu} u^\mu u^\nu d\tau, \quad L = g_{\mu\nu} u^\mu u^\nu \quad \text{and} \quad \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau} \frac{\partial L}{\partial u^\mu} = 0$$

$$\frac{\partial L}{\partial x^\mu} = g_{\alpha\beta, \mu} u^\alpha u^\beta \quad \text{and} \quad \frac{\partial L}{\partial u^\mu} = g_{\alpha\mu} u^\alpha + g_{\mu\beta} u^\beta$$

$$\frac{d}{d\tau} \frac{\partial L}{\partial u^\mu} = g_{\alpha\mu, \nu} u^\nu u^\alpha + g_{\alpha\mu} \frac{du^\alpha}{d\tau} + g_{\mu\beta, \rho} u^\rho u^\beta + g_{\mu\beta} \frac{du^\beta}{d\tau}$$

$$g_{\alpha\beta, \mu} u^\alpha u^\beta - g_{\alpha\mu, \nu} u^\nu u^\alpha - g_{\mu\beta, \rho} u^\rho u^\beta - 2 g_{\mu\beta} \frac{du^\beta}{d\tau} = 0$$

$$\text{Hence the geodesic equation : } \Gamma^\mu_{\alpha\beta} u^\alpha u^\beta + \frac{du^\mu}{d\tau} = 0$$

Conclusion and synthesis

Let's come back to the notion of inertial frame of reference.

We have a circular definition: the postulates are true in inertial frames of reference, and a reference frame is inertial if the postulates are verified.

If a particle in a reference frame has a curved trajectory, is it due to a force or to the non-inertial nature of the frame?

In Newtonian mechanics, if we know beforehand the nature of the forces, we can determine whether a reference frame is Galilean. Let's take the electromagnetic and gravitational forces: if there are no charges and masses present, and the trajectory is nevertheless curved, you can deduce that the reference frame is non-Galilean. You have to imagine such a region of empty space, far enough away from all matter that the remote action of the forces is negligible.

Do you know the Olbers' paradox?

In cosmology, the universe is like a fluid homogeneous and isotropic of galaxies. You see the stars in the dark night, the resulting brightness is low, but logically the night should be white. Indeed, the further away you look, the weaker is the light received by the observer from each luminous object, but at the same time their number increases in the same proportions. The night finally is dark because the Universe is expanding.

But back to the reference frames, if we apply the Olbers' Paradox to gravitation, we have the same result, the gravitational field would tend towards infinity at all points in the Universe... Here we want to illustrate how the foundations of classical mechanics are not trivial. Moreover, can we determine the nature of forces without the help of Newton's laws?

In relativity, the situation is much simpler, we use geometry. The behavior of spacetime alone makes it possible to determine if the frame of reference is inertial – without using the notion of force.

Beforehand, it is sufficient to have a set of clocks at rest and synchronized on the region being studied. If, during the experiment, the clocks do not go out of sync, the reference frame is inertial.

Minkowski metric

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Inertial frame of reference

Minkowskian metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad \text{with} \quad v_{\text{light}} = c$$

Inertial frame/ Maxwell's equations

Flat spacetime

Non-inertial frame / Metric effects

Accelerated rocket / Rotating disk

Zero curvature tensor

Back to the Minkowski metric by a change of coordinates

S P E C I A L R E L A T I V I T Y

G E N E R A L R E L A T I V I T Y

Equivalence principle / Einstein's equations

Curved spacetime in vacuum

Gravitation / Spatiotemporal waves

Energy-momentum and Ricci tensors are zero

Non-zero curvature tensor

Curved spacetime

Matter / Sources of the gravitational field

Energy-momentum and Ricci tensors are non-zero

Non-zero curvature tensor

Exercises

1. ▲▲▲ Change of basis

Let consider the basis \tilde{e}'_{μ} of the inertial frame.

1 - Determine the basis \tilde{e}_{μ} of the uniformly accelerated reference frame of the rocket as the function of \tilde{e}'_{μ} .

Place some examples of vectors from this base on a Minkowski diagram.

2 - Determine the basis \tilde{e}_{μ} of the uniformly rotating reference frame of the disk as the function of \tilde{e}'_{μ} .

Represent this base on a Minkowski diagram.

Answers p409

2. ▲▲▲ Riemann curvature tensor

We give here the curvature tensor without justification. We will apply the formulas to show that for the accelerated rocket, as for the rotating disk, we are in flat space-time despite the non-inertial nature of the reference frames. If all the components of the tensor are zero the spacetime is flat, if even one of the components is non-zero the spacetime is curved.

Riemann tensor as a function of the connections:

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}{}_{\beta\delta,\gamma} - \Gamma^{\alpha}{}_{\beta\gamma,\delta} + \Gamma^{\alpha}{}_{\sigma\gamma}\Gamma^{\sigma}{}_{\beta\delta} - \Gamma^{\alpha}{}_{\sigma\delta}\Gamma^{\sigma}{}_{\beta\gamma}$$

Connection coefficients³⁶ :

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(\partial_{\mu}g_{\beta\nu} + \partial_{\nu}g_{\beta\mu} - \partial_{\beta}g_{\mu\nu})$$

Notation: $\frac{\partial}{\partial x^{\mu}} = \partial_{\mu} = ,_{\mu}$ so $\Gamma^{\alpha}{}_{\beta\delta,\gamma} = \partial_{\gamma}\Gamma^{\alpha}{}_{\beta\delta}$.

The curvature tensor is antisymmetric in the last two indices. The connection coefficient is symmetric in the last two indices.

1 - Rocket: uniformly accelerated reference frame.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

b- Determine all the connection coefficients. You must identify the non-zero coefficients for the calculation of the curvature.

Helps: you can set $g(x) = \left(1 + \frac{ax}{c^2}\right)^2$.

Help yourself as much as possible with the symmetries. Identify the non-zero terms of $g_{\mu\nu}$ and $g^{\mu\nu}$. Are they constant? Which coordinates do they depend on? Which terms $\partial_{\mu}g_{\beta\nu}$ are non-zero?

c- Show that all the components of the curvature tensor are zero.

Help: what is the consequence of antisymmetry?

2 - Disk : uniformly rotating reference frame.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

³⁶ Also called Christoffel symbols.

b- Determine all the connection coefficients.

c- Demonstrate that all the components of the curvature tensor are zero.

3 - Spherical body: reference frame studied with Schwarzschild coordinate system. To compare with a situation where spacetime is curved.

We invite you to set $g = 1 - \frac{r_S}{r} = e^f$.

a- Determine $g_{\mu\nu}$ and $g^{\mu\nu}$.

b- Determine all the non-zero connection components.

c- To show that the spacetime is curved calculate the component R^0_{101} .

Prove that $R_{0101} = \frac{r_S}{r^3}$.

Answers p412

3. ▲▲▲ A non-uniformly rotating Disk

In the previous exercise we demonstrated that the curvature tensor was null in the uniformly rotating frame of the disc. We will continue the demonstration in the case of any rotational motion of the disk. We had for the inertial observer as a function of the coordinates of the observer at rest with respect to the disc: $\theta' = \theta + \omega t$. We now take the general expression: $\theta' = \theta + \lambda(t)$, where $\lambda(t)$ is any function of

time. Thus are included the possible phases of acceleration, deceleration, oscillation, etc.

- 1 - Determine the connection coefficients.
- 2 - Calculate the Riemann curvature tensor.
- 3 - Was the result expected?

Answers p419

4. ▲▲▲ Spatial curvatures

The Riemann curvature tensor applies to any space, space-time and sub-space regardless of the number of dimensions. We have calculated the curvature of 4-dimensional space-time and we will calculate the curvatures for the spatial parts. We take the three examples of the uniformly accelerated, the Schwarzschild and the uniformly rotating frames.

Let us detail the method and explain the general approach to measure times and distances³⁷.

For the time, we determine the proper time interval $d\tau$ by setting the $dx^i=0$ ($i=1, 2$ or 3) :

$$d\tau = \frac{1}{c} \sqrt{g_{00}} dx^0 \quad \text{and} \quad \tau = \frac{1}{c} \int \sqrt{g_{00}} dx^0 \quad (x^0 = ct)$$

For the space, if the reference system is synchronous $g_{0i}=0$ and: $ds^2 = g_{00} c^2 dt^2 - dl^2$

$$\text{with } dl^2 = -g_{ij} dx^i dx^j = \gamma_{ij} dx^i dx^j$$

The curvature tensor is then calculated with the three-dimensional metric tensor γ_{ij} as before. Here, we run the indices from 1 to 3.

³⁷ Landau / Lifchitz, *The Classical Theory of Field*, § Distances and time intervals.

If the reference system is not synchronous, the temporal coordinate is not directly separated from the spatial coordinates, and, we show that:

$$\gamma_{ij} = -g_{ij} + \frac{g_{0i}g_{0j}}{g_{00}} \quad \text{and} \quad dl^2 = \gamma_{ij} dx^i dx^j$$

We can then calculate dl with the three-dimensional metric tensor. On the other hand, we cannot, in general, determine the distance between two bodies. Also, the curvature tensor cannot be directly calculated in the form previously given³⁸. Nevertheless, in the particular case where the reference frame is stationary, metric coefficients $g_{\mu\nu}$ independent of time, we can integrate the element dl and the curvature tensor is in the usual form :

$$\text{Stationary frame: } \frac{\partial g_{\mu\nu}}{\partial t} = 0, \quad l = \int dl \quad \text{and} \quad R^i_{jkl}.$$

1 - Rocket: Is the reference system synchronous?
Is the space curved?

2 - Spherical body:
Is the reference system synchronous?
Is the space curved?

3 - Disk:
a- Is the reference system synchronous?
b- Determine γ_{ij} .
c- Is the reference frame stationary? What is the ratio of the perimeter of a circle to its diameter?

³⁸ Cattaneo's projection technique.

Rizzi / Ruggiero, *Space geometry of rotating platforms*, 2008.

(circle centered on the axis of rotation)

Does the observer attached to the rotating disc experience a curvature?

d- Calculate R^i_{jkl} .

e- It is shown that, for a two-dimensional space, there is only one independent component of the curvature tensor R_{ijkl} ($i=1, 2$)³⁹.

Calculate the Gaussian curvature K of the surface:

$$K = \frac{1}{R_1 R_2} = \frac{R_{1212}}{\gamma_{11} \gamma_{22} - \gamma_{12}^2}$$

where R_1 and R_2 are the radii of curvature at a point of the disk. You can compare it to the Gaussian curvature of a sphere.

Answers p419

5.+ ▲△△ **Pair production**

A high-energy particle can under certain conditions create a particle-antiparticle pair. Let's take the example of the collision of two protons. In the barycentric reference frame they arrive face to face with the same velocity. When their kinetic energy is just sufficient, we say at the *threshold*, they create four particles at rest:

$$p + p \rightarrow p + p + p + \bar{p}$$

39 Landau, § *Properties of the curvature tensor*.

Draw the Minkowski diagram at the threshold in the barycentric frame where $\sum \vec{p}_i = \vec{0}$.

Answers p422.

6. ▲▲▲ Wave equation

The wave equation describes the behavior of a multitude of waves: waves on water, sound waves, seismic waves, electromagnetic waves, etc. These waves, although of different physical natures, all obey the same equation. The amplitude of the wave $\varphi(\vec{r}, t)$ is the solution to the following differential equation:

$$\Delta \varphi - \frac{1}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = 0 \quad \text{so} \quad \square \varphi = 0$$

c is the celerity of the wave which depends on the type of wave and the medium.

Definition of the Laplacian in Cartesian coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

d'Alembert operator : $\square = \Delta - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$

1 - Demonstrate that the wave equation is not invariant under the Galilean transformation.

Help: In classical mechanics, the amplitude of the wave is a physical quantity that should not depend on the chosen coordinate system. At a point M and at a given time: $\varphi'(x', t') = \varphi(x, t)$. Such as, for example, the wave height, or the sound pressure. By identifying $d\varphi$ and $d\varphi'$ deduce the relations between the partial derivatives.

2 - Show that the electromagnetic wave equation in vacuum is invariant under the Lorentz transformation: $\square \vec{E} = 0$ and $\square \vec{B} = 0$. In this case the amplitude of the wave depends on the reference frame, the transformation formulas are given on page 427.

Answers p422.

7. ▲▲△ Schrödinger equation

In quantum physics, the wave function obeys the following equation of evolution:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi$$

The probability density of presence of a particle is obtained by multiplying the wave function by its complex conjugate:

$$\rho = \frac{dP}{dV} = \Psi \Psi^*$$

We can limit the study to the motion in one dimension of a free particle of mass m :

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2},$$

and a standard Galilean transformation: $\vec{v}_{R'/R} = v \vec{i}$

1 - The probability of presence of a particle in a given volume should not depend on the reference frame. On the other hand, the wave function is not unique and the probability density is not modified if we multiply the wave function by a complex number of modulus one.

Show that the Schrödinger equation is invariant under a Galilean transformation with:

$$\Psi' = e^{\frac{i}{\hbar}(Et - px)} \Psi \quad \text{where} \quad E = \frac{1}{2}mv^2 \quad \text{and} \quad p = mv.$$

2 - Show why the Schrödinger equation cannot be invariant under the Lorentz transformation.

Answers p424.

8. ▲▲▲ The electromagnetic field

Electric and magnetic fields are not written as fourvectors but as components of a rank-2 tensor:

$$F = F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}$$

The \vec{E} and \vec{B} fields are in fact one and only one physical entity and their components depend on the observational inertial frame of reference. We are here in the inertial frame R , and we will also consider the frame R' in uniform rectilinear translation along x :

$$\vec{v}_{R'/R} = \vec{v} = v \vec{u}_x.$$

The tensor of the electromagnetic field is antisymmetric: $F^{\mu\nu} = -F^{\nu\mu}$.

1 - Like mass, electric charge is an attribute of the particle that does not depend on the reference frame. We can simply build a four-vector for the charge and its motion:

$$\tilde{j} = q \tilde{u} \quad (\text{4-vector current})$$

We will demonstrate that the 4-vector $\tilde{F} \tilde{j}$ is identified with the electromagnetic 4-force:

$$\frac{d\tilde{p}}{d\tau} = F\tilde{j} \quad \text{and for the components} \quad \frac{dp^\mu}{d\tau} = F^{\mu\nu} j_\nu$$

By developing the components, temporal then spatial, show that we find the electromagnetic power, as well as the expression of the Lorentz force.

2 - Give the expression of the components of \vec{E}' and \vec{B}' in R' as a function of those of \vec{E} and \vec{B} in R .

3 - Determine the components of the tensor $F_{\mu\nu}$.

4 - Find the expressions of the two Lorentz invariants of electromagnetic fields. They are scalar invariants functions of \vec{E} and \vec{B} . The first one is obtained by contracting all components of the electromagnetic tensor with itself: $F^{\mu\nu}F_{\mu\nu}$. The second use the *completely antisymmetric unit tensor* of fourth rank: $\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$. $\epsilon^{\mu\nu\alpha\beta}$ components are zero if two indices are the same and ± 1 else. The tensor alternates sign under interchange of any pair of indices. We set: $\epsilon^{0123} = 1$.

5 - In the reference frame of the laboratory R , we have two planar metallic plates separated by a distance e and respective plate charge densities σ and $-\sigma$. The capacitor plates are assumed to be infinite and we will take the z -axis from the negative plate to the positive plate.

We will use the Gauss's and Ampère's circuital laws:

$$\oiint_S \vec{E} \cdot d\vec{S} = \frac{Q_{in}}{\epsilon_0} \quad \oint_\Gamma \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad (\epsilon_0 \mu_0 c^2 = 1)$$

The use of these tools is not explained here. A book in itself on this subject would be necessary. Refer to a undergraduate level course on electrostatics and magnetostatics.

a- Determine the electric field at any point in the space. Write the matrix $F^{\mu\nu}$ in R .

b- We are now in the frame of reference R' in uniform rectilinear translation along the x -axis at the velocity \vec{v} . For a classical observer of this frame of reference the charge density remains the same on the plates and the electric field $\vec{E}' = \vec{E}$. On the other hand, as the charges are in motion, a surface current density appears: determine the magnetic field at any point. Write the matrix $F'^{\mu\nu}$ in R' .

c- Starting from the tensor $F^{\mu\nu}$ do you find $F'^{\mu\nu}$ with the change of basis lambda matrices? Do we well have the invariance of the two Lorentz invariants?

6 - In the reference frame of the laboratory R , we have a homokinetic beam of protons of velocity \vec{v} , radius r and density n . We call R' the proper referential of protons.

a- Determine the electric field outside the beam in R' .

b- By general considerations, determine the structure of this same field in R with few calculations.

Answers on page 425.

9. ▲▲▲ Maxwell's equations

James Clerk Maxwell established in 1864 the theory of electromagnetism which unifies Michael Faraday's theory of electricity and André-Marie Ampère's theory of magnetism through the following equations:

In vacuum:

$$\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \qquad \vec{\nabla} \cdot \vec{B} = 0$$

With sources:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad \vec{\nabla} \wedge \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

The fields are derived from a potential V and a vector potential \vec{A} according to:

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t} \quad \text{and} \quad \vec{B} = \vec{\nabla} \wedge \vec{A}$$

Lorentz gauge condition: $\frac{1}{c^2} \frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = 0$

Charge conservation: $\vec{\nabla} \cdot \vec{j} + \frac{\partial \rho}{\partial t} = 0$

Definition of operators in the Cartesian coordinate system:

Gradient of f :
$$\vec{\nabla} f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

Divergence of \vec{C} :
$$\vec{\nabla} \cdot \vec{C} = \frac{\partial C_x}{\partial x} + \frac{\partial C_y}{\partial y} + \frac{\partial C_z}{\partial z}$$

Curl of \vec{C} :

$$\vec{\nabla} \wedge \vec{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \right) \vec{i} + \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \right) \vec{j} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \vec{k}$$

1 - Galilean transformation:

a- Show that Newton's second law is invariant under the Galilean transformation.

b- Lorentz's force is considered invariant under this same transformation. From this, deduce the Galilean transformation laws of \vec{E} and \vec{B} as a function of $\vec{v}_e = \vec{v}_{R'/R}$. Check that they well correspond to the non-relativistic limit of the Lorentz transformation of these same fields.

c- Show that the first two Maxwell's equations

$\vec{\nabla} \cdot \vec{B} = 0$ and $\vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ remain invariant under a Galilean transformation.

Help to do the calculations in vector form:

Partial derivatives: $\vec{\nabla} = \vec{\nabla}'$ and $\frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \vec{v}_e \cdot \vec{\nabla}'$

Useful formula:

$$\vec{\nabla} \wedge (\vec{A} \wedge \vec{B}) = \vec{A} (\vec{\nabla} \cdot \vec{B}) - \vec{B} (\vec{\nabla} \cdot \vec{A}) + (\vec{B} \cdot \vec{\nabla}) \vec{A} - (\vec{A} \cdot \vec{\nabla}) \vec{B}.$$

d- Show that the following two Maxwell's equations are not invariant under a Galilean transformation (to simplify the calculations, we can consider the case without the sources ρ and \vec{j}).

Useful formula: $\vec{\nabla} \cdot (\vec{A} \wedge \vec{B}) = \vec{B} \cdot (\vec{\nabla} \wedge \vec{A}) - \vec{A} \cdot (\vec{\nabla} \wedge \vec{B}).$

2 - Lorentz transformation: Let us show that from 1905 the Maxwell equations could incorporate their natural relativistic framework.

a- Show that Maxwell's equations are invariant under the Lorentz transformation.

b- We introduce the 4-vector current density $\tilde{j} = \rho_p \tilde{u}$ where ρ_p is the charge volume density in the proper frame of reference. Show that by using the 4-vector gradient $\partial_\mu = \vec{\nabla} = \left(\frac{\partial}{\partial ct}, \vec{\nabla} \right)$ we obtain a charge conservation equation in covariant form.

c- We propose to introduce the potential 4-vector $\tilde{A} = (V/c, \vec{A})$. Show that the Lorentz gauge condition is simply written in tensor form with A^μ and the 4-vector gradient ∂_μ . Show that by judiciously combining the four-vectors A^α and ∂^β , we obtain the tensor $F^{\mu\nu}$.

d- Show that the covariant equation $\partial_\mu F^{\mu\nu} = \mu_0 j^\nu$ gives back the Maxwell equations with sources.

e- Show that the equation $\partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0$ gives back the first two Maxwell equations.

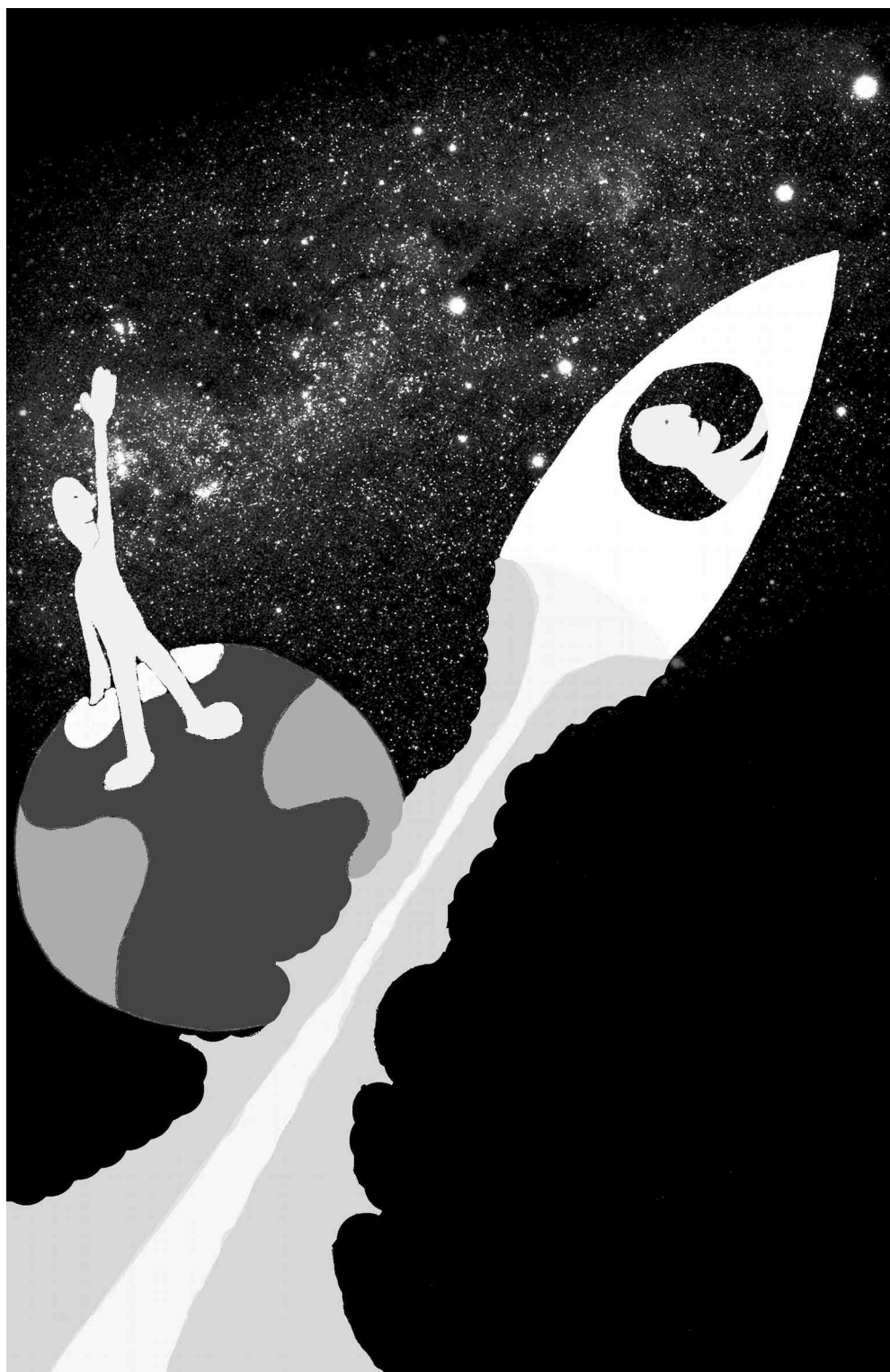
f- Find the expression of the propagation wave equations of V and \vec{A} .

3 - Show that the fields are not modified by the following gauge change:

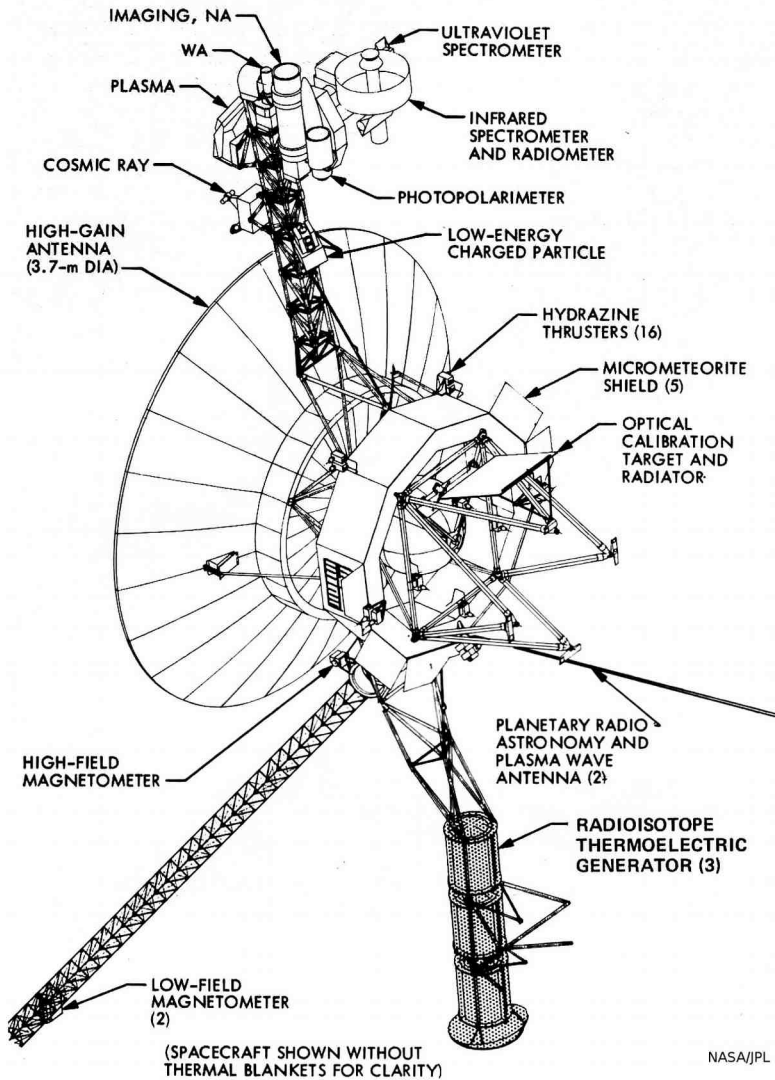
$$\forall f \quad \begin{cases} V' = V - \frac{\partial f}{\partial t} \\ \vec{A}' = \vec{A} + \vec{\nabla} f \end{cases}$$

This is called gauge invariance. The Lorentz gauge condition corresponds to a particular gauge choice that gives the potential propagation equations a simpler form. Above all, A^μ then behaves like a 4-vector, and the invariance of Maxwell's equations becomes immediate.

Answers p431.



Voyager 1 and 2 probes



INTERSTELLAR TRAVEL AND ANTIMATTER

∞ INTRODUCTION

Who says travel, says to leave his place of life for several reasons:

- by necessity, for reasons of survival
- in the spirit of adventure and discovery
- to conquer and colonize

For all these reasons, we have for centuries:

- explored our planet Earth
- we are right now exploring our solar system
- and, one day, surely, we will leave our system to explore other stars

Our planet is fragile, and even if we managed to live on it in harmony, it may seem risky to stay in only one place.

SURVIVE / DISCOVER / CONQUER



A representation of a picture of our galaxy, the Milky Way. At night on a beautiful starry night without clouds and without Moon, we clearly see a milky band arching the celestial vault, the cross section of our galaxy. Our Sun is at the center of the small circle, and most of the stars we see at night are our neighbors and are contained in this zone.

Of course, this is not a real picture, we have never placed a camera in a place outside our own galaxy. This is a computer-generated reconstruction from real photos.

For example, it is very likely that a meteorite, like the one responsible for the disappearance of the dinosaurs, will hit the Earth again one day, in a few years, or, millions of years, we don't know. Hence the idea of a multi-planetary humanity, with as a starting point the establishment of autonomous colonies and extraterrestrial bases.

Some, such as Elon Musk are targeting the planet Mars with a manned mission planned for the near future, and subsequently the establishment of a Martian base and the terraforming of the planet. This project is exciting, but before a group of humans can live on Mars without being dependent on freight arrivals from Earth, it may take several centuries.

The planet Mars is perhaps the best candidate among the eight planets that orbit our Sun. But probably not among the thousands of exoplanets already discovered that orbit other stars!

The idea is to join an exoplanet that has a greater similarity to Earth than Mars, a twin planet of Earth, so, despite a longer journey, the colony could establish itself much faster.

Some will tell you that the other stars are far too far away and that interstellar travel is unrealistic, when in fact we are already making interstellar travel with *Voyager* probes.

They were built with the technologies of the 70s. They have already crossed the heliopause, the limit of our solar system, and are now traveling through the interstellar medium. These probes were designed to explore only the solar system, but, simply, with current technologies, they could be adapted to reach other stars. For example, the radioisotope thermoelectric generator will stop in 2025 and the transmission with. They can easily be replaced by batteries with an isotope with a much longer lifetime. The *Voyager* probes travel at about 61,000 km/h and would reach the closest star to our Sun, Proxima Centauri located 4 light-years away, in 70,000 years⁴⁸.

This is a lot compared to the life span of an individual, but very little compared to the age of mankind. As we will see, the spaceship can be large and reach this speed on the same principle. We can then design, still with currently accessible technologies, a *seedship*.

A manned journey over such a length of time is difficult to conceive, people would be born and die in the vessel over several generations, this type of vessel is called a *generation ship*.

On the other hand, the seedship contains only

⁴⁸ In fact, over such periods of time we can no longer consider the stars motionless from one another. Nevertheless, in order not to complicate the presentation unnecessarily and to get to the point, we will consider the star Proxima Centauri fixed at 4 light-years.

frozen ovocytes and spermatozoa (no risk of them hitting each other!). Once close to Earth's twin planet, an automated process starts the incubators and the first generation of children will be raised by robots with artificial intelligence.

At this rate, an extraterrestrial human civilization can establish itself and re-launch a new interstellar seeding ark in 100,000 years. Thus, step by step, in small leaps of 10 light-years, humanity can colonize the entire galaxy in less than a billion years. Reasonable duration, compared to the age of our Sun, 4.5 billion years, and the appearance of the first cells 3.8 billion years ago.

We will first talk about the *Voyager* probes and then detail other technologies that would allow us to reach the other stars much faster.

∞ **VOYAGER PROBES**

The two *Voyager 1* and *Voyager 2* probes were built identically and were launched in 1977. They each have a mass of 820 kg including 90 kg of propellants.

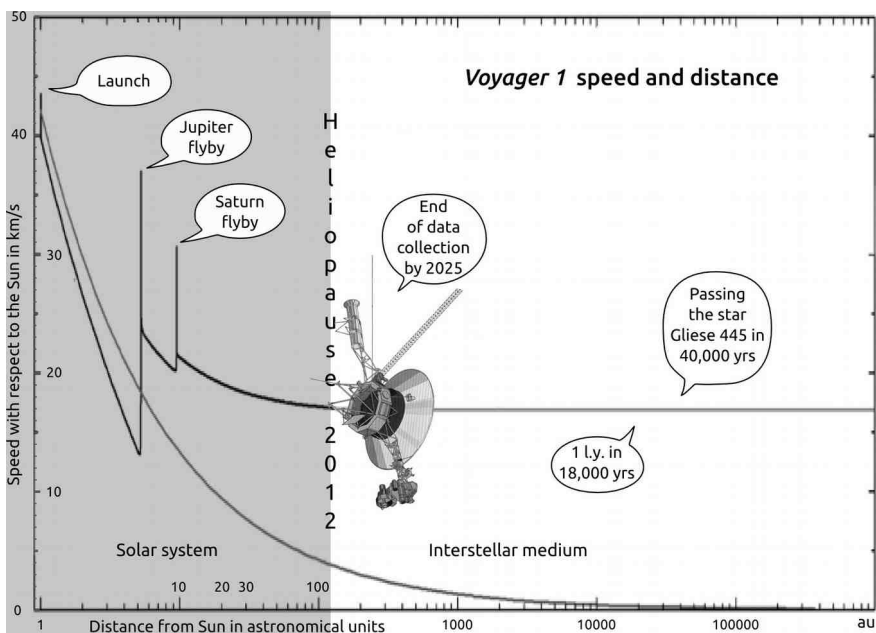


In astronautics, the term propellant, refers to the chemical substance that allows the propulsion of the rocket. For your car to work you must regularly take your vehicle to the pump to fill the fuel tank. But your car would not be able to run on the Moon, because for the combustion of the fuel it also needs the oxygen naturally present in the Earth's atmosphere. A rocket operates in vacuum and therefore has to carry both the fuel (the reductant), and the oxidant, the combination of the two is called propellant.

From the ground the probes left the terrestrial attraction on board Titan rockets containing tons of propellants. In addition to the speed thus gained, is added the speed of the Earth in its orbit around the Sun. But even so the speed of the probes was insufficient to break away from the solar attraction. And it is not the few kg of propellants carried by the probe that would allow it, they are used for trajectory corrections. The *Voyager* probes cleverly used the gravity assist of the planets to escape from the Sun's gravitational well.

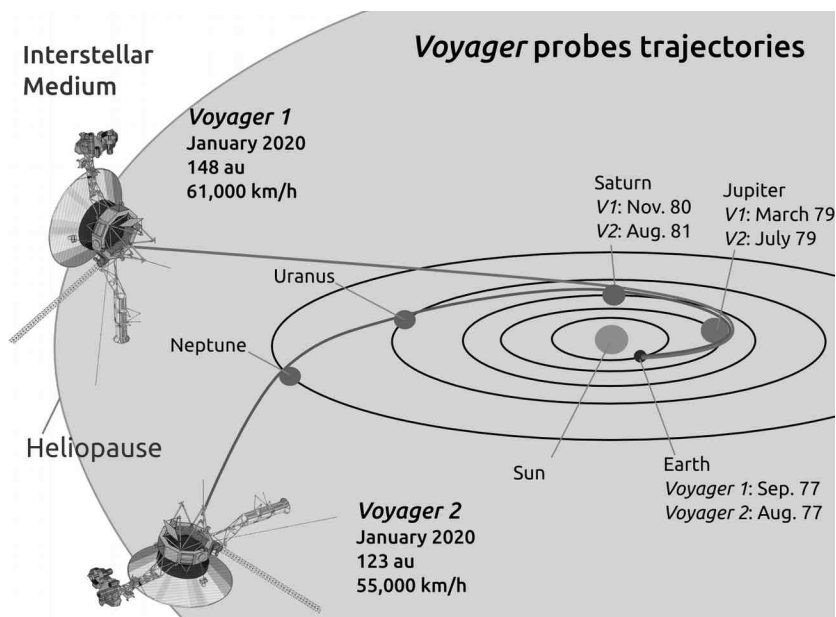
∞ SLING EFFECT

We use the speed of revolution of the planets around the Sun. For example, Jupiter orbits at 13 km/s around the Sun and the *Voyager 1* spacecraft after its deflection by the planet has gained more than 12 km/s.



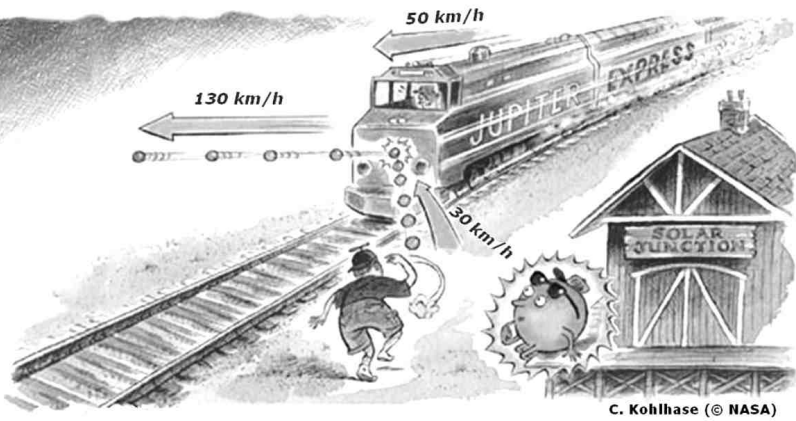
The black line represents the speed of the probe as a function of the distance to the Sun (multiplicative scale). By flying over Jupiter, the probe escapes its orbit around the Sun. The shaded line crossed corresponds to the speed necessary to escape from our stellar system. The astronomical unit corresponds to the distance Earth-Sun, one light-year is about 60,000 au.

The *Voyager 2* probe even took advantage of the slingshot effect of four planets: Jupiter, Saturn, Uranus and Neptune.

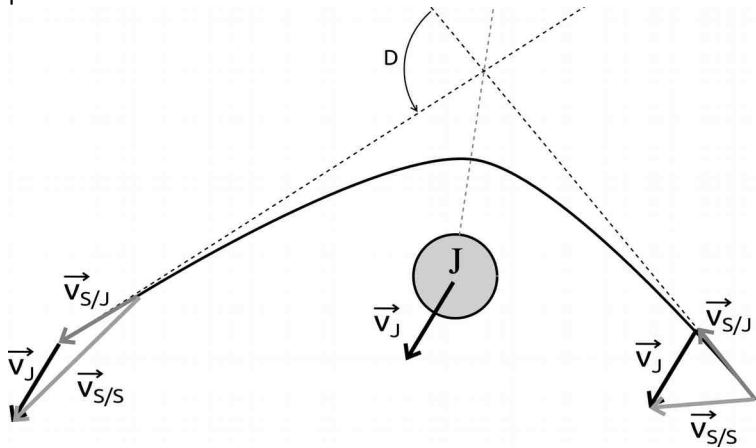


We have a small drawing, that follows, which allows us to understand simply the sling effect. A train moves towards you at 50 km/h and you throw a ball at 30 km/h to make it bounce on the front of the locomotive. Let's now put ourselves in the position of the train driver, he sees by additivity of velocities the ball arriving faster, at 80 km/h, the sum of the velocities, with respect to the ground, of the train and the ball. If the collision is perfectly elastic, the ball starts again, with respect to the train, with the same speed and in the opposite direction. So the ball thrower sees the ball bounce back with a speed of 130 km/h with respect to the ground. By throwing the

ball frontally, the speed of the ball increases by twice the speed of the train.



If now you throw the ball at a certain angle, the effect will be weaker but the principle remains the same. The same happens with the probe and the planets.



Jupiter in the center and the hyperbolic trajectory of the probe in the frame of reference which has for origin Jupiter. The velocity of the spacecraft $\vec{v}_{S/J}$ with respect

to Jupiter changes in direction but not in magnitude. The velocity of Jupiter with respect to the Sun must be added \vec{v}_J to obtain the velocity of the probe \vec{v}_{SIS} with respect to the Sun. We see in our figure that this speed increases, this is the slingshot effect. In the example of the train, there was a half-turn of the ball and the deviation D was 180° . For the passage of the Voyager 1 probe in March 1979, the deviation was 80° and the heliocentric speed of the probe increased by 12.5 km/s^{49} . The object which benefits from the gravitational assistance can have an important mass without modifying the effect (its mass must remain small in front of the mass of Jupiter...).

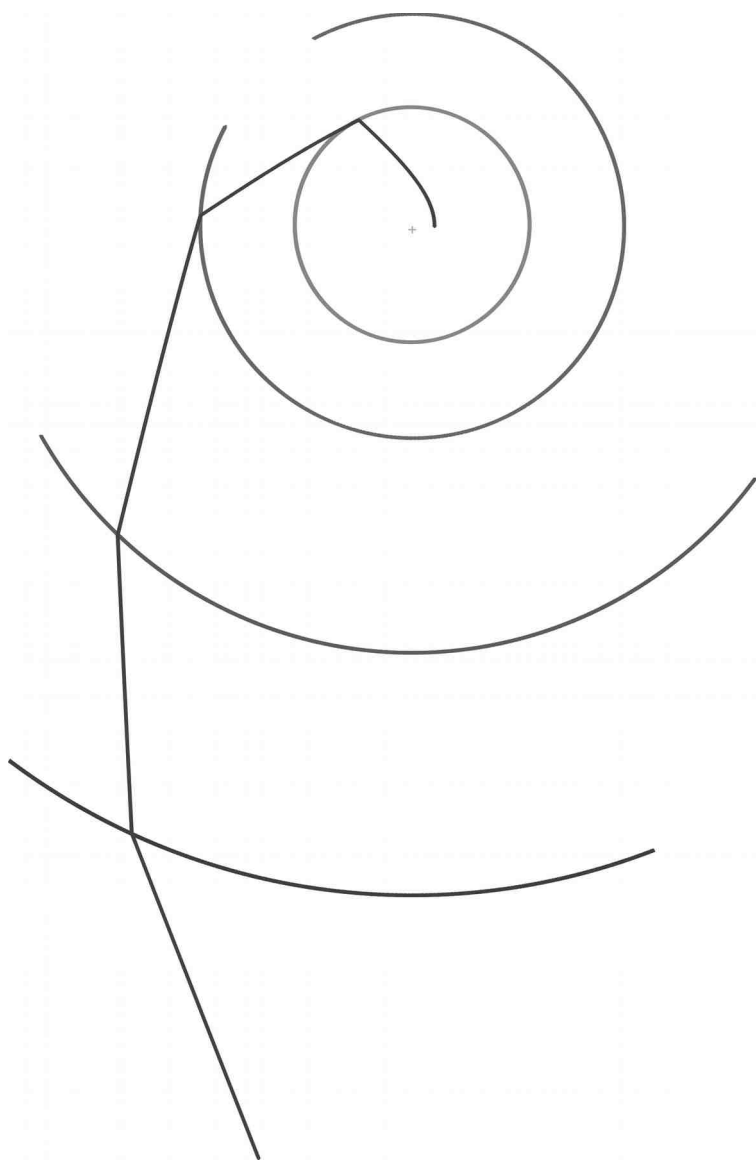
∞ VOYAGER 3 PROJECT

The Voyager probes were not designed for interstellar travel, but to explore the solar system. For the Voyager 3 project, we are optimizing the slingshots to gain speed and reach nearby stars. For example, we could take advantage of an opportunity: in 25,000 years, Proxima will be as close as possible to the Sun, 3 light-years away instead of 4.

This is a great project for mankind that also allows humanity to project itself into the future.

Next page, a numerical simulation of the trajectory of the spacecraft with the successive deviations at the flyby of Jupiter, Saturn, Uranus and Neptune.

49 Document: *La fronde gravitationnelle*, Pierre MAGNIEN, 2019.
Real time position of the Voyager probes: voyager.jpl.nasa.gov.

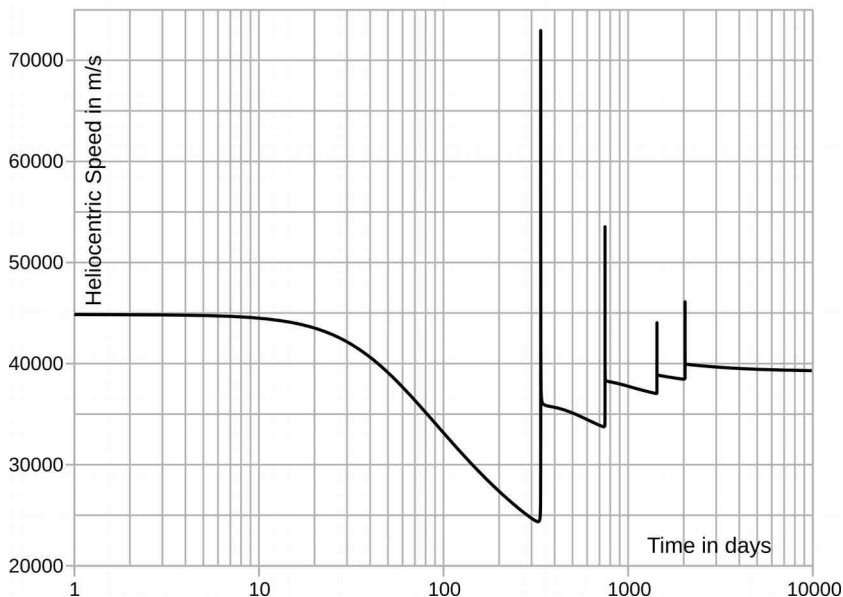


Voyager 3: the probe is propelled at the level of the Earth's orbit and it then chains four slings around the gas giants. The final speed is 140,000 km/h. Two differences compared to the historical Voyager probes: additional propellant is used and the effect of the slings is optimized.

The mass of the whole, the probe and the propellant, is very reasonable: only about ten tons, which can be sent into space with the current rockets.

Below is the speed profile of the probe. We see an initial velocity surplus of 5 km/s given by the propellants. Each slingshot borders the upper atmospheres of the gaseous planets for maximum speed gain.

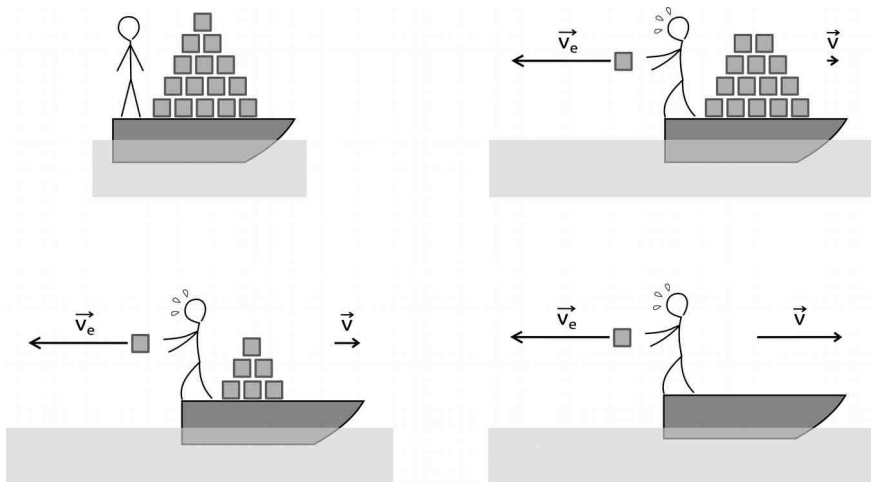
Speed of Voyager 3



∞ ROCKET EQUATION

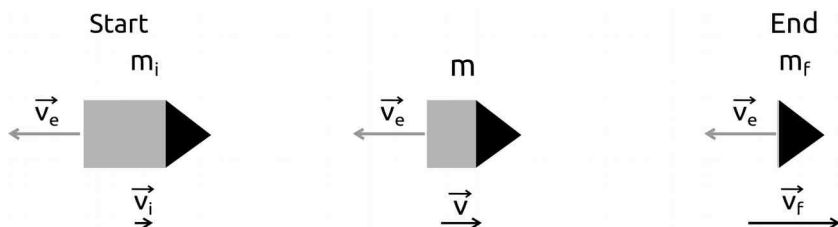
We would like to go even faster towards the stars by thrusting the probe with propellants. The propellants burn and the resulting gases are ejected backwards and allow the rocket to gain speed by reaction. The law of astronautics gives the speed increase Δv of the rocket as a function of the initial mass m_i of the rocket, of its final mass m_f and of the speed of ejection v_e of the gases.

We can begin by illustrating this law with the example of a small boat on which a person throws stones backwards as far as possible with all his strength:



The boat is at first immobile with all its reserve of stones. The person on the boat throws a first stone backwards. The boat then starts to move slightly. This is the conservation of momentum. The friction with the water is

neglected: the acquired speed is preserved. The person throws the stones until the stock is consumed and the speed of the boat increases with each throw. The last stone increases the speed much more than the first one because at the end the boat is much lighter. The first stones are not very effective because the boat is initially very heavy and they are used above all to move the stock of stones in waiting.



The initial mass of the rocket is that of the probe and the propellants, the final mass corresponds to the probe alone. The speed variation Δv is the difference between the final speed and the initial speed. The mass of propellant required increases very quickly, much faster than the speed reached.

Rocket equation:

$$\Delta v = v_e \times \left(\frac{m_i}{m_f} \right)$$

The crocodile illustrates that in spite of a mass ratio made important by the increase in the quantity of propellants, this ratio is massively crushed by the need to also increase the speed of these same propellants before their combustion.

For a conventional chemical propellant we have an ejection speed of approximately 4 km/s. Let's imagine that we want to go twice as fast to reach Proxima with a Voyager-type probe. How much propellant would we have to take on board?

We then have $\Delta v = 60,000$ km/h, or 16 km/s. The mass of fuel to be embarked increases exponentially and it would take 40 tons of propellants to get to Proxima in 35,000 years... To get there in 50 years, we would far exceed the mass of the Universe!

Duration of a trip to 4 light-years (current Sun-Proxima distance) with a Voyager type probe using traditional propellants (chemical energy / probe with a mass of 800 kg):

Duration of the trip	Mass of propellants required	$\frac{m_i}{m_f}$	$\ln\left(\frac{m_i}{m_f}\right)$
70 000 yrs	0 ton	1	0
35 000 yrs	40 tons	50	4
1 000 yrs	Mass greater than that of the observable Universe	∞	140
50 yrs		∞	2800

Once the star system is reached we can slow down the probe by sling effect. For the journey twice as fast, if we don't want to simply fly over the distant star system, the gravity assistance will not be sufficient to put ourselves in orbit around the star and we must also bring fuel to slow down the probe. As we have a factor of 50, we need 2000 tons of propellants at the departure from Earth to be able to be in orbit at the level of the exoplanet at the arrival!

To get around this monstrous increase in mass, the ejection speed would have to be increased instead. We would then have to use other technologies. We can use nuclear energy or mass energy. For one kilogram of propellant, which substance allows the maximum release of energy?

Let's compare energy efficiencies. It is the energy released compared to mass energy. For example, one gram of antimatter releases more energy than a thousand tons of chemical propellants:

Propellant	Efficiency		Details
Chemical	1 / 6 billions	0.00000002 %	Oxygen-Hydrogen
Fission	1 / 1000	0.1 %	Uranium 235
Fusion	1 / 250	0.4 %	Deuterium-Tritium
Antimatter	1	100 %	$E=mc^2$

In the current state of scientific knowledge, antimatter appears to be the ideal fuel. The entire mass is then converted into energy and motion of the rocket.

Duration of a one-way trip for Proxima Centauri for a Voyager-type probe using an antimatter reactor (10% efficiency):

Travel time to Proxima	Antimatter mass required
70 000 yrs	0
35 000 yrs	230 grams ⁵⁰
10 000 yrs	1.4 kg
1 000 yrs	16 kg
50 yrs	333 kg

Calculations for a distance of 4 light-years. In fact, Proxima Centauri will be closest to the Sun at 3 ly in 25,000 years. For an equivalent quantity of propellants, we gain 10,000 years.

We see that the problem of the mass of propellants to carry has disappeared. We will therefore focus on antimatter: its nature, its collection and its storage.

∞ ANTIMATTER

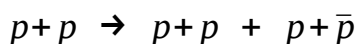
Paul Dirac in 1928 constructed a theory to unify special relativity and quantum physics. It was then that antimatter imposed itself in the equations, it was later discovered experimentally as early as 1932 with the positron. Theoretical prediction appears as symmetry in the Dirac equation. In nature, to each elementary particle corresponds a "twin" particle, a particle with exactly the same mass but with an opposite electric charge.

⁵⁰ One gram of antimatter releases as much energy as an atomic bomb.

For example, to the electron corresponds the anti-electron commonly called positron, or positon. In 1955, the antiproton was discovered by creating it with a particle accelerator. In 1995, the first atom of antimatter was created, the atom of anti-hydrogen. When a matter particle meets its antimatter counterpart, the two disappear and annihilate each other in pure energy. Hence perhaps the name antimatter, but, to avoid any confusion related to this name, let us specify that antimatter is matter.

We can produce antimatter artificially with a particle accelerator, but it also exists - although in much smaller quantities than matter - in nature.

The production of antimatter in the laboratory requires a lot of time and energy. For example, to create antiprotons, protons are accelerated and when they collide at high energy, they create proton/antiproton pairs:



You create a proton for nothing and the productivity is low. It is very interesting and precious to understand the secrets of matter on a small scale, but, to produce the propellant for a rocket, it is perhaps not the most judicious⁵¹.

51 In 2020, world energy production corresponds to the energy released by the annihilation of 3.5 tons of antimatter, however, with the existing current means, even to produce just one gram of antimatter would be prohibitively expensive.

It would be simpler to collect it in the nature. Positrons are released by beta-positive radioactivity, by cosmic rays or even storms. Antiprotons are a fuel of choice because they have a mass energy much higher than positrons. However, unlike positrons, antiprotons are not directly produced in our solar system. The Sun, the most powerful source of energy in our star system, only rises in energy to the level of fusion and the solar wind does not contain antiprotons.

We must, therefore, look for a source of antimatter outside our system. This source exists, it was discovered in 1912, it is the cosmic rays. It is made up of particles of very high energy capable of creating antiprotons. The precise sources of this radiation are not yet known, but it is now believed that they are mainly located in our galaxy. This galactic radiation is constantly passing through the solar system, and it is estimated that 200,000 tons of antimatter crosses the heliosphere every year⁵².

The density of antiprotons is higher in the planetary magnetospheres. For example, around the Earth, there is an antimatter belt with a zone a thousand times denser than the surrounding cosmic rays⁵³. Cosmic antiprotons are trapped, and moreover,

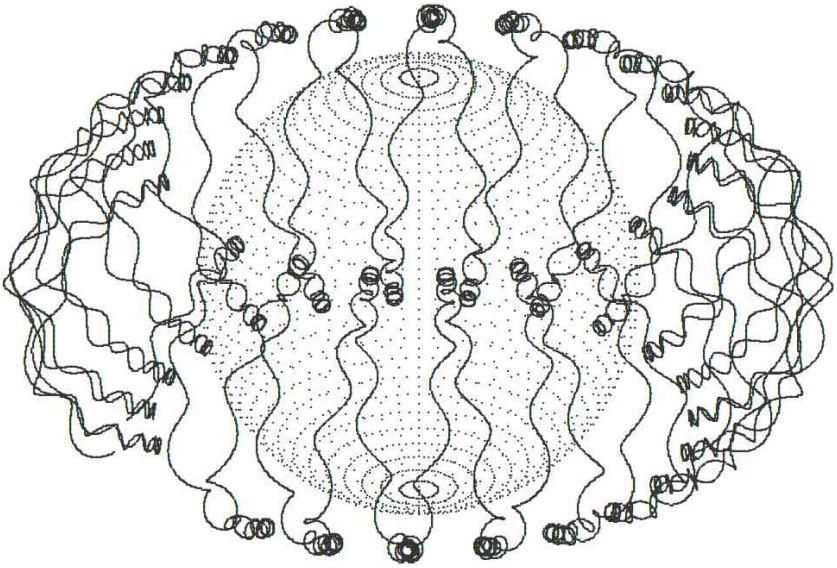
52 A lot of data is taken from a very comprehensive article from the Draper Laboratory: *Extraction of antiparticles concentrated in planetary magnetic fields*, 77 pages, 2006.

53 Analysis of results from the PAMELA detector installed on a satellite in Earth orbit: *The discovery of geomagnetically trapped cosmic ray antiprotons*, 2011.

others are directly created by the interaction of cosmic rays with the upper layer of the Earth's atmosphere. The Earth's antiproton belt is located several hundred kilometers above sea level in the Van Allen radiation belt.

∞ **JUPITER: THE SOLAR SYSTEM GAS PUMP**

The Earth generates a magnetic field that traps charged particles at altitude, such as electrons contained in the solar wind. Sometimes during a destabilization of the magnetosphere, for example following a solar flare, electric particles are released at the poles and create beautiful polar auroras. The magnetosphere acts as a giant magnetic bottle that stores all kinds of charged particles. The Earth's magnetosphere is subjected to a flux of about 4 grams of antiprotons per year. But it is mainly the large gas giant planets, and, without a doubt, the gigantic magnetosphere of Jupiter that could contain the largest amount of antimatter with a flux estimated at 9 kg per year.

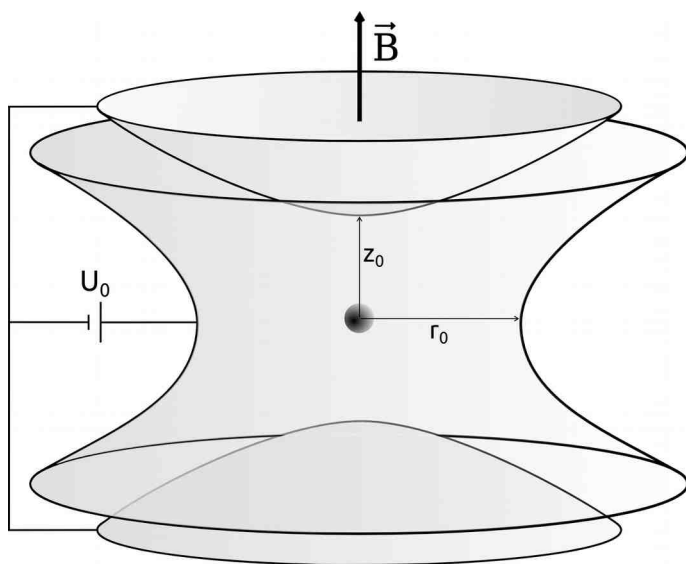


A picture of the antiproton belt around the Earth. Here, an antiproton moving at 70% of the speed of light. The Earth's magnetic field curves its trajectory and traps it using three types of combined motions: the fastest, a cyclotron rotation that makes it make small circles, then, an oscillation between the poles, and finally, a slower drift that makes it go around the Earth.

Satellites could collect and store this antimatter. The ships would then refuel at Jupiter before leaving for the stars.

∞ ANTIMATTER STORAGE

We currently know how to store antiprotons for more than a year. The temperature is maintained below one Kelvin and the measurements of the characteristics of the antiproton are extremely accurate⁵⁴. Nevertheless, the quantities are very small and the mass of the trap is very large compared to the mass of antimatter stored.



Penning trap. By combining a magnetic field and an electric field, charged particles can be trapped in the laboratory.

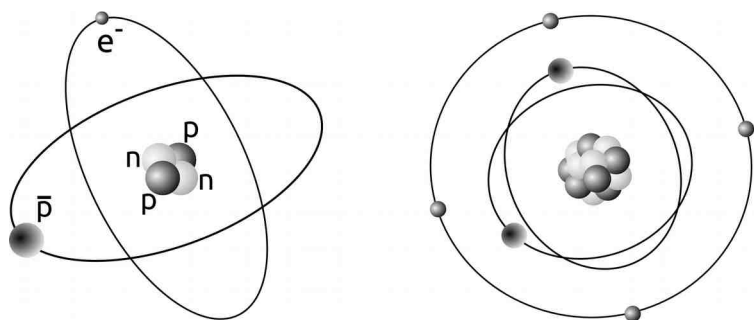
The ideal would be to store antimatter on a microscopic scale. The antimatter thus trapped and

⁵⁴ BASE experiment: *A parts-per-billion measurement of the antiproton magnetic moment*, review Nature, 2017.

confined at the atomic or molecular scale could then be stored like matter. We would have a flexible and versatile use of this new fuel, both for space travel and in our daily lives. For example, a car could travel around the Earth on a single tank of a few milligrams of antimatter.

Let's call *Proximum* this hypothetical fuel of the future. A luminal fuel that would allow us to reach the stars and bring us into a new energy era. Could this dream come true? Only experimentation will allow us to make progress on this question. Let's start by letting our imagination consider different options.

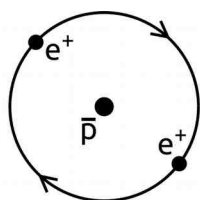
1 - Exotic atoms where an electron would be replaced by an antiproton:



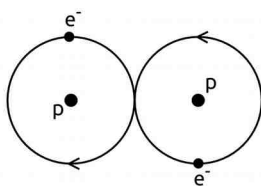
Examples of helium and carbon atoms where one or more e^- have been substituted by a \bar{p} . Antimatter density of the structures: 20% and 14%. The first compound, sometimes called antiprotonic helium and noted $\bar{p}\text{He}^+$, was discovered by serendipity at the Japanese CEC laboratory in 1991, and then studied at the CERN antiproton decelerator. Normally an antiproton is stopped

by matter and annihilates on a nucleus in a time of the order of a picosecond. In this experiment, where a beam of slow antiprotons encounters a liquid helium target, we naturally obtain the metastable $\bar{p}\text{He}^+$ state in which the trapped antiproton can be stored for several microseconds⁵⁵.

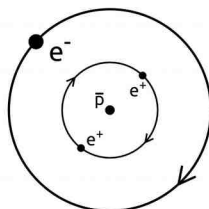
2 - An antihydrogen atom ionized with an additional positron \bar{H}^+ , could replace the nucleus of a hydrogen atom. Two such exotic atoms would constitute a Proximum molecule:



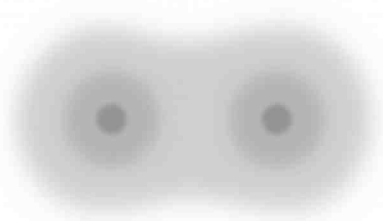
anti-hydride \bar{H}^+
Stable



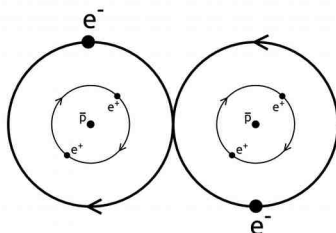
dihydrogen
Stable



\bar{H}^+ and e^- Stable?



Proximum

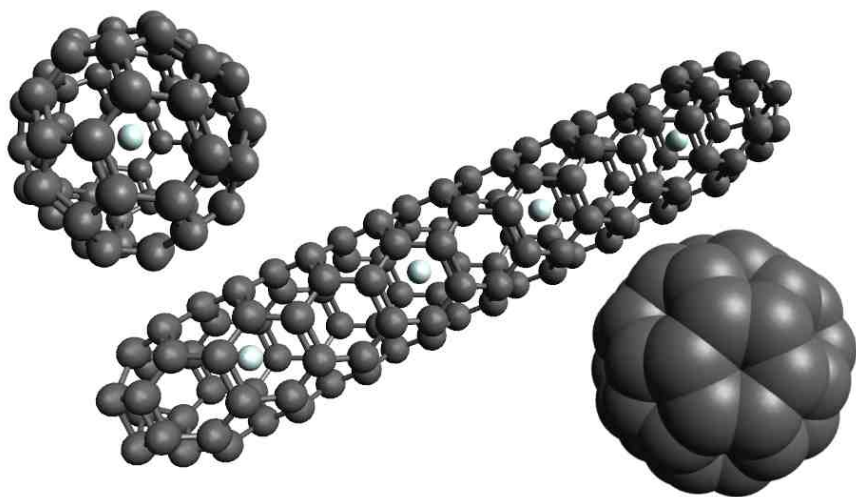


Stable?

The storage density in this case would be almost 100%. Experimental research can first focus on the synthesis of an anti-proximum molecule. Experiment easier to implement for a molecule that has the same stability.

⁵⁵ Article of HAYANO *Spectroscopy of antiprotonic helium atoms and its contribution to the fundamental physical constants*, Japan, 2010.

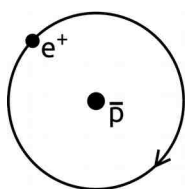
3 - A cage molecule. There are many cage molecules in chemistry that allow the encapsulation of molecules. We can imagine such a molecule that contains an antiproton as in a microscopic Penning trap. We have, for example, fullerene-type molecules and nanotubes:



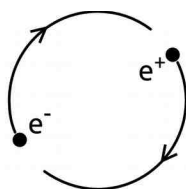
Different carbon-based structures. In the top left corner, we represented the C₆₀ fullerene. Different types of atoms have already been trapped in these structures. Fullerene can easily be negatively ionized and could thus be a good antiproton trap. Bottom right, the same structure using a model showing the electrostatic spheres of influence of electronic clouds. Diagonal, a nanotube with 4 confined antiprotons.

And so on... We can start by measuring the life span of such structures, and maybe one day we will have the pleasant surprise of finding a stable one. Scientific research makes it possible to test multiple combinations. It's worth the effort because even if we don't find what we're looking for, we'll have learned a lot about matter.

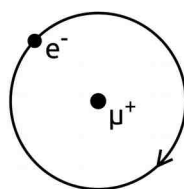
Scientists have already studied different exotic atoms. We have created and studied anti-hydrogen atoms that have proven to be stable. Another hydrogen derivative, positronium, which consists of an electron and a positron that revolve around each other, has a stability of 100 nanoseconds. The muonium, on the other hand, replaces the nucleus of a hydrogen atom by a muon, the stability is 2 microseconds.



Anti-Hydrogen
Stable



Positronium
100 ns



Muonium
2 μ s

Stability can also depend on the context. For example, a neutron in the nucleus of an atom is stable, whereas in its free, isolated state, the neutron has a lifetime of only 10 minutes.

∞ CONCLUSION

By learning to master antimatter we could reach the first stars in 50 years and explore the entire galaxy in a few million years. This type of vessel could be manned and would quickly overtake the previously sent seed ships. Both scenarios deserve to be developed in parallel over the next decades.

Elon Musk projects a colony on Mars of one million humans by 2050 and a progressive empowerment. Also planned are microprobes for Proxima propelled by giant lasers placed on Earth.

Often for interstellar travel, nuclear fission or fusion are proposed as a source of energy and antimatter is little considered. The aim of this conference is to show the important potential of antimatter as a key element for the future.

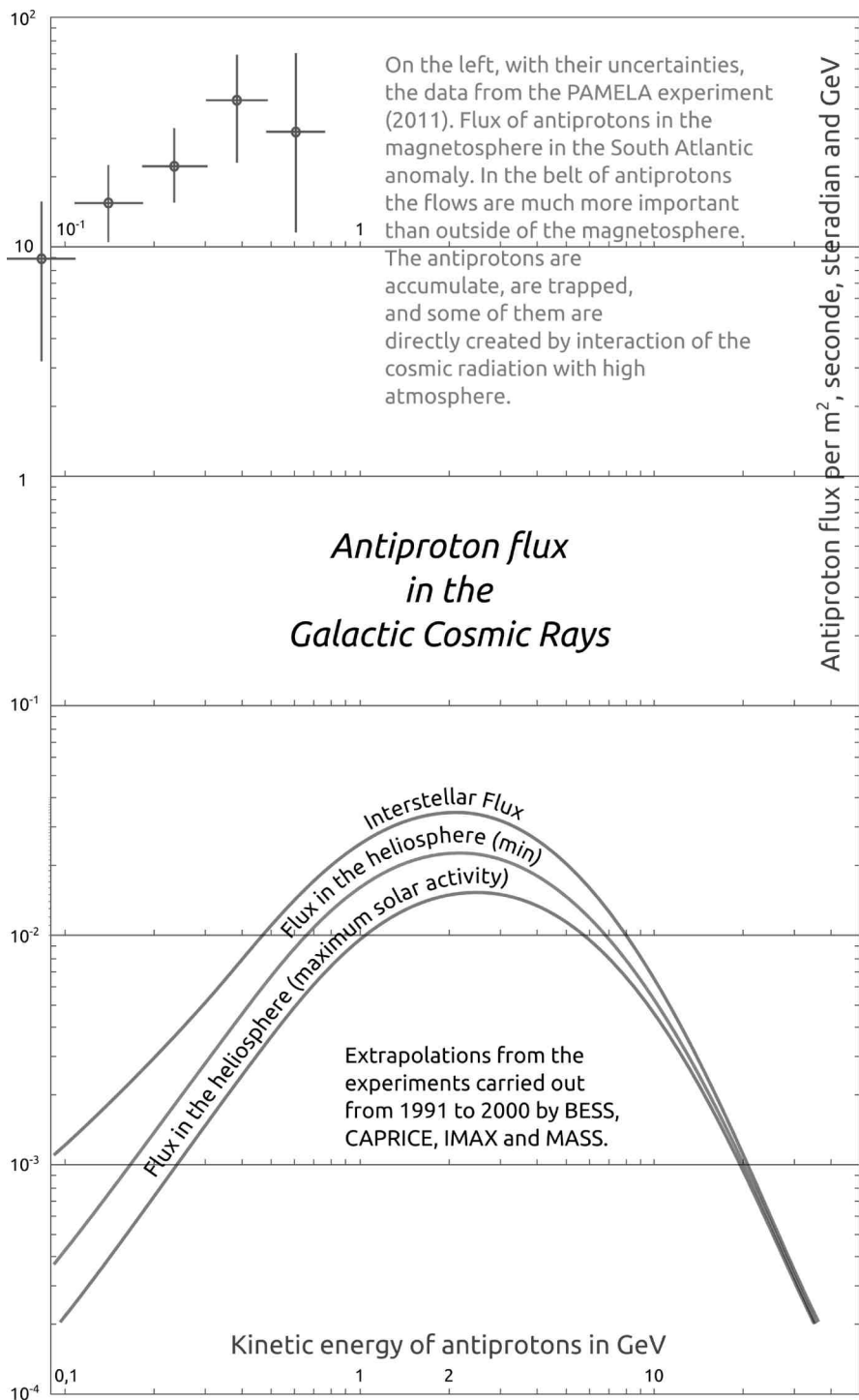
Exercises

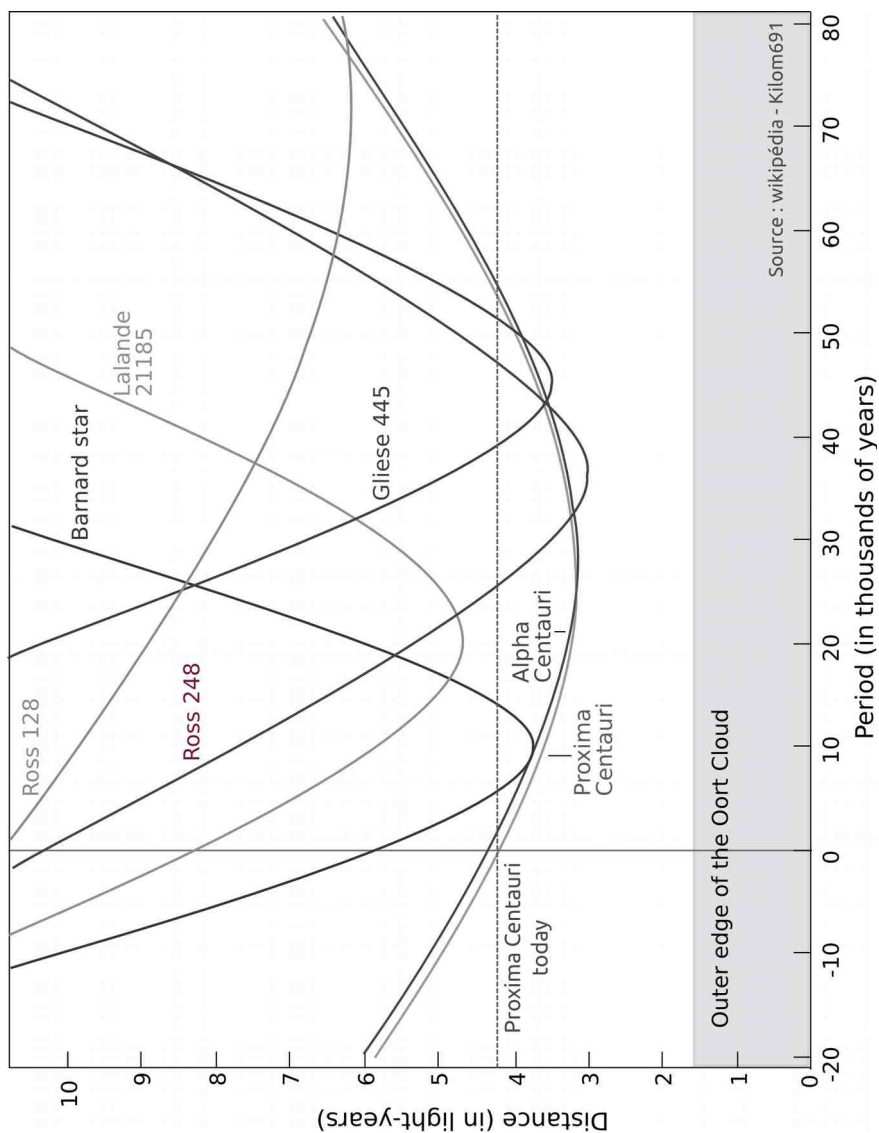
1. ▲△△ Figures

Find the numerical values of the conference:

- A probe goes at 61,000 km/h to a 4 ly star. Do you find 70,000 years of travel?
- World energy consumption is estimated at 15,000 Mtoe in 2020. The toe (ton of oil equivalent) is worth 42 GJ. Show that this energy is equivalent to the energy released by the annihilation of 3.5 tons of antimatter.
- Using the data in the table on page 34 of the article *Extraction of antiparticles concentrated in planetary magnetic fields*, find the 200,000 tonnes of antimatter that crosses the heliosphere each year. For example, for Jupiter the flux is 9.1 kg of antiprotons for a cross section of $45 R_J$ radius (zone of influence of the Jovian magnetosphere with R_J the radius of Jupiter). The effective radius of the Sun is taken at heliopause, limit of the influence zone of the solar magnetic field. If we now take the interstellar flux of cosmic radiation, external to the heliosphere, evaluate how much the antimatter flux is by using the following curve.

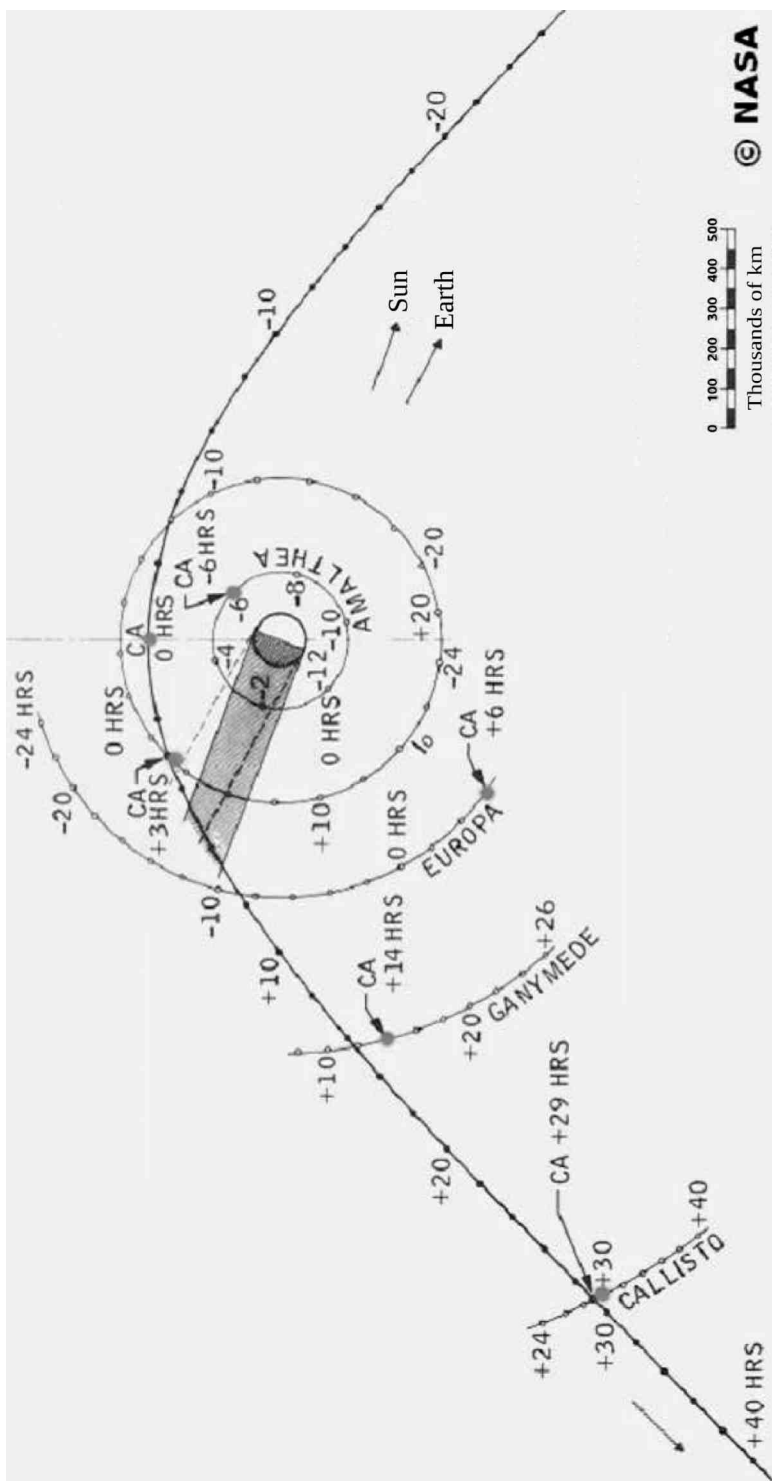
Answers on page 441.





Source : wikipédia - Kilom691

Over significant periods of time, several thousand years, the stars can no longer be considered fixed to one another. The three stars of the Alpha Centauri system will be closest to the Sun in 25,000 years at three light years.



2. \ ▲▲▲ The distances of stars over time

In the conference the distance Sun-Proxima is set to 4 light-years. For fast journeys the stars can be considered fixed, but for slow journeys of more than 10,000 years the variations in distance are no longer negligible. We have placed the curve in the previous pages. Show that the *Voyager 1* and 2 probes could not reach Proxima Centauri. What should be the minimum speed of the probes? How fast does a probe have to go to reach the Alpha Centauri system when it is closest?

Answers p442.

3. \ ✓ ▲▲▲ Sling effect

We consider the flyby of *Voyager 1* at the level of Jupiter.

a - With an initial probe speed of 12.6 km/s and a Jovian speed of 12.8 km/s, find the speed variation of *Voyager 1* (heliocentric velocities). The motions are assumed to be coplanar and the trajectory of Jupiter in the heliocentric reference frame circular. You will estimate the required angles using the curve on the previous page.

Help: it is not easy to visualize the asymptotes, trajectories at a great distance from the probe, the view is too close. Two indications: the inner angle between the two

asymptotes of the hyperbola is 82° and the impact parameter b is $13 R_J$ (b : distance between the barycenter of Jupiter and the asymptotes – R_J : radius of Jupiter).

Definition of angles : $\alpha_i = \widehat{(\vec{v}_J, -\vec{v}_i)}$ and $\alpha_f = \widehat{(\vec{v}_f, \vec{v}_J)}$.

b - Evaluate on the NASA graph the maximum speed of the probe at the *periastron*. Does the result correspond to the peak on the graph page 283 ?

Estimate the speed of the probe 38 hours after its passage at the periastron. Deduce, by calculation, the speed of the probe to infinity. Evaluate the minimum approach distance of *Voyager*, and deduce by calculation the *impact parameter* b of the probe.

Help: For an isolated system, in a Galilean frame of reference, there is conservation of mechanical energy and angular momentum.

c - Conic parameters.

Find the *semi-latus rectum* p , the *eccentricity* e and the *deviation* D .

Aids: The general solution of the Kepler problem provides the polar equation of a conic (hyperbola, parabola and ellipse):

$$r = \frac{p}{1 + e \cos \theta} \quad p = \frac{L^2}{\alpha m} \quad \alpha = G m M$$

Origin of the reference system: center of mass of Jupiter.

Angles origin: main axis of the hyperbola.

p : *semi-latus rectum* of the conic.

e : *eccentricity*. L : *angular momentum* of the probe.

$M = M_J = 1.90 \times 10^{27}$ kg. m : *mass* of the probe.

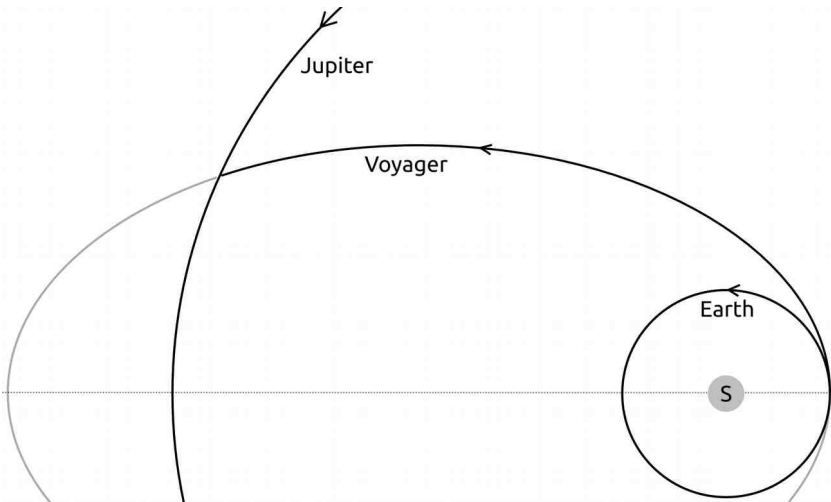
Distance Sun-Jupiter: 800×10^6 km. $M_s = 2 \times 10^{30}$ kg.

d - We want to increase the sling effect.

- All else being equal, for what value of α_f do we get a maximum v_f ? Determine the corresponding Δv . If the probe then left the solar system directly, what would be its interstellar speed?

- The trajectory of the probe from the Earth is considered to correspond to an orbit of the Hohmann transfer elliptic orbit type.

What is the semi-major axis a of this ellipse?



We can also find again the angle of approach. How could we increase the interstellar speed of the probe? We must not get too close to Jupiter. The equatorial radius of Jupiter is 71,492 km and an altitude of 1,000 km places the probe as close as possible, in an atmosphere sufficiently tight that its influence can be neglected.

Aids: Mechanical energy for a conic:

$$E_m = \frac{\alpha}{2p}(e^2 - 1), \quad \text{Ellipse: } E_m = -\frac{\alpha}{2a} \quad \text{and} \quad p = \frac{b^2}{a}$$

- Explain why Mars does not allow to have a consequent slingshot effect despite its high orbital speed.
- Retrieve the characteristics of the speed profile of *Voyager 1* by considering the two slings one after the other (Jupiter then Saturn). A spreadsheet can be used for a systematic calculation for n slings. Conservation of the angular momentum and mechanical energy between the slings, properties of the hyperbola during a sling.
- Model the succession of the four fronds from Jupiter to Neptune. Show that it is possible to obtain, by optimizing the successive effects, an interstellar speed of 100,000 km/h (on the principle of *Voyager* probes and using only gravitational assistance). Show that by giving, at the level of the Earth's orbit, a speed surplus of 4.8 km/s using propellants, the probe reaches an interstellar speed of more than 137,000 km/h.
- Globally, all the planets revolve around the Sun in the same plane, called the ecliptic. In our model for the succession of the fronds, the probe leaves the solar system in this plane. However, most of the stars are out of the ecliptic. For example, at the closest, in 25,000 years, the star Proxima will be located 39°

below the ecliptic plane⁵⁶. The velocity given by the gravity assist has a value but also a specific direction. The direction of the velocity is just as important as its magnitude: what's the point of going fast if it's not to the right place? Do you have a proposal to have a correctly directed velocity without using huge quantities of propellants?

- The probe at the end of its 25,000-year journey flies over the Alpha Centauri star system. How should we proceed to slow down the probe in order to trap it in the star system? Should additional propellant reserves be provided for this purpose?

Answers p442.

4.10⁹¹ √ ▲▲▲▲ Numerical simulations of the slings

The simulations make it possible to recover the results established in the previous exercise, which used Kepler's formulas. Also, simulations give a great deal of freedom and help to envisage a number of situations. The counterpart is the necessary computing power. We will use that of a personal computer. This will be sufficient for a first approach and to explain the basic principles.

⁵⁶ Calculations in the exercise *Motion of the stars* on page 322.

Current ecliptic coordinates of the stars: heasarc.gsfc.nasa.gov/cgi-bin/Tools/convcoord/convcoord.pl. Often only equatorial coordinates are given, all conversions on this site.

We will study the problem of N bodies in gravitational interaction. The modeling is very ambitious and the computation time can be very long: the number of interactions evolves in N factorial and from $N=3$ we can have chaotic regimes. Each body has 6 degrees of freedom, three for the position and three for the velocity components. We will, therefore, simplify with a set of reasonable hypotheses.

For the *Voyager* probes the motions will be considered in the same plane: indeed, it is a reality, basically all the planets orbit in the plane of the ecliptic, moreover, it is shown that the two-body motion is done in one plane.

We will assume that the Sun is motionless. This way we have one less body to consider. The heliocentric reference frame is then Galilean. No need to consider the center of mass of the solar system and the Copernican frame of reference, because the mass of the Sun is very large in front of those of the other bodies.

We will not consider the forces between the planets. Always to simplify the equations, reduce the number of relations, and the computation time. Only the Sun exerts its force on a planet. Only the probe remains connected to all the bodies.

Newton's equations of motion give a system of coupled differential equations:

$$\frac{d\vec{OM}_i}{dt} = \vec{v}_i \quad \text{and} \quad \frac{d\vec{v}_i}{dt} = \sum_{j \neq i} Gm_j \frac{\vec{M}_i \vec{M}_j}{r_{ij}^3}$$

For each body, we have two vectorial differential equations of order one. For a 2D motion, we have four variables per body: x_i , y_i , v_{xi} and v_{yi} . Finally, for the probe and the four gas giants we have 20 equations. It is already a lot.

The principle of digital resolution is simple, it is a step-by-step method. We have the initial conditions at $t=0$, positions and velocities of all bodies. After a small interval of time Δt , we evaluate the new velocities and positions using differential equations. We thus pass, step by step, from t_n to t_{n+1} :

$$x_{i,n+1} = x_{i,n} + v_{x,i,n} \Delta t, \dots, \\ v_{x,i,n+1} = v_{x,i,n} + F_{x,i}(x_{j,n}, y_{j,n}) \Delta t, \dots$$

This is the *Euler method*. We will then study the much more precise *Runge-Kutta method*.

Mechanically, as in a line of dominoes that fall one after the other, we move causally from one stage to the next. At each step, we make a small local error that accumulates to the one of the previous step. We will take a step small enough to be able to properly linearize each segment and minimize the global error. Since we are not mathematicians, in this initiation exercise we will be content to control, as good physicists, the conservation of mechanical energy and angular momentum.

We will use a spreadsheet program. No need to download any special programming software, a worksheet will be enough.

Let's start by practicing on simple models for which the analytical solutions are known.

1 - *Revolution of the Earth around the Sun:*

Let us take as initial conditions the Earth at its perihelion: $r_{min}=147,098,074$ km and $v_{max}=30,287$ m/s.

Sun mass: $M_s=1.9891 \times 10^{30}$ kg.

Gravitational constant: $G=6.6743 \times 10^{-11}$ N.m²/kg².

a- Kepler law's: The previous data comes from Wikipedia. Determine, from them, the semi-latus rectum p of the conic, the eccentricity e , r_{max} , v_{min} , the semi-major axis a and the period T .

b- First simulation with a step $h=1$ day.

Do you get a satisfactory simulation on a revolution? What is the percentage of error on the radius after one revolution? How does this percentage change for $h/2$, $h/4$ and $h/8$?

Do you find the right values for the period of revolution and the values at aphelion ?

Even already on the first step from $t=0$ to $t=h$, do you notice an anomaly ?

How to explain it?

We have calculated the values at t_{n+1} from those at t_n . For example, the velocity $v_{x,n+1}$ is calculated with $v_{x,n}$, x_n , and y_n . On the same principle, the position x_{n+1} is calculated with $v_{x,n}$ and x_n . But it would also be quite possible to determine the positions x_{n+1} and y_{n+1}

with the velocities at rank $n+1$. Indeed, it is no more false to take the velocity at the end of the interval than at the beginning. Run the simulation again for $h=1$ day with this modification for the calculation of the positions. Do you now find better estimates for the period and the aphelion? What is then the global error for the radius after one revolution? What is the value of the variation of mechanical energy over 365 days? Conclusion.

2 - Runge-Kutta method of order 4 (RK4):

The global error with the Euler method was of the order of h , with the midpoint method (for example, the modified Euler method seen previously) according to h^2 , and with RK4 in h^4 . Although the calculation for one step will be a little longer, the total calculation time for the same global error will be immensely shorter. Rather than using only one slope, the one at the beginning of the interval, as for the Euler method, we will use four slopes judiciously distributed and weighted over the interval.

We give the general Runge-Kutta scheme for two degrees of freedom, and let you generalize. The degrees of freedom are named X and Y . For example, in physics, for a one-body motion in one direction, we would have $X=x$ and $Y=v_x$.

$X(t)$ and $Y(t)$ obey the following differential equations:

$$\frac{dX}{dt} = A(X, Y) \quad \text{and} \quad \frac{dY}{dt} = B(X, Y)$$

With the initial conditions $X(0)$ and $Y(0)$ known.

We determine the values X_{n+1} and Y_{n+1} from those of the previous rank X_n and Y_n over the interval $[nh, (n+1)h]$ with the following iterative method. For each degree of freedom we have four slopes to calculate. For example, for X , A_1 corresponds to the slope at the beginning of the interval, A_2 and A_3 are estimates of the slope in the middle of the interval, and A_4 is an estimate at the end of the interval:

$$A_1 = A(X_n, Y_n) \quad B_1 = B(X_n, Y_n)$$

$$A_2 = A\left(X_n + \frac{h}{2} A_1, Y_n + \frac{h}{2} B_1\right)$$

$$B_2 = B\left(X_n + \frac{h}{2} A_1, Y_n + \frac{h}{2} B_1\right)$$

$$A_3 = A\left(X_n + \frac{h}{2} A_2, Y_n + \frac{h}{2} B_2\right)$$

$$B_3 = B\left(X_n + \frac{h}{2} A_2, Y_n + \frac{h}{2} B_2\right)$$

$$A_4 = A(X_n + h A_3, Y_n + h B_3)$$

$$B_4 = B(X_n + h A_3, Y_n + h B_3)$$

$$X_{n+1} = X_n + \frac{h}{6} (A_1 + 2 A_2 + 2 A_3 + A_4)$$

$$Y_{n+1} = Y_n + \frac{h}{6} (B_1 + 2 B_2 + 2 B_3 + B_4)$$

We take again the case of the revolution of the Earth around the Sun with this method.

a- Establish the RK4 scheme to solve this problem: define the variables, write the differential equations of order 1 while naming the functions and the slopes.

b- Start the numerical calculation for a step of one day and compare the precision of the method with the previous simulations.

The RK4 method will now be the preferred method.

3 - Voyager 1: Establish the Runge-Kutta scheme (here we have 48 slopes to calculate for each iteration). Find the characteristics of the speed profile, the approach distances and check the values and the conservation of mechanical energy and angular momentum between two slings.

It will be necessary to adapt the step at the moment of the slings because the curvature is then important. The motion is plane and on each step you can calculate the angular variation on the osculating circle to check a good tracking of the trajectory.

4 - Voyager 3 Project: retrieve the speed profile. Adjusting the initial conditions to perfectly chain the four slings can be tedious. It can be judicious to proceed as in reality, with, for example, the use of a bit of propellant for a trajectory correction at the Uranus periastron (minimum energy consumption: powered flyby and Oberth effect).

Answers p458.

5.√ ▲▲△ Calculation of propellant masses

The aim is to retrieve all the values given during the conference.

1 - You are out for some repairs outside your space station. But a small loss of attention and you are detached from your rope drifting freely in space with your adjustable wrench in your hand. You slowly move away from the station. How could you get back?

By throwing the one kilo wrench with all your strength, it can reach a speed of 36 km/h. Your mass, including your suit, is 100 kg. What will be your speed after the throw? What quantity is conserved before and after? Is energy a quantity that is conserved? Is the kinetic energy acquired by the key the same as yours?

2 - Resume the calculation for a rocket. In this case the mass varies over time and must be integrated. The gas ejection speed is considered constant. Show how the formula fits for antimatter.

3 - In the relativistic case of *Voyage to Proxima*, calculate, for an ideal photon rocket, the antimatter masses for a round trip.

Duration of the outward journey: 3 years of proper time. Constant acceleration: 1 g.

4 - Calculate the mass of propellants required for the *Voyager 3 Project*.

Answers p473.

6. ▲△△ Planetary alignments

For the slings, the planets must have particular relative positions. We can use the alignments as markers. For example, for a slingshot around Jupiter after a departure from Earth, we start by looking for the Sun-Earth-Jupiter alignment dates. The alignments searched are approximate. Perfect alignments are very rare or do not exist. For example, the global alignment of the Earth with the Moon and the Sun happens twice a lunar month. On the other hand, exact alignments occur only at eclipse times. We consider circular and coplanar trajectories. Periods of revolution of gas giants:

$$T_{Jupiter} \simeq 11.86 \text{ yrs}$$

$$T_{Saturn} \simeq 29.44 \text{ yrs}$$

$$T_{Uranus} \simeq 84.05 \text{ yrs}$$

$$T_{Neptune} \simeq 164.86 \text{ yrs}$$

1 - Show that two planets A and B are aligned according to the period:

$$T_{AB} = \frac{T_A T_B}{T_B - T_A}$$

where B is further from the Sun than A. T_{AB} is the synodic period.

2 - Determine the Jupiter-Earth synodic period and the next alignment date with the help of ephemerides⁵⁷.

⁵⁷ Institut de Mécanique Céleste et de Calcul des Éphémérides de l'Obs. de Paris / CNRS : vo.imcce.fr/webservices/miriade/?forms
Form. : p:Earth, p:Jupiter / heliocenter / Ecliptic.

- 3 - Set a date for the Earth-Jupiter-Saturn alignment.
- 4 - How often does the alignment of the four gas giants with the Earth take place?

Answers p477.

7.√ ▲△△ **Motion of the stars**

For a quick trip to the nearby stars we can consider them fixed. In the case of slow travel over 25,000 years, we must anticipate the motion of the star to launch the probe in the direction it will be at the time of arrival. The velocity of a star is divided into its transverse and radial parts. The transverse components are known with good resolution thanks to the Hipparcos satellite, and now with the even more precise Gaia satellite, which took over in 2013. The Gaia spectrometer allows, by Doppler method, to improve the accuracy on the radial part.

1 - *Determination of the velocity of a star:*

The databases give the current distance d_0 of the star, the *proper motion* μ , and, the *radial velocity* v_r . The proper motion indicates the angular displacement per unit time. This angular change is itself split into two orthogonal components, along longitude

and latitude in equatorial coordinates: μ_α and μ_δ .

α : right ascension / δ : declination

units: milliarcseconds per year

Proxima Centauri :

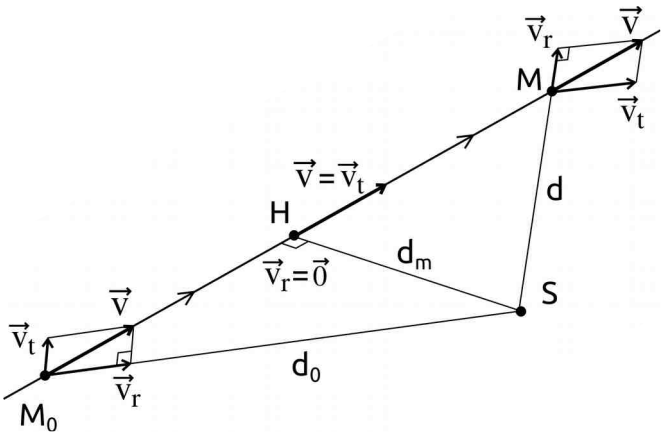
d_0 (ly)	V_{r0} (km/s)	$\mu_{\alpha 0}$ (mas/yr)	$\mu_{\delta 0}$ (mas/yr)	α_0	δ_0
4.244	-22.2	-3781.3	769.8	14h29 ^m 43 ^s	-62°40'46"

Determine μ , the tangential velocities $v_{t\alpha}$, $v_{t\delta}$, v_t , and the velocity v of the star Proxima Centauri.

What will be the equatorial coordinates of Proxima in a century?

2 - ▲▲ *Linear motion approximation:*

We neglect the Sun gravity and the galactic gravitational potential⁵⁸. At first order, the velocity vector of the star can be considered as constant. The motion of the star is then rectilinear and uniform:



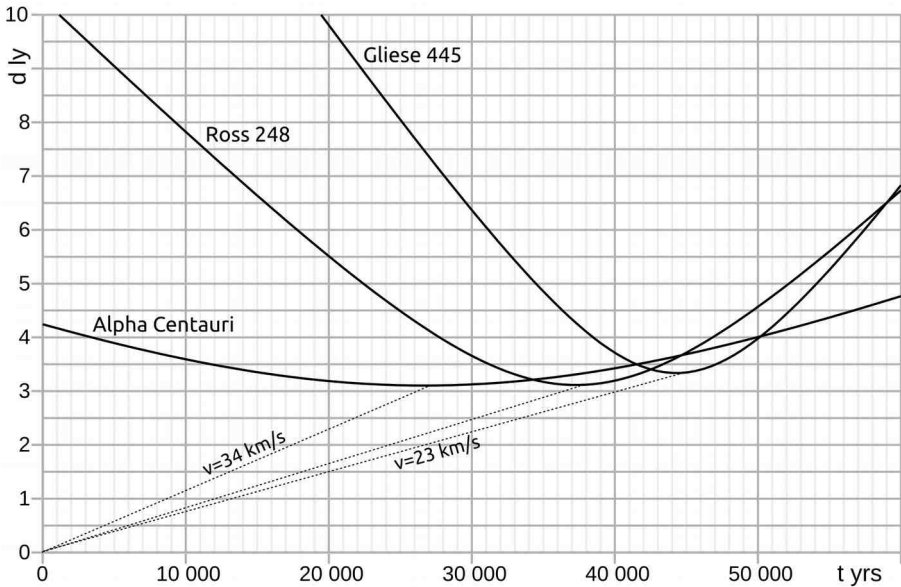
⁵⁸ *The Close Approach of Stars in the Solar Neighbourhood*, Matthews, 1993. *Close encounters of the stellar kind*, Bailer-Jones, 2014.

a- Determine the distance d of the star from the Sun as a function of time.

b- Determine the minimum approach distance d_m and the corresponding date t_m .

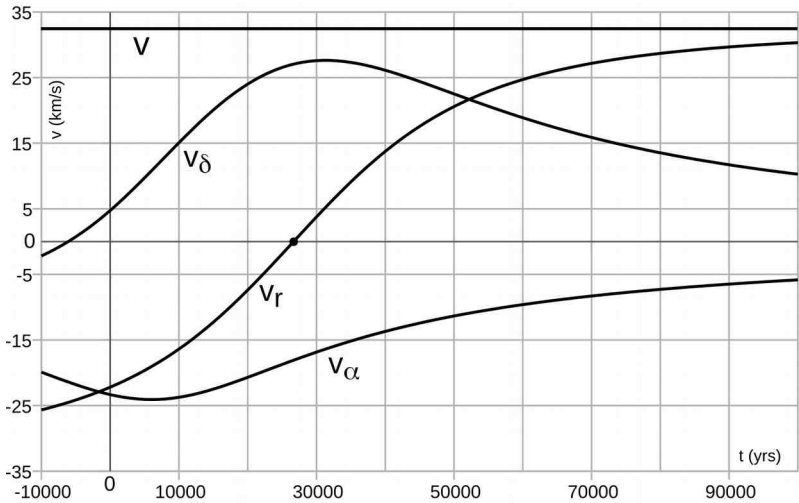
c- What are the coordinates of the star at the closest approach distance?

Distance of stars over time:



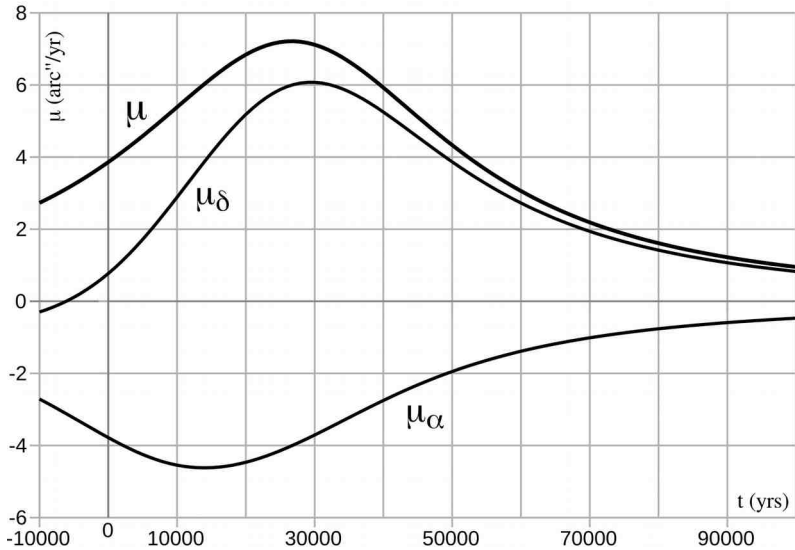
Three stars that can be reached in less than 50,000 years by a probe that uses gravitational assistance.

Radial and tangentials velocities of Proxima Centauri



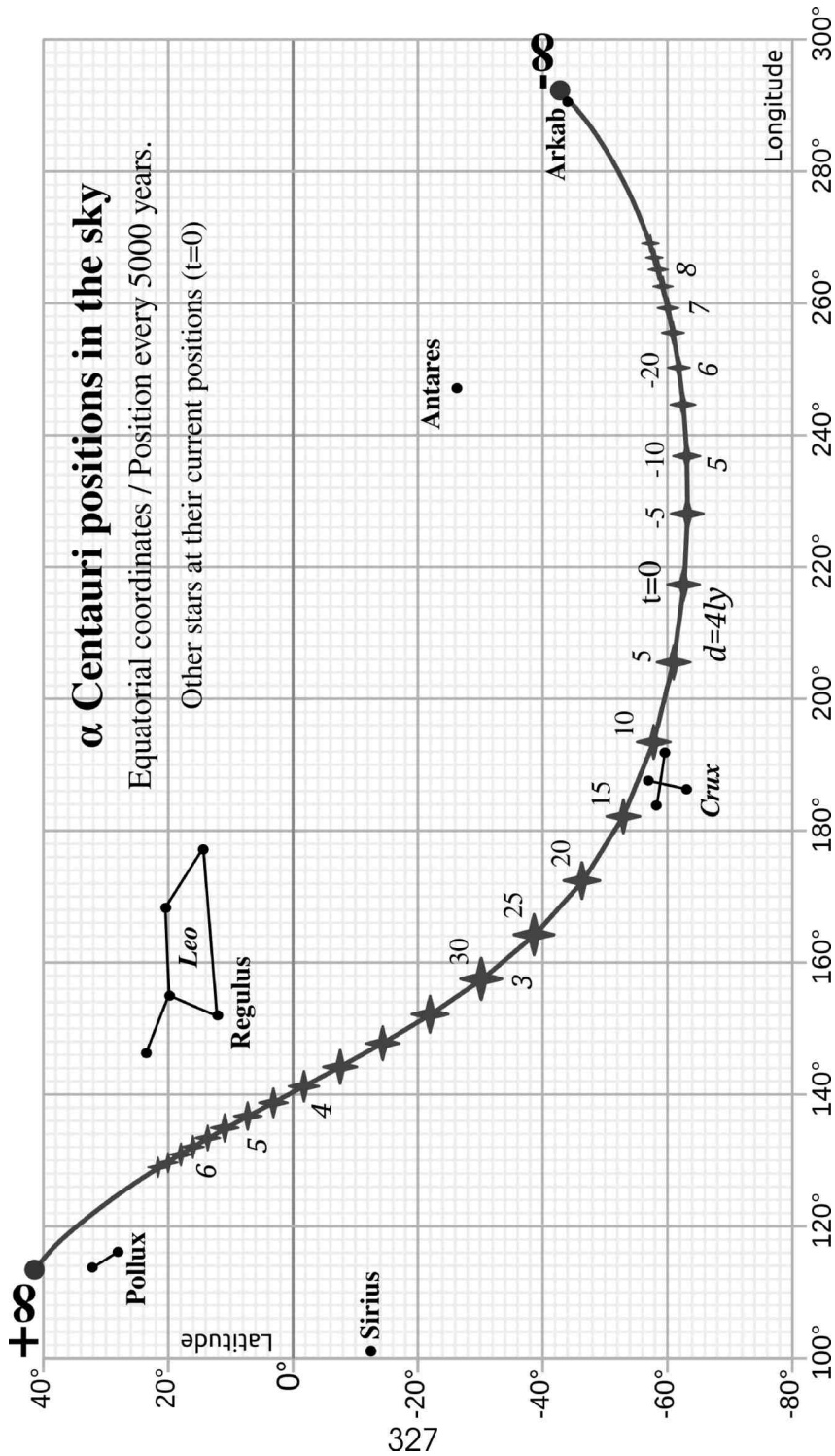
The motion of a star is rectilinear and its speed v is constant. However, its three components, normal to each other, vary with time. At the perihelion time, the radial velocity is zero and the tangential velocity is maximum: $v_t = \sqrt{v_\alpha^2 + v_\delta^2}$. At infinite times, the velocity becomes purely radial and the tangential components tend towards zero.

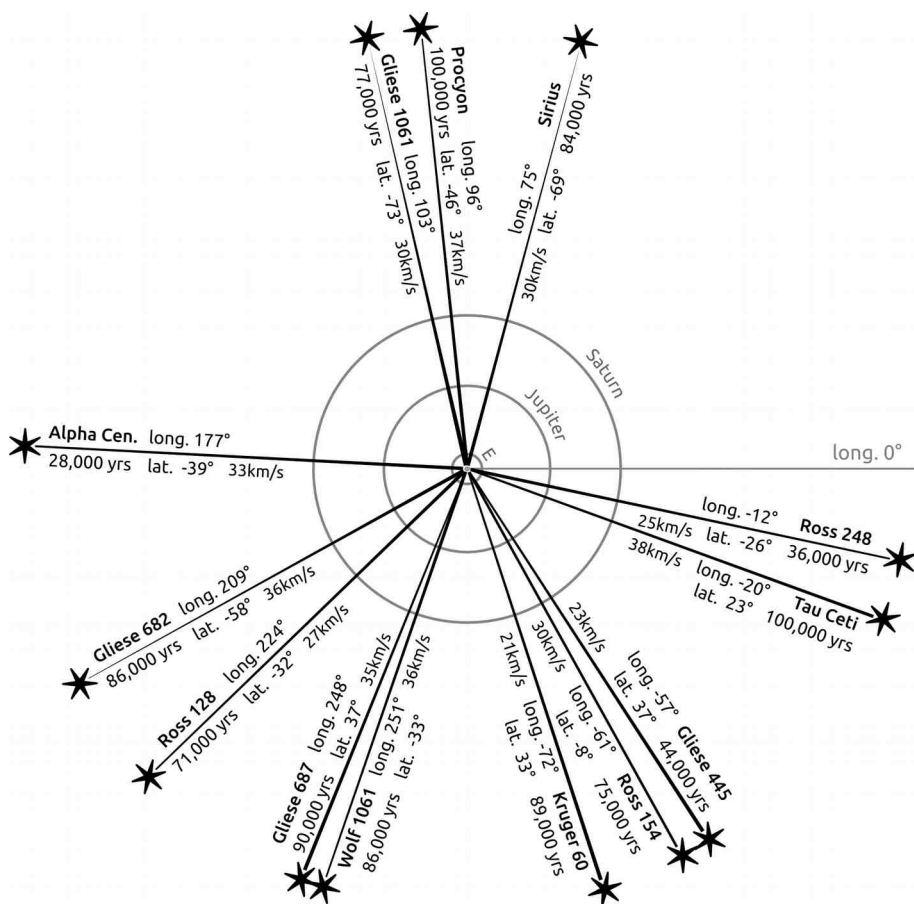
Proper motions of Proxima Centauri



Proper motion of a star for a terrestrial observer. We have the annual angular variations on the celestial sphere in equatorial coordinates of the position of a star. These proper motions are not constant and vary over the millennia. The distant stars can be considered fixed and the closer they are to our Sun, the more apparent their motion becomes.

One second of arc = one 3600th of a degree.





The position of the stars in ecliptic coordinates at the time when the spacecraft will have joined the distant star system. 13 stars at less than 100 000 years and 40 km/s.

Answers p478.

8.√ ▲▲△ Can a pair of primordial black holes be used as a stargate?

Researchers explain in a 2019 paper⁵⁹ how the existence of primordial black holes beyond Neptune's orbit would explain, both, the anomalous orbits observed for transneptunian objects, and, an excess in gravitational microlensing events observed by the OGLE experiment⁶⁰. The primordial black holes (PBH) would have been created in the first moments of the Big Bang. They could explain the origin of gamma-ray bursts and part of the dark matter. These small black holes have not yet been observed, they would be the size of a fist and a few earth masses.

In this exercise we assume the existence of such black holes beyond Neptune, and we imagine that they sometimes form pairs in rapid rotation around their barycenter.

Characteristic data for PBHs: Radius $R=4.5$ cm.

Mass $M=5 M_{\oplus}$. Distance from Sun $D=300$ au.

1 - Show how such a pair of primordial black holes could help to reach dizzying speeds by gravity assist. Could we, from there, reach Proxima in less than 50 years?

2 - As we get closer to primordial black holes, the tidal forces increase. Would a manned mission be viable?

Answers p485.

⁵⁹ *What if Planet 9 is a Primordial Black Hole?* J. Scholtz, J. Unwin.

⁶⁰ *Optical Gravitational Lensing Experiment* is a Polish astronomy project based at the University of Warsaw.

9.√ ▲▲△ Antiproton-proton collision

1 - In a particle accelerator, what must be the minimum speed of protons incident on a hydrogenated target to create a pair $p \bar{p}$?

Mass of a proton: $938 \text{ MeV}/c^2$.

2 - The same thing can happen when an antiproton collides with a proton. Do the antiprotons of cosmic rays have sufficient kinetic energies to create pairs? The quantity of cosmic protons is much greater than that of antiprotons. Could we obtain a consequent flux of \bar{p} using energetic p ?

Data on page 7 of "The discovery of geomagnetically..." and on page 13 of "Extraction of particles...": there are about 10,000 times more protons than antiprotons in this energy range.

Answers p488.

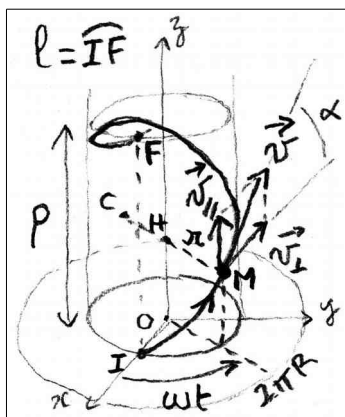
10.√ ▲▲△ Helical motion

This kinematic and geometric study will help us to interpret the dynamics of the antiproton in the Earth's magnetic field.

Parametric equations of the trajectory in Cartesian coordinates for uniform helical motion:

$$\begin{cases} x(t) = r \cos \omega t \\ y(t) = r \sin \omega t \\ z(t) = v_z t \end{cases} \quad r = cst > 0 \quad \omega = cst \quad v_z = cst$$

- 1 - Write the equations in cylindrical coordinates.
- 2 - Determine the components of the velocity \vec{v} and the acceleration \vec{a} .
- 3 - Calculation of v , a , dv/dt and the radius of curvature R .
- 4 - Relation between R , the radius $r=HM$ of the helix and the pitch p ($|\Delta z|$ for one complete helix turn).
- 5 - Calculation of the arc length l traveled by the particle on one turn as a function of: r and p , then of, v and v_{\perp} , and eventually, of R and α (angle between \vec{v} and the horizontal).



Answers p489.

11.√ ⁰¹/₁₀ ▲▲▲ The magnetosphere

The field lines of the Earth's magnetosphere are similar to that of a giant bar magnet with its south magnetic pole close to the geographic north pole.

- 1 - Show that in a magnetic field the speed of a particle is constant.

Help: In relativistic mechanics $\vec{f} = \frac{d\vec{p}}{dt} = \frac{dm\gamma\vec{v}}{dt}$ and we have, here, for the Lorentz force $\vec{f} = q(\vec{E} + \vec{v} \wedge \vec{B})$.

For the energy aspect $\vec{f} \cdot \vec{v} = \frac{dE}{dt}$ with $E = T + mc^2$.

2 - Give the trajectory of a charged particle in a uniform magnetic field.

3 - Give the shape of the field lines of a magnetic dipole. Characteristics and components of the magnetic field of a dipole in spherical coordinates.

4 - Show the mirror effect on the example of a narrowing field tube.

5 - Show the drift phenomenon in the simple case of two areas with uniform magnetic fields of different intensities.

6 - Trapped antiproton: We will carry out a numerical simulation with the Runge-Kutta method of order 4 (method described page 313).

a- Establish the expression of the components of the magnetic field of a dipole in Cartesian coordinates.

b- Give the equations of motion of a charged particle in a magnetic field.

c- Write the RK4 scheme.

d- Carry out the numerical simulation. On a spreadsheet it can be too computationally intensive. In this case we preferred to program in php and to make the calculations on server.

Answers p490.

12.✓ ▲▲▲ Penning trap

This charged particle trap, designed in 1936, uses a quadrupole electric field and a uniform magnetic field. Penning traps are commonly used at CERN to store antiprotons. The electric field is created by a set of electrodes that follow the hyperboloidal equipotentials of the quadrupole. The globally uniform magnetic field in the storage area is the one created inside a solenoid.

1 - Expression of the electric field:

$$\vec{E} = \frac{U_0}{r_0^2} (-x \vec{i} - y \vec{j} + 2z \vec{k})$$

Show that \vec{E} derives from a potential that we will determine.

2 - Show that the origin O is an equilibrium position. Discuss the stability along the (Oz) axis and then in the plane (Oxy). Calculate the pulsation ω_z of the oscillations along Oz.

3 - To stabilize the trajectory of the antiproton we add a uniform magnetic field:

$$\vec{B} = B_0 \vec{k}$$

a - Is the motion along (Oz) modified?

b - According to (xOy): show that the antiproton is trapped if B_0 is greater than a critical value B_c to be determined (to do this, establish the differential equation verified by $\rho = x + jy$, $j^2 = -1$, with $\omega_c = e B_0 / m$).

c - Solve and highlight two angular frequencies ω_c and ω_m (magnetron frequency).

Numerical Applications: $U_0=9.3 \text{ V}$, $r_0=29.1 \text{ cm}$, $B_0=0.55 \text{ T}$, $e=1.6 \times 10^{-19} \text{ C}$, $m_p=1.67 \times 10^{-27} \text{ kg}$.

d - Plot the trajectory.

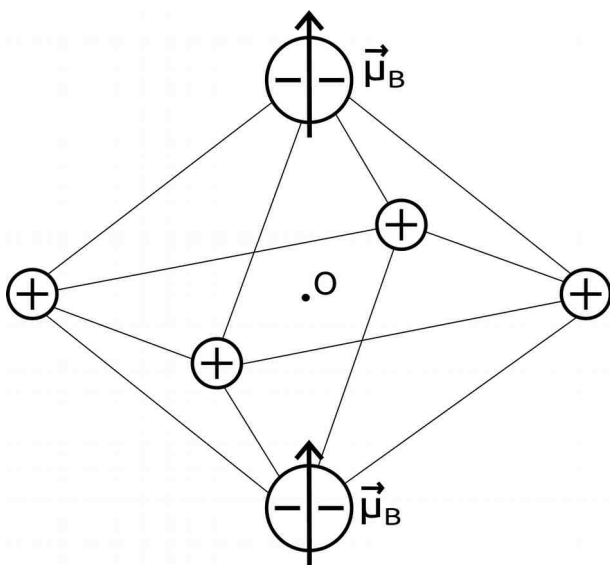
4 - Microscopic cage: Could we create a Penning trap at the microscopic scale? We are going to propose a model to try to give elements of an answer. For the quadrupole electric field we can use cations and anions. For the magnetic field we have paramagnetic atoms which have a permanent magnetic moment (iron is an example among many others). Let's take six atoms arranged in a bipyramid with a square base. The two atoms at the vertices have a charge $2\ominus$ and an elementary magnetic moment μ_B . The four atoms at the base are cations of elemental charge \oplus .

Data (usual order of magnitude):

Edges of the regular octahedron equal to: $a=100 \text{ pm}$.

Elementary charge: $e=1.6 \times 10^{-19} \text{ C}$. $\epsilon_0=8.85 \times 10^{-12} \text{ C}^2 \cdot \text{m}^{-2} \cdot \text{N}^{-1}$.

Elementary magnetic moment: that created by a classical electron orbiting in a hydrogen atom, called Bohr magneton: $\mu_B=9.27 \times 10^{-24} \text{ A} \cdot \text{m}^2$. All atomic magnetic moments are equivalent to a few elementary magnetons (orbital and spin moments combined).



Representation of a hypothetical microscopic Penning trap within a crystal lattice or molecular structure. The paramagnetic atoms placed at the top and bottom create a globally uniform magnetic field around the center O. These atoms at the apexes of the bipyramid correspond to the upper and lower caps of a macroscopic Penning trap, and the cations at the square base, to the ring electrode.

a - Show that this atomic structure is not a monopole, nor an electric dipole.

b - Evaluate the magnetic field B_0 created at the center of the bipyramid. You can use the expressions on page 491.

c - Estimate the factor U_0/r_0^2 .
You can consider the Oz axis to identify the expressions.

d - Is the magnetic field sufficient to trap an antiproton?
Conclusions.

Answers p507.

Answers

1

1. The Crystals of the Pop Exomoon (Barnard system)

Exercise p25.

Distance and relative time
measured in the galactic
frame R' .

Arrival at the galactic
year **2120**

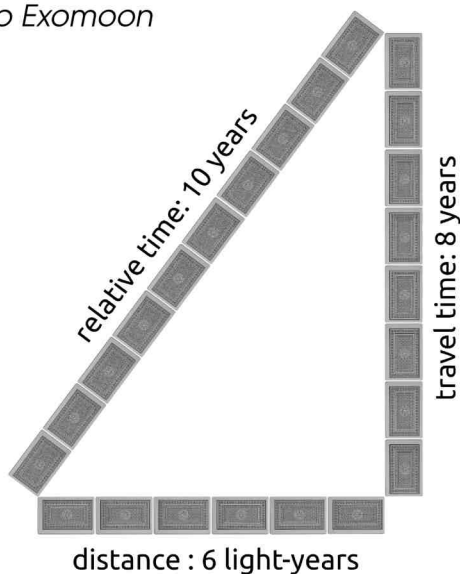
(=2010 + 10).

Rocket speed in R' :

$v = 6/10 c$

= **60% of c** .

It is a double
triangle of the 3-4-5.



2. One-way ticket for Sirius with an old $\beta 6$

(Exercise p26)

speed = (distance / time) $_{R'}$

then: relative time = 9 l.y. / 60% (See vessel
characteristics), so:

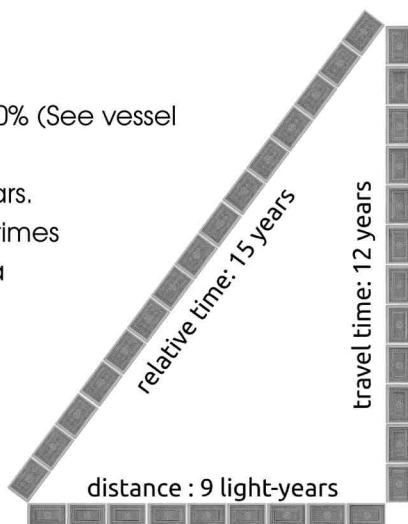
$\Delta t' = 9 / 0.6 = 9 \times 10 / 6 = 15$ years.

We try to build a triangle of times
with a base of 9 cards and a
hypotenuse of 15 cards. The
proper time is 12 years.

Arrival at **42 years** (=30+12).

Arrival date: 2169 (2154 +

15). You arrive one
year after the first



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Articles and docs: www.voyagepourproxima.fr/docs/

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Graphics: *Gimp*, *Inkscape* and *Blender*.

Operating system: *Ubuntu*.

Programming in *php* on *Linux* server.

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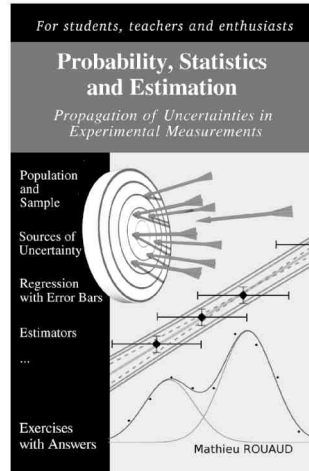
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